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*Research Article*

# Strong Convergence Theorems for Maximal Monotone Operators with Nonspreading Mappings in a Hilbert Space

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We prove the strong convergence theorems for finding a common element of the set of fixed points of a nonspreading mapping $T$ and the solution sets of zero of a maximal monotone mapping and an $\alpha$-inverse strongly monotone mapping in a Hilbert space. Manaka and Takahashi (2011) proved weak convergence theorems for maximal monotone operators with nonspreading mappings in a Hilbert space; there we introduced new iterative algorithms and got some strong convergence theorems for maximal monotone operators with nonspreading mappings in a Hilbert space.

## 1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let $C$ be a nonempty closed convex subset of $H$. We denote by $F(T)$ the set of fixed point of $T$. Then, a mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The mapping $T : C \to C$ is said to be firmly nonexpansive if $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle$ for all $x, y \in C$; see, for instance, Browder [1] and Goebel and Kirk [2]. The mapping $T : C \to C$ is said to be firmly nonspreading [3] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \tag{1.1}$$

for all $x, y \in C$. Iemoto and Takahashi [4] proved that $T : C \to C$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \tag{1.2}$$

for all $x, y \in C$. It is not hard to know that a nonspreading mapping is deduced from a firmly nonexpansive mapping; see [5, 6], and a firmly nonexpansive mapping is a nonexpansive mapping.

Many studies have been done for structuring the fixed point of nonexpansive mapping $T$. In 1953, Mann [7] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \tag{1.3}$$

where the initial guess $x_1 \in C$ is arbitrary and $\{a_n\}$ is a real sequence in $[0, 1]$. It is known that under appropriate settings, the sequence $\{x_n\}$ converges weakly to a fixed point of $T$. However, even in a Hilbert space, Mann iteration may fail to converge strongly, for example see [8].

Some attempts to construct iteration method guaranteeing the strong convergence have been made. For example, Halpern [9] proposed the following so-called Halpern iteration:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \tag{1.4}$$

where $u, x_1 \in C$ are arbitrary and $\{a_n\}$ is a real sequence in $[0, 1]$ which satisfies $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$. Then, $\{x_n\}$ converges strongly to a fixed point of $T$; see [9, 10].

In 1975, Baillon [11] first introduced the nonlinear ergodic theorem in Hilbert space as follows:

$$S_n x = \sum_{k=0}^{n-1} T^k x \tag{1.5}$$

converges weakly to a fixed point of $T$ for some $x \in C$.

Recently, in the case when $T : C \to C$ is a nonexpansive mapping, $A : C \to H$ is an $\alpha$-inverse strongly monotone mapping, and $B \in H \times H$ is a maximal monotone operator, Takahashi et al. [12] proved a strong convergence theorem for finding a point of $F(T) \cap (A + B)^{-1}(0)$, where $F(T)$ is the set of fixed points of $T$ and $(A + B)^{-1}(0)$ is the set of zero points of $A + B$.

In 2011, Manaka and Takahashi [13] for finding a point of the set of fixed points of $T$ and the set of zero points of $A + B$ in a Hilbert space, they introduced an iterative scheme as follows:

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) T(J_{\lambda_n}(I - \lambda_n A) x_n), \tag{1.6}$$

where $T$ is a nonspreading mapping, $A$ is an $\alpha$-inverse strongly monotone mapping, and $B$ is a maximal monotone operator such that $J_\lambda = (I - \lambda B)^{-1}$; $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences which satisfy $0 < c \leq \beta_n \leq d < 1$ and $0 < a \leq \lambda_n \leq b < 2\alpha$. Then they proved that $\{x_n\}$ converges weakly to a point $p = \lim_{n\to\infty} P_{F(T)\cap(A+B)^{-1}(0)} x_n$.

Motivated by above authors, we generalize and modify the iterative algorithms (1.5) and (1.6) for finding a common element of the set of fixed points of a nonspreading mapping $T$ and the set of zero points of monotone operator $A + B$ ($A$ is an $\alpha$-inverse strongly monotone mapping, and $B$ is a maximal monotone operator). First, we prove that the sequence generated by our iterative method is weak convergence under the property conditions. Then, we prove that the strong convergence in a Hilbert space. As expected, we get some weak and strong convergence theorems about the common element of the set of fixed points of a nonspreading mapping and the set of zero points of an $\alpha$-inverse strongly monotone mapping and a maximal monotone operator in a Hilbert space.

## 2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $C$ be a nonempty closed convex subset of $H$. A set-valued mapping $B : D(B) \subseteq H \rightarrow H$ is said to be monotone if for any $x, y \in D(B)$ and $x^* \in Bx$ and $y^* \in By$, it holds that

$$\langle x - y, x^* - y^* \rangle \geq 0. \tag{2.1}$$

A monotone operator $B$ on $H$ is said to be maximal if $B$ has no monotone extension, that is, its graph is not properly contained in the graph of any other monotone operator on $H$. For a maximal monotone operator $B$ on $H$ and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} : 2^H \rightarrow D(B)$, which is called the resolvent of $B$ for $r > 0$. Let $B$ be a maximal monotone operator on $H$, and let $B^{-1}(0) = \{x \in H : 0 \in Bx\}$. For a constant $\alpha > 0$, the mapping $A : C \rightarrow H$ is said to be an $\alpha$-inverse strongly monotone if for any for all $x, y \in C$,

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2. \tag{2.2}$$

*Remark 2.1.* It is not hard to know that if $A$ is an $\alpha$-inverse strongly monotone mapping, then it is $1/\alpha$-Lipschitzian and hence uniformly continuous. Clearly, the class of monotone mappings include the class of an $\alpha$-inverse strongly monotone mappings.

*Remark 2.2.* It is well known that if $T : C \rightarrow C$ is a nonexpansive mapping, then $I - T$ is $1/2$-inverse strongly monotone, where $I$ is the identity mapping on $H$; see, for instance, [14]. It is known that the resolvent $J_r$ is firmly nonexpansive and $B^{-1}(0) = F(J_r)$ for all $r > 0$.

For a single-valued mapping $T$, a point $p$ is called a fixed point of $T$ if $p = Tp$. For a multivalued mapping $T$, a point $p$ is called a fixed point of $T$ if $p \in Tp$. The set of fixed points of $T$ is denoted by $F(T)$.

Let $E$ be a uniformly convex real Banach space, $K$ be a nonempty closed convex subset of $E$. A multivalued mapping $T : K \rightarrow CB(K)$ is said to be as follows.

(i) Contraction if there exists a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq k\|x - y\|, \quad \forall x, y \in K. \tag{2.3}$$

(ii) Nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, \quad \forall x, y \in K. \tag{2.4}$$

(iii) Quasinonexpansive if $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq \|x - p\|, \quad \forall x \in K, \ \forall p \in F(T). \tag{2.5}$$

It is well known that every nonexpansive multivalued mapping $T$ with $F(T) \neq \emptyset$ is multivalued quasinonexpansive. But there exist multivalued quasi-nonexpansive mappings that are not multivalued nonexpansive. It is clear that if $T$ is a quasi-nonexpansive multivalued mapping, then $F(T)$ is closed.

A Banach space $E$ is said to satisfy Opials condition if whenever $\{x_n\}$ is a sequence in $E$ which converges weakly to $x$, then

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \quad \forall y \in E, \ x \neq y. \tag{2.6}$$

**Lemma 2.3** (Manaka and Takahashi [13]). *Let $H$ be a real Hilbert space, and let $C$ be a nonempty closed convex subset of $H$. Let $\alpha > 0$. Let $A$ be an $\alpha$-inverse strongly monotone mapping of $C$ into $H$, and let $B$ be a maximal monotone operator on $H$ such that the domain of $B$ is included in $C$. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of $B$ for any $\lambda > 0$. Then, the following hold*

(i) *if $u, v \in (A + B)^{-1}(0)$, then $Au = Av$;*

(ii) *for any $\lambda > 0$, $u \in (A + B)^{-1}(0)$ if and only if $u = J_\lambda(I - \lambda A)u$.*

**Lemma 2.4** (Schu [15]). *Suppose that $E$ is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers $n$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of $E$ such that $\limsup_{n \to \infty} \|x_n\| \leq r$, $\limsup_{n \to \infty} \|y_n\| \leq r$, and $\lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r \geq 0$. Then, $\lim_{n \to \infty} \|x_n - y_n\| = 0$.*

**Lemma 2.5** (Liu [16] and Xu [17]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property as follows*

$$a_{n+1} \leq (1 - t_n) a_n + b_n + t_n c_n, \tag{2.7}$$

*where $\{t_n\}$, $\{b_n\}$, and $\{c_n\}$ satisfy the restrictions as follows*

(i) $\sum_{n=0}^{\infty} t_n = \infty$,

(ii) $\sum_{n=0}^{\infty} b_n < \infty$,

(iii) $\limsup_{n \to \infty} c_n \leq 0$.

*Then, $\{a_n\}$ converges to zero as $n \to \infty$.*

## 3. Strong Convergence Theorem

In this section, we prove the strong convergence theorems for finding a common element in common set of the fixed sets of a nonspreading mapping and the solution sets of zero of a maximal monotone operator and an $\alpha$-inverse strongly monotone operator and in a Hilbert space.

**Theorem 3.1.** *Let $C$ be a nonempty convex closed subset of a real Hilbert space $H$, let $A : C \to H$ be an $\alpha$-inverse strongly monotone, let $B : D(B) \subseteq C \to 2^H$ be maximal monotone, let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of $B$ for any $\lambda > 0$, and let $T : C \to C$ be a nonspreading mapping. Assume that $F := F(T) \cap (A+B)^{-1}(0) \neq \emptyset$. We define*

$$\begin{aligned} x_1 &= x \in C, \text{ arbitrarily}, \\ z_n &= J_{\lambda_n}(I - \lambda_n A)x_n, \\ y_n &= \frac{1}{n}\sum_{k=1}^{n} T^k z_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n, \end{aligned} \tag{3.1}$$

*where $\{\alpha_n\}$ is sequences in $[0,1]$ such that $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$. There exists $a$, $b$ such that $0 < a \le \lambda_n \le b < 2\alpha$ for each $n \in N$. Then, $\{x_n\}$ converges strongly to $Pu$, and $P$ is the metric projection of $H$ onto $F$.*

*Proof.* First, we prove that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in F(T)$. In fact, from Lemma 2.3, we have $p = J_{\lambda_n}(I - \lambda_n A)p$, together with (3.1) and $A$ is an $\alpha$-inverse strongly monotone, we get that

$$\begin{aligned} \|z_n - p\|^2 &= \|J_{\lambda_n}(I - \lambda_n A)x_n - J_{\lambda_n}(I-\lambda_n A)p\|^2 \\ &\le \|(I-\lambda_n A)x_n - (I - \lambda_n A)p\|^2 \\ &= \|x_n - p\|^2 - 2\lambda_n\langle x_n - p, Ax_n - Ap\rangle + \lambda_n^2\|Ax_n - Ap\|^2 \\ &\le \|x_n - p\|^2 - 2\lambda_n\alpha\|Ax_n - Ap\|^2 + \lambda_n^2\|Ax_n - Ap\|^2 \\ &= \|x_n - p\|^2 - \lambda_n(2\alpha - \lambda_n)\|Ax_n - Ap\|^2 \\ &\le \|x_n - p\|^2. \end{aligned} \tag{3.2}$$

From the definition of $y_n$ and $T$ is nonspreading mapping, we obtain that

$$\begin{aligned} \|y_n - p\| &= \left\|\frac{1}{n}\sum_{k=0}^{n-1} T^k z_n - p\right\| \le \frac{1}{n}\sum_{k=0}^{n-1}\left\|T^k z_n - p\right\| \le \frac{1}{n}\sum_{k=0}^{n-1}\|z_n - p\| \\ &= \|z_n - p\| \le \|x_n - p\|. \end{aligned} \tag{3.3}$$

Together with (3.1), we have that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n u + (1-\alpha_n)y_n - p\| \\ &\le \alpha_n\|u - p\| + (1-\alpha_n)\|y_n - p\| \\ &\le \alpha_n\|u - p\| + (1-\alpha_n)\|x_n - p\|. \end{aligned} \tag{3.4}$$

Hence, we get that

$$\|x_{n+1} - p\| \leq \max\{\|u - p\|, \|x_n - p\|\}, \tag{3.5}$$

for all $n \in N$. This means that $\{x_n - p\}$ is bounded, so $\{x_n\}$ is bounded. From $T$ is nonspreading, (3.3), and (3.2), we get that $\{y_n\}$, $\{z_n\}$, and $\{T^n z_n\}$ are all bounded.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty}\|x_{n_k} - p\|$ exists. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}} \rightharpoonup w \in C$ as $i \to \infty$. Now, we prove that $w \in F$. First, we prove that $w \in F(T)$. Since $\|x_{n+1} - y_n\| = \alpha_n\|u - y_n\|$, replacing $n$ by $n_{k_i}$, we have $\|x_{n_{k_i}+1} - y_{n_{k_i}}\| = \alpha_{n_{k_i}}\|u - y_{n_{k_i}}\|$. Together with $\alpha_n \to 0$ and $\{y_n\}$ is bounded, we obtain that $\lim_{i\to\infty}\|x_{n_{k_i}+1} - y_{n_{k_i}}\| = 0$, so we have $y_{n_{k_i}} \rightharpoonup w$.

Let $n \in N$. Since $T$ is nonspreading, we have that for all $y \in C$ and $k = 0, 1, 2, \ldots, n-1$,

$$\begin{aligned} \left\|T^{k+1}z_n - Ty\right\|^2 &\leq \left\|T^k z_n - y\right\|^2 + 2\left\langle T^k z_n - T^{k+1}z_n, y - Ty\right\rangle \\ &= \left\|T^k z_n - Ty\right\|^2 + \|Ty - y\|^2 + 2\left\langle T^k z_n - Ty, Ty - y\right\rangle \\ &\quad + 2\left\langle T^k z_n - T^{k+1}z_n, y - Ty\right\rangle. \end{aligned} \tag{3.6}$$

Summing these inequalities from $k = 0$ to $n - 1$ and dividing by $n$, we have

$$\frac{1}{n}\left(\|T^n z_n - Ty\|^2 - \|z_n - Ty\|^2\right) \leq \|Ty - y\|^2 + 2\langle y_n - Ty, Ty - y\rangle + \frac{2}{n}\langle z_n - T^n z_n, y - Ty\rangle. \tag{3.7}$$

Replacing $n$ by $n_{k_i}$, we have

$$\begin{aligned} &\frac{1}{n_{k_i}}\left(\left\|T^{n_{k_i}}z_{n_{k_i}} - Ty\right\|^2 - \left\|z_{n_{k_i}} - Ty\right\|^2\right) \\ &\quad\leq \|Ty - y\|^2 + 2\left\langle y_{n_{k_i}} - Ty, Ty - y\right\rangle \\ &\qquad + \frac{2}{n_{k_i}}\left\langle z_{n_{k_i}} - T^{n_{k_i}}z_{n_{k_i}}, y - Ty\right\rangle. \end{aligned} \tag{3.8}$$

Since $\{z_n\}$ and $\{T^n z_n\}$ are bounded, we have that

$$0 \leq \|Ty - y\|^2 + 2\langle w - Ty, Ty - y\rangle \tag{3.9}$$

as $i \to \infty$. Putting $y = w$, we have

$$0 \leq \|Tw - w\|^2 + 2\langle w - Tw, Tw - w\rangle = -\|Tw - w\|^2. \tag{3.10}$$

Hence, $w \in F(T)$.

Next, we prove that $w \in (A+B)^{-1}(0)$. From (3.2) and (3.3) we have that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \alpha_n\|u-p\|^2+(1-\alpha_n)\|y_n-p\|^2 \\
&\leq \alpha_n\|u-p\|^2+(1-\alpha_n)\|z_n-p\|^2 \\
&\leq \alpha_n\|u-p\|^2+(1-\alpha_n)\left(\|x_n-p\|^2-\lambda_n(2\alpha-\lambda_n)\|Ax_n-Ap\|^2\right) \\
&= \alpha_n\left(\|u-p\|^2-\|x_n-p\|^2\right)+\|x_n-p\|^2-(1-\alpha_n)\lambda_n(2\alpha-\lambda_n)\|Ax_n-Ap\|^2.
\end{aligned} \tag{3.11}$$

We rewrite above inequality as follows:

$$(1-\alpha_n)\lambda_n(2\alpha-\lambda_n)\|Ax_n-Ap\|^2 \leq \alpha_n\left(\|u-p\|^2-\|x_n-p\|^2\right)+\|x_n-p\|^2-\|x_{n+1}-p\|^2. \tag{3.12}$$

Replacing $n$ by $n_k$, we have

$$\begin{aligned}
&(1-\alpha_{n_k})\lambda_{n_k}(2\alpha-\lambda_{n_k})\|Ax_{n_k}-Ap\|^2 \\
&\quad\leq \alpha_{n_k}\left(\|u-p\|^2-\|x_{n_k}-p\|^2\right) \\
&\qquad+\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2.
\end{aligned} \tag{3.13}$$

Together with $\lim_{n\to\infty}\alpha_n=0$, $0<a\leq\lambda_n\leq b<2\alpha$ and since $\lim_{k\to\infty}\|x_{n_k}-p\|$ exists, we obtain that

$$\lim_{k\to\infty}\|Ax_{n_k}-Ap\|=0. \tag{3.14}$$

Since $J_{\lambda_n}$ is firmly nonexpansive, and from (3.2), we have that

$$\begin{aligned}
\|z_n-p\|^2 &= \|J_{\lambda_n}(I-\lambda_nA)x_n-J_{\lambda_n}(I-\lambda_nA)p\|^2 \\
&\leq \langle z_n-p,(I-\lambda_nA)x_n-(I-\lambda_nA)p\rangle \\
&= \frac{1}{2}\Big\{\|z_n-p\|^2+\|(I-\lambda_nA)x_n-(I-\lambda_nA)p\|^2 \\
&\qquad -\|z_n-p-(I-\lambda_nA)x_n+(I-\lambda_nA)p\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|z_n-p\|^2+\|x_n-p\|^2-\|z_n-p-(I-\lambda_nA)x_n+(I-\lambda_nA)p\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|z_n-p\|^2+\|x_n-p\|^2-\|z_n-x_n\|^2-2\lambda_n\langle z_n-x_n,Ax_n-Ap\rangle-\lambda_n^2\|Ax_n-Ap\|^2\Big\}.
\end{aligned} \tag{3.15}$$

This means that

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \|z_n - x_n\|^2 - 2\lambda_n\langle z_n - x_n, Ax_n - Ap\rangle - \lambda_n^2\|Ax_n - Ap\|^2. \tag{3.16}$$

Together with (3.1) and (3.3), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\|u - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\le \alpha_n\|u - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\
&\le \alpha_n\|u - p\|^2 + (1-\alpha_n) \\
&\quad \times \left\{\|x_n - p\|^2 - \|z_n - x_n\|^2 - 2\lambda_n\langle z_n - x_n, Ax_n - Ap\rangle - \lambda_n^2\|Ax_n - Ap\|^2\right\} \\
&\le \alpha_n\|u - p\|^2 + \|x_n - p\|^2 - \|z_n - x_n\|^2 \\
&\quad - 2\lambda_n\langle z_n - x_n, Ax_n - Ap\rangle - \lambda_n^2\|Ax_n - Ap\|^2.
\end{aligned} \tag{3.17}$$

Therefore, we have

$$\begin{aligned}
\|z_n - x_n\|^2 \le \alpha_n\|u - p\|^2 &+ \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&- 2\lambda_n\langle z_n - x_n, Ax_n - Ap\rangle - \lambda_n^2\|Ax_n - Ap\|^2.
\end{aligned} \tag{3.18}$$

Replacing $n$ by $n_k$, we have

$$\begin{aligned}
\|z_{n_k} - x_{n_k}\|^2 \le \alpha_{n_k}\|u - p\|^2 &+ \|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2 \\
&- 2\lambda_{n_k}\langle z_{n_k} - x_{n_k}, Ax_{n_k} - Ap\rangle - \lambda_{n_k}^2\|Ax_{n_k} - Ap\|^2.
\end{aligned} \tag{3.19}$$

Since $\lim_{k\to\infty}\|x_{n_k} - p\|$ exists, from (3.14) and $\lim_{n\to\infty}\alpha_n = 0$, we obtain

$$\lim_{n\to\infty}\|z_{n_k} - x_{n_k}\| = 0. \tag{3.20}$$

Since $A$ is Lipschitz continuous, we also obtain

$$\lim_{n\to\infty}\|Az_{n_k} - Ax_{n_k}\| = 0. \tag{3.21}$$

By the definition of $J_{\lambda_n}$ and (3.1), we have that

$$\begin{aligned} z_n &= (I-\lambda_n B)^{-1}(I-\lambda_n A)x_n \\ &\Longleftrightarrow (I-\lambda_n A)x_n \in (I-\lambda_n B)z_n = z_n + \lambda_n B z_n \\ &\Longleftrightarrow x_n - z_n - \lambda_n A x_n \in \lambda_n B z_n \\ &\Longleftrightarrow \frac{1}{\lambda_n}(x_n - z_n - \lambda_n A x_n) \in B z_n. \end{aligned} \tag{3.22}$$

Since $B$ is monotone, so for $(e,f) \in B$, we have that

$$\left\langle z_n - e, \frac{1}{\lambda_n}(x_n - z_n - \lambda_n A x_n) - f \right\rangle \geq 0, \tag{3.23}$$

and hence

$$\langle z_n - e, x_n - z_n - \lambda_n (A x_n + f) \rangle \geq 0. \tag{3.24}$$

Replacing $n$ by $n_{k_i}$, we have that

$$\left\langle z_{n_{k_i}} - e, x_{n_{k_i}} - z_{n_{k_i}} - \lambda_{n_{k_i}} \left( A x_{n_{k_i}} + f \right) \right\rangle \geq 0. \tag{3.25}$$

Since $A$ is an $\alpha$-inverse strongly monotone, we have

$$\left\langle x_{n_{k_i}} - w, A x_{n_{k_i}} - A w \right\rangle \geq \alpha \left\| A x_{n_{k_i}} - A w \right\|^2. \tag{3.26}$$

This means that $A x_{n_{k_i}} \to A w$ as $i \to \infty$. From (3.20) and $x_{n_{k_i}} \rightharpoonup w$, we get that $z_{n_{k_i}} \rightharpoonup w$, together with (3.25), we have that

$$\langle w - e, -A w - f \rangle \geq 0. \tag{3.27}$$

Since $B$ is maximal monotone, so $(-Aw) \in Bw$. That is, $w \in (A+B)^{-1}(0)$.

Now, we prove that $x_n \to Pu$ as $n \to \infty$. Without loss of generality, we may assume that there exists a subsequence $\{x_{n_{k_i}+1}\}$ of $\{x_{n+1}\}$ such that

$$\limsup_{n\to\infty} \langle u - Pu, x_{n+1} - Pu \rangle = \lim_{i\to\infty} \left\langle u - Pu, x_{n_{k_i}+1} - Pu \right\rangle. \tag{3.28}$$

Since $P$ is the metric projection of $H$ onto $F$ and $x_{n_{k_i}+1} \rightharpoonup w \in F$, we have

$$\lim_{i\to\infty} \left\langle u - Pu, x_{n_{k_i}+1} - Pu \right\rangle = \langle u - Pu, w - Pu \rangle \leq 0. \tag{3.29}$$

This implies that

$$\lim_{n\to\infty} \langle u - Pu, x_{n+1} - Pu \rangle \leq 0. \tag{3.30}$$

From (2.1), (3.1), and (3.3), we have

$$\begin{aligned} \|x_{n+1} - Pu\|^2 &= \left\|(1-\alpha_n)(y_n - Pu) + \alpha_n(u - Pu)\right\|^2 \\ &\leq (1-\alpha_n)^2\left\|y_n - Pu\right\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle \\ &\leq (1-\alpha_n)\|x_n - Pu\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle. \end{aligned} \tag{3.31}$$

From Lemma 2.5 and (3.30), we have

$$\lim_{n\to\infty} \|x_n - Pu\| = 0. \tag{3.32}$$

This means that $x_n \to Pu$ as $n \to \infty$. □

## References


[1] F. E. Browder, “Convergence theorems for sequences of nonlinear operators in Banach spaces,” *Mathematische Zeitschrift*, vol. 100, pp. 201–225, 1967.

[2] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, vol. 28 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.

[3] F. Kohsaka and W. Takahashi, “Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces,” *Archiv der Mathematik*, vol. 91, no. 2, pp. 166–177, 2008.

[4] S. Iemoto and W. Takahashi, “Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space,” *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 71, no. 12, pp. e2082–e2089, 2009.

[5] T. Igarashi, W. Takahashi, and K. Tanaka, “Weak convergence theorems for nonspreading mappings and equilibrium problems,” in *Nonlinear Analysis and Optimization*, S. Akashi, W. Takahashi, and T. Tanaka, Eds., pp. 75–85, Yokohama Publishers, Yokohama, Japan, 2009.

[6] W. Takahashi, “Fixed point theorems for new nonlinear mappings in a Hilbert space,” *Journal of Nonlinear and Convex Analysis*, vol. 11, no. 1, pp. 79–88, 2010.

[7] W. R. Mann, “Mean value methods in iteration,” *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.

[8] A. Genel and J. Lindenstrauss, “An example concerning fixed points,” *Israel Journal of Mathematics*, vol. 22, no. 1, pp. 81–86, 1975.

[9] B. Halpren, “Fixed points of nonexpansive maps,” *Bulletin of the American Mathematical Society*, vol. 73, pp. 957–961, 1967.

[10] R. Wittmann, “Approximation of fixed points of nonexpansive mappings,” *Archiv der Mathematik*, vol. 58, no. 5, pp. 486–491, 1992.

[11] J. B. Baillon, “Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert,” *Comptes Rendus De L'Académie Des Sciences A-B*, vol. 280, pp. A1511–A1514, 1975.

[12] S. Takahashi, W. Takahashi, and M. Toyoda, “Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces,” *Journal of Optimization Theory and Applications*, vol. 147, no. 1, pp. 27–41, 2010.

[13] H. Manaka and W. Takahashi, “Weak convergence theorems for maximal monotone operators with nonspreading mappings in a Hilbert space,” *Cubo*, vol. 13, no. 1, pp. 11–24, 2011.

[14] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.

[15] J. Schu, "Weak and strong convergence to fixed points of asymptotically nonexpansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 43, no. 1, pp. 153–159, 1991.

[16] L. S. Liu, "Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 194, no. 1, pp. 114–125, 1995.

[17] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.