Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 176470, 4 pages
http://dx.doi.org/10.1155/2013/176470

Research Article

Some Properties of the q-Extension of the p-Adic Gamma Function

Hamza Menken[1] and Adviye Körükçü[2]

[1] *Mathematics Department, Science and Arts Faculty, Mersin University, Ciftlikkoy Campus, 33343 Mersin, Turkey*
[2] *Mathematics Graduate Program, Institute of Science, Mersin University, Ciftlikkoy Campus, 33343 Mersin, Turkey*

Correspondence should be addressed to Hamza Menken; hmenken@mersin.edu.tr

Received 10 October 2012; Accepted 7 February 2013

Academic Editor: Giovanni P. Galdi

We study the q-extension of the p-adic gamma function $\Gamma_{p,q}$. We give a new identity for the q-extension of the p-adic gamma $\Gamma_{p,q}$ in the case $p = 2$. Also, we derive some properties and new representations of the q-extension of the p-adic gamma $\Gamma_{p,q}$ in general case.

1. Introduction

Let p be a prime number and let $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. It is well known that the analogous of the classical gamma function Γ in p-adic context depends on modifying the factorial function $n!$ [1]. The factorial function $(n!)_p$ in $\mathbb{Q}_p$ is defined as

$$(n!)_p = \prod_{\substack{j<n \\ (p,j)=1}} j. \tag{1}$$

The p-adic gamma function Γ_p is defined by Morita [2] as the continuous extension to $\mathbb{Z}_p$ of the function $n \to (-1)^n (n!)_p$. That is, $\Gamma_p(x)$ is defined by the formula

$$\Gamma_p(x) = \lim_{n\to x} (-1)^n \prod_{\substack{j<n \\ (p,j)=1}} j \tag{2}$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers. The p-adic gamma function $\Gamma_p(x)$ had been studied by Diamond [3], Barsky [4], and others. The relationship between some special functions and the p-adic gamma function $\Gamma_p(x)$ were investigated by Gross and Koblitz [5], Cohen and Friedman [6]. and Shapiro [7].

The q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$ is defined by Koblitz as follows.

Definition 1 (see [8]). Let $q \in \mathbb{C}_p$, $|q-1|_p < 1$, $q \neq 1$. The q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$ is defined by formula

$$\Gamma_{p,q}(x) = \lim_{n\to x} (-1)^n \prod_{\substack{j<n \\ (p,j)=1}} \frac{1-q^j}{1-q} \tag{3}$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers. We recall that $\lim_{q\to 1} \Gamma_{p,q} = \Gamma_p$.

The q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$ was studied by Koblitz [8, 9], Nakazato [10], Kim et al. [11], and Kim [12].

2. Main Results

In the present work, we give a new identity for the q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$ in special case $p = 2$. Also, we derive some properties and representations for the q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$.

Theorem 2. *If $p = 2$, then for all $x \in \mathbb{Z}_2$*

$$\Gamma_{2,q}(x)\,\Gamma_{2,q}(1-x) = (-1)^{1+\sigma_1(x)} \lim_{n\to x} \prod_{\substack{j<n \\ (2,j)=1}} q^j, \tag{4}$$

where σ_1 is defined by the formula

$$\sigma_1\left(\sum_{j=0}^{\infty} a_j 2^j\right) = a_1. \quad (5)$$

Proof. Let $p = 2$ and $n \in \mathbb{N}$. From Proposition 3 in [12] we known that

$$\Gamma_{2,q}(n+1)\,\Gamma_{2,q}(-n) = (-1)^{n+1-[n/2]} \prod_{\substack{j<n+1\\(2,j)=1}} q^j. \quad (6)$$

Here, $[\cdot]$ is the greatest integer function. Taking $n-1$ in place of n, the relation becomes

$$\Gamma_{2,q}(n)\,\Gamma_{2,q}(1-n) = (-1)^{n-[(n-1)/2]} \prod_{\substack{j<n\\(2,j)=1}} q^j. \quad (7)$$

Now, let $n = a_0 + a_1 2 + a_2 2^2 + \cdots$ in base 2. If $a_0 \neq 0$, then $a_1 = 1$ in base 2 and

$$\begin{aligned}\left[\frac{n-1}{2}\right] &= \left[\frac{\left(a_0 - 1 + a_1 2 + a_2 2^2 + \cdots\right)}{2}\right]\\ &= \left[a_1 + a_2 2 + \cdots\right] \equiv a_1 \pmod 2.\end{aligned} \quad (8)$$

Thus, we get

$$\begin{aligned}(-1)^{n-[(n-1)/2]} &= (-1)^n(-1)^{-[(n-1)/2]} = (-1)^1(-1)^{-a_1}\\ &= (-1)^{1-a_1} = (-1)^{1+a_1} = (-1)^{1+\sigma_1}.\end{aligned} \quad (9)$$

If $a_0 = 0$, then

$$\begin{aligned}\left[\frac{n-1}{2}\right] &= \left[\frac{\left(-1 + a_1 2 + a_2 2^2 + \cdots\right)}{2}\right]\\ &= \left[\frac{\left(1 + (a_1 - 1)\,2 + a_2 2^2 + \cdots\right)}{2}\right]\\ &\equiv a_1 - 1 \pmod 2.\end{aligned} \quad (10)$$

Hence,

$$\begin{aligned}(-1)^{n-[(n-1)/2]} &= (-1)^n(-1)^{-[(n-1)/2]}\\ &= (-1)^2(-1)^{-(a_1-1)}\\ &= (-1)^{2+a_1-1}\\ &= (-1)^{1+a_1}\\ &= (-1)^{1+\sigma_1(n)}.\end{aligned} \quad (11)$$

Thus, we have

$$\Gamma_{2,q}(n)\,\Gamma_{2,q}(1-n) = (-1)^{1+\sigma_1(n)} \prod_{\substack{j<n\\(2,j)=1}} q^j \quad (12)$$

and thus, we obtain

$$\Gamma_{2,q}(x)\,\Gamma_{2,q}(1-x) = (-1)^{1+\sigma_1(x)} \lim_{n\to x} \prod_{\substack{j<n\\(2,j)=1}} q^j. \quad (13)$$

□

We recall that the q-factorial $[n;q]!$ is defined in [13] by the formula

$$[n;q]! = [n;q]\,[n-1;q]\cdots[2;q]\,[1;q] \quad (14)$$

for $n \geq 1$, where

$$[x;q] = \frac{1-q^x}{1-q}. \quad (15)$$

Note that for $n = 0$, we can define $[0;q]! = 1$.

We use the following theorem to prove our results.

Theorem 3 (see [12]). *Let $n \in \mathbb{N}$. Then,*

$$\Gamma_{p,q}(n+1) = (-1)^{n+1} \frac{[n;q]!}{[p;q]^{[n/p]}\,[[n/p]\,;q^p]!}, \quad (16)$$

where $[\cdot]$ is the greatest integer function. In particular,

$$[p^n - 1;q]! = (-1)^p[p;q]^{p^{n-1}-1}\left[p^{n-1}-1;q^p\right]!\Gamma_{p,q}(p^n). \quad (17)$$

Theorem 4. *Let $n \in \mathbb{N}$ and let s_n be the sum of the digits of $n = \sum_{j=0}^{s} a_j p^j$ $(a_s \neq 0)$ in base p. Then*

(a) $[[n/p^s];q]! = (-1)^{n+1-s}(-[p;q])^{(n-s_n)/(p-1)}$ $\prod_{j=0}^{s-1}[[n/p^{j+1}];q^p]!/[[n/p^j];q]!\prod_{j=0}^{s}\Gamma_{p,q}([n/p^j]+1)$

(b) $[n;q]! = (-1)^{n+1-s}(-[p;q])^{(n-s_n)/(p-1)}[[n/p];q^p]!$ $\prod_{j=1}^{s}[[n/p^{j+1}];q^p]!/[[n/p^j];q]!\prod_{j=0}^{s}\Gamma_{p,q}([n/p^j]+1)$.

Proof. From the Theorem 3 we know that

$$[n;q]! = (-1)^{n+1}[p;q]^{[n/p]}\left[\left[\frac{n}{p}\right];q^p\right]!\Gamma_{p,q}(n+1). \quad (18)$$

By taking $[n/p^0], [n/p^1], \ldots, [n/p^s]$ instead of n, respectively, we get the relations

$$\begin{aligned}\left[\left[\frac{n}{p^0}\right];q\right]! &= (-1)^{[n/p^0]+1}[p;q]^{[n/p^1]}\\ &\quad\times\left[\left[\frac{n}{p^1}\right];q^p\right]!\Gamma_{p,q}\left(\left[\frac{n}{p^0}\right]+1\right),\\ \left[\left[\frac{n}{p^1}\right];q\right]! &= (-1)^{[n/p^1]+1}[p;q]^{[n/p^2]}\\ &\quad\times\left[\left[\frac{n}{p^2}\right];q^p\right]!\Gamma_{p,q}\left(\left[\frac{n}{p^1}\right]+1\right),\\ &\;\;\vdots\\ \left[\left[\frac{n}{p^s}\right];q\right]! &= (-1)^{[n/p^s]+1}[p;q]^{[n/p^{s+1}]}\\ &\quad\times\left[\left[\frac{n}{p^{s+1}}\right];q^p\right]!\Gamma_{p,q}\left(\left[\frac{n}{p^s}\right]+1\right).\end{aligned} \quad (19)$$

By multiplying of the equalities above, we can easily obtain

$$\begin{aligned}\left[\left[\frac{n}{p^s}\right];q\right]! &= (-1)^{[n/p^0]+\cdots+[n/p^s]+s+1}[p;q]^{[n/p^1]+\cdots+[n/p^{s+1}]}\\ &\quad\times\left[\left[\frac{n}{p^{s+1}}\right];q^p\right]!\prod_{j=0}^{s-1}\frac{\left[\left[n/p^{j+1}\right];q^p\right]!}{\left[\left[n/p^j\right];q\right]!}\\ &\quad\times\prod_{j=0}^{s}\Gamma_{p,q}\left(\left[\frac{n}{p^j}\right]+1\right)\\ &= (-1)^{(n-s_n)/(p-1)}(-1)^{n+1-s}[p;q]^{(n-s_n)/(p-1)}\\ &\quad\times\left[\left[\frac{n}{p^{s+1}}\right];q^p\right]!\prod_{j=0}^{s-1}\frac{\left[\left[n/p^{j+1}\right];q^p\right]!}{\left[\left[n/p^j\right];q\right]!}\\ &\quad\times\prod_{j=0}^{s}\Gamma_{p,q}\left(\left[\frac{n}{p^j}\right]+1\right).\end{aligned}\tag{20}$$

Therefore, we get the relation (a)

$$\begin{aligned}\left[\left[\frac{n}{p^s}\right];q\right]! &= (-1)^{n+1-s}\left(-[p;q]^{(n-s_n)/(p-1)}\right)\\ &\quad\times\prod_{j=0}^{s-1}\frac{\left[\left[n/p^{j+1}\right];q^p\right]!}{\left[\left[n/p^j\right];q\right]!}\prod_{j=0}^{s}\Gamma_{p,q}\left(\left[\frac{n}{p^j}\right]+1\right),\\ [n;q]! &= (-1)^{[n/p^0]+\cdots+[n/p^s]+s+1}[p;q]^{[n/p^1]+\cdots+[n/p^{s+1}]}\\ &\quad\times\left[\left[\frac{n}{p}\right];q^p\right]!\prod_{j=1}^{s}\frac{\left[\left[n/p^{j+1}\right];q^p\right]!}{\left[\left[n/p^j\right];q\right]!}\\ &\quad\times\prod_{j=0}^{s}\Gamma_{p,q}\left(\left[\frac{n}{p^j}\right]+1\right)\\ &= (-1)^{(n-s_n)/(p-1)}(-1)^{n+1-s}[p;q]^{(n-s_n)/(p-1)}\\ &\quad\times\left[\left[\frac{n}{p}\right];q^p\right]!\prod_{j=1}^{s}\frac{\left[\left[n/p^{j+1}\right];q^p\right]!}{\left[\left[n/p^j\right];q\right]!}\\ &\quad\times\prod_{j=0}^{s}\Gamma_{p,q}\left(\left[\frac{n}{p^j}\right]+1\right).\end{aligned}\tag{21}$$

Therefore, we get the relation (b)

$$\begin{aligned}[n;q]! &= (-1)^{n+1-s}(-[p;q])^{(n-s_n)/(p-1)}\left[\left[\frac{n}{p}\right];q^p\right]!\\ &\quad\times\prod_{j=1}^{s}\frac{\left[\left[n/p^{j+1}\right];q^p\right]!}{\left[\left[n/p^j\right];q\right]!}\prod_{j=0}^{s}\Gamma_{p,q}\left(\left[\frac{n}{p^j}\right]+1\right).\end{aligned}\tag{22}$$

□

Theorem 5. *Let $n\in\mathbb{N}$ and let $n=\sum_{j=0}^{s}a_jp^j$ $(a_s\neq 0)$. Then*

$$\begin{aligned}\left[p^n-1;q\right]! &= (-1)^p(-[p;q])^{(p^n-1)/(p-1)}\\ &\quad\times[p;q]^{-n}\left[p^{n-1}-1;q^p\right]!\\ &\quad\times\prod_{j=0}^{n-2}\frac{\left[p^j-1;q^p\right]!}{\left[p^{j+1}-1;q\right]!}\prod_{j=0}^{n}\Gamma_{p,q}\left(p^j\right).\end{aligned}\tag{23}$$

Proof. From Theorem 3 it follows that

$$\left[p^j-1;q\right]! = (-1)^p[p;q]^{p^{j-1}-1}\left[p^{j-1}-1;q^p\right]!\Gamma_{p,q}\left(p^j\right).\tag{24}$$

Taking of $0,1,\ldots,n$ instead of j, respectively, we have the equalities

$$\begin{aligned}\left[p^0-1;q\right]! &= 1 = (-1)\,\Gamma_{p,q}\left(p^0\right),\\ \left[p^1-1;q\right]! &= (-1)^p[p;q]^{p^0-1}\left[p^0-1;q^p\right]!\Gamma_{p,q}\left(p^1\right),\\ \left[p^2-1;q\right]! &= (-1)^p[p;q]^{p^1-1}\left[p^1-1;q^p\right]!\Gamma_{p,q}\left(p^2\right),\\ &\vdots\\ \left[p^n-1;q\right]! &= (-1)^p[p;q]^{p^{n-1}-1}\left[p^{n-1}-1;q^p\right]!\Gamma_{p,q}\left(p^n\right).\end{aligned}\tag{25}$$

By multiplying of the equalities above, we can easily obtain

$$\begin{aligned}\left[p^n-1;q\right]! &= (-1)^{np+1}[p;q]^{p^0+p^1+\cdots+p^{n-1}-n}\left[p^{n-1}-1;q^p\right]!\\ &\quad\times\prod_{j=0}^{n-2}\frac{\left[p^j-1;q^p\right]!}{\left[p^{j+1}-1;q\right]!}\prod_{j=0}^{n}\Gamma_{p,q}\left(p^j\right).\end{aligned}\tag{26}$$

Thus,

$$\begin{aligned}\left[p^n-1;q\right]! &= (-1)^p(-[p;q])^{(p^n-1)/(p-1)}\\ &\quad\times[p;q]^{-n}\left[p^{n-1}-1;q^p\right]!\\ &\quad\times\prod_{j=0}^{n-2}\frac{\left[p^j-1;q^p\right]!}{\left[p^{j+1}-1;q\right]!}\prod_{j=0}^{n}\Gamma_{p,q}\left(p^j\right).\end{aligned}\tag{27}$$

□

Lemma 6. *Let $n\in\mathbb{Z}^+$, $n=\sum_{j=0}^{s}a_jp^j$ $(a_s\neq 0)$, and let p be a prime number. Then, for $j=0,1,\ldots,s$*

$$\frac{\left[\left[n/p^j\right];q\right]!}{[p;q]^{[n/p^j]}\left[\left[n/p^j\right];q^p\right]!} = \prod_{k=1}^{[n/p^j]}\frac{1-q^k}{1-q^{kp}}\quad(0\leq k\leq s).\tag{28}$$

Proof. For $j = 0$

$$\begin{aligned}\frac{[n;q]!}{[p;q]^n[n;q^p]!} &= \frac{[1;q][2;q]\cdots[n,q]}{[p;q]^n[1;q^p][2;q^p]\cdots[n;q^p]}\\ &= \left(\frac{1-q}{1-q}\frac{1-q^2}{1-q}\cdots\frac{1-q^n}{1-q}\right)\\ &\quad\times\left(\left(\frac{1-q^p}{1-q}\right)^n\frac{1-q^p}{1-q^p}\cdots\frac{1-q^{np}}{1-q^p}\right)^{-1}\\ &= \left(\frac{1-q}{1-q}\frac{1-q^2}{1-q}\cdots\frac{1-q^n}{1-q}\right)\\ &\quad\times\left(\frac{1-q^p}{1-q}\frac{1-q^{2p}}{1-q}\cdots\frac{1-q^{np}}{1-q}\right)^{-1}\\ &= \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q^p)(1-q^{2p})\cdots(1-q^{np})}.\end{aligned} \tag{29}$$

For $1 \le j \le s$ it follows that

$$\begin{aligned}&\frac{\left[\left[n/p^j\right];q\right]!}{[p;q]^{[n/p^j]}\left[[n/p^j];q^p\right]!}\\ &\quad= \frac{[1;q][2;q]\cdots\left[\left[n/p^j\right],q\right]}{[p;q]^{[n/p^j]}[1;q^p][2;q^p]\cdots\left[[n/p^j];q^p\right]}\\ &\quad= \left(\frac{1-q}{1-q}\frac{1-q^2}{1-q}\cdots\frac{1-q^{[n/p^j]}}{1-q}\right)\\ &\qquad\times\left(\left(\frac{1-q^p}{1-q}\right)^{[n/p^j]}\frac{1-q^p}{1-q^p}\cdots\frac{1-q^{[n/p^j]p}}{1-q^p}\right)^{-1}\\ &\quad= \frac{(1-q)(1-q^2)\cdots\left(1-q^{[n/p^j]}\right)}{(1-q^p)(1-q^{2p})\cdots\left(1-q^{[n/p^j]p}\right)}.\end{aligned} \tag{30}$$

Then, we obtain

$$\frac{\left[\left[n/p^j\right];q\right]!}{[p;q]^{[n/p^j]}\left[[n/p^j];q^p\right]!} = \prod_{k=1}^{[n/p^j]}\frac{1-q^k}{1-q^{kp}}. \tag{31}$$

□

Theorem 7. *Let $n \in \mathbb{N}$ and let s_n be the sum of the digits of $n = \sum_{j=0}^{s} a_j p^j$ $(a_s \neq 0)$ in base p. Then*

$$\begin{aligned}[n;q]! &= (-1)^{((n-s_n)/(p-1))+n+1-s}\prod_{k=1}^{[n/p^1]}\frac{(1-q^{kp})}{(1-q^k)}\cdots\\ &\quad\prod_{k=1}^{[n/p^s]}\frac{(1-q^{kp})}{(1-q^k)}\prod_{j=0}^{s}\Gamma_{p,q}\left(\left[\frac{n}{p^j}\right]+1\right).\end{aligned} \tag{32}$$

Proof. This theorem can be proved by using Theorem 4 and Lemma 6. □

Acknowledgments

This work is supported by Mersin University and the Scientific and Technological Research Council of Turkey (TÜBİTAK). The authors would like to thank the reviewers for their useful comments and suggestions.

References

[1] W. H. Schikhof, *Ultrametric Calculus: An Introduction to* p-*adic Analysis*, vol. 4, Cambridge University Press, Cambridge, UK, 1984.

[2] Y. Morita, "A p-adic analogue of the Γ-function," *Journal of the Faculty of Science*, vol. 22, no. 2, pp. 255–266, 1975.

[3] J. Diamond, "The p-adic log gamma function and p-adic Euler constants," *Transactions of the American Mathematical Society*, vol. 233, pp. 321–337, 1977.

[4] D. Barsky, "On Morita's p-adic gamma function," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 89, no. 1, pp. 23–27, 1981.

[5] B. H. Gross and N. Koblitz, "Gauss sums and the p-adic Γ-function," *Annals of Mathematics*, vol. 109, no. 3, pp. 569–581, 1979.

[6] H. Cohen and E. Friedman, "Raabe's formula for p-adic gamma and zeta functions," *Annales de l'Institut Fourier*, vol. 58, no. 1, pp. 363–376, 2008.

[7] I. Shapiro, "Frobenius map and the p-adic gamma function," *Journal of Number Theory*, vol. 132, no. 8, pp. 1770–1779, 2012.

[8] N. Koblitz, "q-extension of the p-adic gamma function," *Transactions of the American Mathematical Society*, vol. 260, no. 2, 1980.

[9] N. Koblitz, "q-extension of the p-adic gamma function. II," *Transactions of the American Mathematical Society*, vol. 273, no. 1, pp. 111–129, 1982.

[10] H. Nakazato, "The q-analogue of the p-adic gamma function," *Kodai Mathematical Journal*, vol. 11, no. 1, pp. 141–153, 1988.

[11] T. K. Kim et al., "A note on analogue of gamma functions," in *Proceedings of the 5th Transcendental Number Theory*, pp. 111–118, Gakushin University, Tokyo, Japan, 1997.

[12] Y. S. Kim, "q-analogues of p-adic gamma functions and p-adic Euler constants," *Korean Mathematical Society*, vol. 13, no. 4, pp. 735–741, 1998.

[13] R. Askey, "The q-gamma and q-beta functions," *Applicable Analysis*, vol. 8, no. 2, pp. 125–141, 1978/79.