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Research Article

On Best Proximity Point Theorems and Fixed Point Theorems for p-Cyclic Hybrid Self-Mappings in Banach Spaces

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This paper relies on the study of fixed points and best proximity points of a class of so-called generalized point-dependent (K, λ)-hybrid p-cyclic self-mappings relative to a Bregman distance D_f, associated with a Gâteaux differentiable proper strictly convex function f in a smooth Banach space, where the real functions λ and K quantify the point-to-point hybrid and nonexpansive (or contractive) characteristics of the Bregman distance for points associated with the iterations through the cyclic self-mapping. Weak convergence results to weak cluster points are obtained for certain average sequences constructed with the iterates of the cyclic hybrid self-mappings.

1. Introduction and Preliminaries

The following objects are considered through the paper.

(1) The Hilbert space H on the field X (in particular, $\mathbf{R}$ or $\mathbf{C}$) is endowed with the inner product $\langle x, y\rangle$ which maps $H \times H$ to X, for all $x, y \in H$ which maps $H \times H$ to X, where $(\mathbf{X}, \|\ \|)$ is a Banach space when endowed with a norm $\|\ \|$ induced by the inner product and defined by $\|x\| = \langle x, x\rangle^{1/2}$, for all $x \in H$. It is wellknown that all Hilbert spaces are uniformly convex Banach spaces and that Banach spaces are always reflexive.

(2) The $p(\geq 2)$-cyclic self-mapping $T : A \to A$ with $A := \bigcup_{i\in\overline{p}} A_i$ is subject to $A_{p+1} \equiv A_p$, where $A_i (\neq \varnothing) \subset H$ are p subsets of H, for all $i \in \overline{p} = \{1, 2, \ldots, p\}$, that is, a self-mapping satisfying $T(A_i) \subseteq A_{i+1}$, for all $i \in \overline{p}$

(3) The function $f : D\ (\equiv \operatorname{dom} f) \subset X \to (-\infty, \infty]$ is a proper convex function which is Gâteaux differentiable in the topological interior of the convex set D; $\operatorname{int} D$, that is, $D := \{x \in X : f(x) < \infty\} \neq \varnothing$ and convex since f is proper with

$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y), \quad \forall x, y \in D,\ \forall \alpha \in [0,1], \tag{1}$$

since f is convex, and for each $x \in D$, there is $x^* = f(x) \in X^*$ (the topological dual of X) such that

$$\exists \lim_{t\to 0} \frac{f(x+ty) - f(x)}{t} = \langle y, f'(x)\rangle, \quad \forall y \in D, \tag{2}$$

since f is Gâteaux differentiable in $\operatorname{int} D$ where $f'(x)$ denotes the Gâteaux derivative of f at x if $x \in \operatorname{int} D$. On the other hand, f is said to be strictly convex if

$$f(\alpha x + (1-\alpha) y) < \alpha f(x) + (1-\alpha) f(y), \quad \forall x, y\ (\neq x) \in D,\ \forall \alpha \in (0,1). \tag{3}$$

(4) The Bregman distance (or Bregman divergence) D_f associated with the proper convex function f $D_f : D \times D \to (-\infty, \infty]$, where $\mathbf{R}_{0+} := \{z \in \mathbf{R} : z \geq 0\} = \mathbf{R}_+ \cup \{0\}$, is defined by

$$D_f(y, x) = f(y) - f(x) - \langle y - x, f'(x)\rangle, \quad \forall x, y \in D, \tag{4}$$

provided that it is Gâteaux differentiable everywhere in $\operatorname{int} D$. If f is not Gâteaux differentiable at $x \in \operatorname{int} D$, then (4) is replaced by

$$D_f(y, x) = f(y) - f(x) + f^0(x, x - y), \tag{5}$$

where $f^0(x, x-y) := \lim_{t\to 0^+}((f(x+t(x-y))-f(x))/t)$ and $D_f(y,x)$ is finite if and only if $x \in D^0 \subset D$, the algebraic interior of D defined by

$$D^0 := \{x \in D : \exists z \in (x,y), [x,z] \subseteq D;\ \forall y \in X \setminus \{x\}\}. \tag{6}$$

The topological interior of D is $\text{int}(D) := \{x \in D : x \in fr(D)\} \subset D^0$, where $fr(D)$ is the boundary of D. It is well known that the Bregman distance does not satisfy either the symmetry property or the triangle inequality which are required for standard distances while they are always nonnegative because of the convexity of the function $f : D\ (\equiv \text{dom}\, f) \subset X \to (-\infty, \infty]$. The Bregman distance between sets $B, C \subset H \subset X$ is defined as $D_f(B,C) := \inf_{x\in B, y\in C} D_f(x,y)$. If $A_i \in A \subset H$ for $i \in \overline{p}$, then $D_{fi} := D_f(A_i, A_{i+1}) = \inf_{x\in A_i, y\in A_{i+1}} D_f(x,y)$. Through the paper, sequences $\{x_n\}_{n\in\mathbf{N}_0} \equiv \{T^n x\}_{n\in\mathbf{N}_0}$ with $\mathbf{N}_0 = \mathbf{N}\cup\{0\}$ are simply denoted by $\{T^n x\}$ for the sake of notation simplicity.

Fixed points and best proximity points of cyclic self-mappings $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ in uniformly convex Banach spaces $(X, \|\ \|)$ have been widely studied along the last decades for the cases when the involved sets intersect or not. See, for instance, [1–3] and references therein. In parallel, interesting results have been obtained for both nonspreading, nonexpansive, and hybrid maps in Hilbert spaces including also to focus the related problems via iterative methods supported by fixed point theory and the use of more general mappings such as nonspreading and pseudocontractive mappings. See, for instance, recent background [4–7] and references therein. Let C be a nonempty subset of a Hilbert space H. On the other hand, it has to be pointed out that the characterization of several classes of iterative computations by invoking results of fixed point theory has received much attention in the background literature. See, for instance, [8–11] and references therein. In [12–18], the existence of fixed points of mappings $T : C \to H$ is discussed when $T : C \to H$ is:

(1.1) nonexpansive; that is, $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$

(1.2) nonspreading; that is, $\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle$, for all $x, y \in C$,

(1.3) λ-hybrid [17]; that is, $\|Tx - Ty\|^2 \le \|x - y\|^2 + \lambda\langle x - Tx, y - Ty\rangle$, for all $x, y \in C$. If $\lambda = 1$, then $T : C \to H$ is referred to as hybrid [14, 15], and if $\lambda = \lambda(y)$ and (1.3) is changed to

(1.4) $D_f(Tx, Ty) \le D_f(x,y) + \lambda(y)\langle x - Tx, f'(y) - f'(Ty)\rangle$, for all $x, y \in C$,

where $f : D\ (\equiv \text{dom}\, f) \subset X \to (-\infty, \infty]$ is a Gâteaux differentiable convex function, then $T : C \to H$ is referred to as being point-dependent λ-hybrid relative to the Bregman distance D_f, [16]. A well-known result is that a nonspreading mapping, and then a nonexpansive one, on a nonempty closed convex subset C of a Hilbert space H has a fixed point if and only it has a bounded sequence on such a subset [18]. The result has been later on extended to λ-hybrid mappings, [17] and to point-dependent λ-hybrid ones [16]. As pointed out in [16], what follows directly from the previous definitions, $T : C \to H$ is nonexpansive if and only if it is 0-hybrid while it is nonspreading if and only if it is 2-hybrid; T is hybrid if and only if it is 1-hybrid.

This paper is focused on the study of fixed points and best proximity points of a class of generalized point-dependent (K,λ)-hybrid $p(\ge 2)$-cyclic self-mappings $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$, relative to a Bregman distance D_f in a smooth Banach space, where $\lambda : \bigcup_{i\in\overline{p}} A_i \to \mathbf{R}$ is a point-dependent real function in (1.4) quantifying the "hybrideness" of the $p(\ge 2)$ cyclic self-mapping and $K = K(y) : \bigcup_{i\in\overline{p}} A_i \to [0,1]$ is added as a weighting factor in the first right-hand-side term of (1.4). Such a function is defined through a point-dependent product of the particular point p-functions while quantifies either the "nonexpansiveness" or the "contractiveness" of the Bregman distance for points associated with the iterates of the cyclic self-mapping in each of the sets $A_i \times A_{i+1} \cup A_{i+1} \times A_i$ for $i \in \overline{p} = \{1, 2, \ldots, p\}$, where A_i $(i \in \overline{p})$ are nonempty closed and convex. Thus, the generalization of the hybrid map studied in this paper has two main characteristics, namely, (a) a weighting point-dependent term is introduced in the contractive condition; (b) the hybrid self-mapping is a cyclic self-mappings. Precise definitions and meaning of those functions are given in Definition 2 of Section 2 which are then used to get the main results obtained in the paper. In most of the results obtained in this paper, the Bregman distance D_f is defined associated with a Gâteaux differentiable proper strictly convex function f whose domain includes the union of the p subsets A_i $(i \in \overline{p})$ of the (K,λ)-hybrid $p(\ge 2)$-cyclic self-mapping which are not assumed, in general, to intersect. Weak convergence results to weak cluster points of certain average sequences built with the iterates of the cyclic hybrid self-mappings are also obtained. In particular, such weak cluster points are proven to be also fixed points of the composite self-mappings on the sets A_i $(i \in \overline{p})$, even if such sets do not intersect, while they are simultaneously best proximity points of the point-dependent (K,λ)-hybrid $p(\ge 2)$-cyclic self-mapping relative to D_f.

2. Some Fixed Point Theorems for Cyclic Hybrid Self-Mappings on the Union of Intersecting Subsets

The Bregman distance is not properly a distance, since it does not satisfy symmetry and the triangle inequality, but it is always nonnegative and leads to the following interesting result towards its use in applications of fixed point theory.

Lemma 1. *If $f : D \times D \to (-\infty, \infty]$ is a proper strictly convex function being Gâteaux differentiable in* int D*, then*

$$D_f(x,x) = 0, \quad \forall x \in \text{int}\, D, \tag{7}$$

$$D_f(y,x) > 0, \quad \forall x, y\ (\ne x) \in \text{int}\, D, \tag{8}$$

$$\begin{aligned} D_f(y,x) + D_f(x,y) &= \langle x - y, f'(x) - f'(y)\rangle \\ &\ge 0, \quad \forall x, y \in \text{int}\, D, \end{aligned} \tag{9}$$

$$D_f(y,x) - D_f(x,y) = 2\left(f(y) - f(x)\right) - \left\langle y - x, f'(x) + f'(y)\right\rangle, \quad \forall x, y \in \operatorname{int} D. \tag{10}$$

Proof. By using (4) for $D_f(y,x)$ and defining $D_f(x,y) = f(x) - f(y) - \langle x - y, f'(y)\rangle$, for all $x, y \in \operatorname{int} D$ by interchanging x and y in the definition of $D_f(y,x)$ in (4),

$$\begin{aligned} D_f(y,x) + D_f(x,y) &= \left\langle x, f'(x)\right\rangle + \left\langle y, f'(y)\right\rangle \\ &\quad - \left\langle x, f'(y)\right\rangle > -\left\langle y, f'(x)\right\rangle \\ &= \left\langle x, f'(x) - f'(y)\right\rangle \\ &\quad + \left\langle y, f'(y) - f'(x)\right\rangle, \end{aligned} \tag{11}$$

which leads to (9) since $D_f(y,x) \geq 0$, for all $x, y \in \operatorname{int} D$, [16, 17], if $f : D \times D \to (-\infty, \infty]$ is proper strictly convex, and the fact that $D_f(x,y) \geq 0$, for all $x, y \in \operatorname{int} D$.

Equation (7) follows from (9) for $x = y$ leading to $2D_f(x,x) = 0$. To prove (8), take $x, y(\neq x) \in \operatorname{int} D$ and proceed by contradiction using (4) by assuming that $D_f(y,x) = 0$ for such $x, y(\neq x) \in \operatorname{int} D$ so that

$$\begin{aligned} 0 = D_f(y,x) &= f(y) - f(x) - \left\langle y - x, f'(x)\right\rangle \\ &= f(y) - f(x) + \left\langle x - y, f'(x) - f'(y)\right\rangle \\ &\quad + \left\langle x - y, f'(y)\right\rangle \\ &> f(y) - f(x) - \left\langle y - x, f'(y)\right\rangle = D_f(y,x), \end{aligned} \tag{12}$$

which contradicts $D_f(y,x) = 0$. Then, $D_f(y,x) > 0$, and, hence, (8) follows

$$\begin{aligned} D_f(y,x) - D_f(x,y) &= f(y) - f(x) - \left\langle y - x, f'(x)\right\rangle - f(x) \\ &\quad + f(y) + \left\langle x - y, f'(y)\right\rangle \\ &= 2\left(f(y) - f(x)\right) + \left\langle x - y, f'(y) + f'(x)\right\rangle, \\ &\qquad \forall x, y \in \operatorname{int} D, \end{aligned} \tag{13}$$

And, hence, (10) via (7) and (9). □

The following definition is then used.

Definition 2. If $D \cap A_i \neq \varnothing$, for all $i \in \overline{p}$, and $f : D\ (\equiv \operatorname{dom} f) \subset X \to (-\infty, \infty]$ is a proper convex function which is Gâteaux differentiable in $\operatorname{int} D$, then the p-cyclic self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$, where $A := \bigcup_{i\in\overline{p}} A_i \subseteq \operatorname{int} D \subset H$ and $A_i \neq \varnothing$, for all $i \in \overline{p}$, is said to be a *generalized* contractive point-dependent (K,λ)-hybrid $p(\geq 2)$-cyclic self-mapping relative to D_f if

$$D_f(Tx,Ty) \leq K_i(y) D_f(x,y) + \lambda(y)\left\langle x - Tx, f'(y) - f'(Ty)\right\rangle, \quad \forall x \in A_i,\ \forall y \in A_{i+1},\ \forall i \in \overline{p}, \tag{14}$$

for some given functions $\lambda : \bigcup_{i\in\overline{p}} A_i \to \mathbf{R}$ and $K_i : A_{i+1} \to (0, a_i]$ with $a_i \in \mathbf{R}_+$, for all $i \in \overline{p}$, where $K : \bigcup_{i\in\overline{p}} A_{i+1} \to (0,1]$ defined by $K(y) = \prod_{j=i}^{i+p-1}[K_j(T^{j-i}y)]$ for any $y \in A_{i+1}$, for all $i \in \overline{p}$.

If, furthermore, $K : \bigcup_{i\in\overline{p}} A_i \to (0,1)$, for all $i \in \overline{p}$, then $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is said to be a generalized point-dependent (K,λ)-hybrid $p(\geq 2)$-cyclic self-mapping relative to D_f.

If $p = 1$, it is possible to characterize $T : A_1 \to A_1$ as a trivial 1-cyclic self-mapping with $A_1 = A_2$ which does not need to be specifically referred to as 1-cyclic.

Although $K_i : A_{i+1} \to (0, a_i]$ depends on $i \in \overline{p}$, the whole $K : \bigcup_{i\in\overline{p}} A_i \to (0,1)$ does not depend on $i \in \overline{p}$ so that the cyclic self-mapping is referred to as *generalized* point-dependent (K,λ)-hybrid in the definition.

The following concepts are useful.

$f : D\ (\equiv \operatorname{dom} f) \subset X \to (-\infty, \infty]$ is said to be totally convex if the modulus of total convexity $v_f : D^0 \times [0,\infty) \to [0,\infty]$; that is, $v_f(x,t) = \inf\{D_f(x,y) : y \in D, \|y - x\| = t\}$ is positive for $t > 0$.

$f : D\ (\equiv \operatorname{dom} f) \subset X \to (-\infty, \infty]$ is said to be uniformly convex if the modulus of uniform convexity $\delta_f : [0,\infty) \to [0,\infty]$; that is, $\delta_f(t) = \inf\{f(x) + f(y) - 2f((x+y)/2) : x, y \in D, \|y - x\| \geq t\}$ is positive for $t > 0$. It holds that $v_f(x,t) \geq \delta_f(t)$, for all $x \in D$ [16]. The following result holds.

Theorem 3. *Assume that*

(1) *$f : D\ (\equiv \operatorname{dom} f) \subset X \to (-\infty, \infty]$ is a lower-semicontinuous proper strictly totally convex function which is Gâteaux differentiable in* $\operatorname{int} D$*;*

(2) *$A_i(\neq \varnothing) \subseteq \operatorname{int} D \subset H$, for all $i \in \overline{p}$, are bounded, closed, and convex subsets of H which intersect and $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a generalized point-dependent (K,λ)-hybrid $p(\geq 2)$-cyclic self-mapping relative to D_f for some given functions $\lambda : \bigcup_{i\in\overline{p}} A_i \to \Lambda \subset \mathbf{R}$ and $K : \bigcup_{i\in\overline{p}} A_i \to (0,1)$, defined by $K(y) = \prod_{j=i}^{i+p-1}[K_j(T^{j-i}y)]$ for any $y \in A_{i+1}$, for all $i \in \overline{p}$, and some functions $K_i : A_{i+1} \to (0, a_i]$, for all $i \in \overline{p}$, with Λ being bounded;*

(3) *there is a convergent sequence $\{T^n x\}$ to some $z \in \bigcap_{i\in\overline{p}} A_i$ for some $x \in \bigcup_{i\in\overline{p}} A_i$.*

Then, $z = Tz$ is the unique fixed point of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ to which all sequences $\{T^n x\}$ converge for any $x \in \bigcup_{i\in\overline{p}} A_i$, for all $i \in \overline{p}$.

Proof. The recursive use of (14) yields

$$\begin{aligned}
&D_f\left(T^2x,T^2y\right)\\
&\le K_{i+1}\left(Ty\right)D_f\left(Tx,Ty\right)+\lambda\left(Ty\right)\\
&\quad\times\left\langle Tx-T^2x,f'\left(Ty\right)-f'\left(T^2y\right)\right\rangle\\
&\le K_{i+1}\left(Ty\right)\left[K_i\left(y\right)D_f\left(x,y\right)+\lambda\left(y\right)\right.\\
&\qquad\left.\times\left\langle x-Tx,f'\left(y\right)-f'\left(Ty\right)\right\rangle\right]\\
&\quad+\lambda\left(Ty\right)\left\langle Tx-T^2x,f'\left(Ty\right)-f'\left(T^2y\right)\right\rangle,\\
&\qquad\forall x\in A_i,\ \forall y\in A_{i+1},\ \forall i\in\overline{p},
\end{aligned}\tag{15}$$

$$\begin{aligned}
&D_f\left(T^px,T^py\right)\\
&\le K_{i+p-1}\left(T^{p-1}y\right)D_f\left(T^{p-1}x,T^{p-1}y\right)\\
&\quad+\lambda\ \left(T^{p-1}y\right)\left\langle T^{p-1}x-T^px,f'\left(T^{p-1}y\right)-f'\left(T^py\right)\right\rangle\\
&\le\left[\prod_{j=1}^{p}K_{p-j+1}\left(T^{p-j+1-i}y\right)\right]D_f\left(x,y\right)\\
&\quad+\sum_{k=1}^{p}\left(\prod_{j=k+1}^{p}\left[K_{p-j+i}\left(T^{p-j+1}\right)y\right]\right)\lambda\left(T^{k-1}y\right)\\
&\quad\times\left\langle T^{k-1}x-T^kx,f'\left(T^{k-1}y\right)-f'\left(T^ky\right)\right\rangle,\\
&\qquad\forall x\in A_i\ ,\ \forall y\in A_{i+1},\ \forall i\in\overline{p},
\end{aligned}\tag{16}$$

with $T^px\in A_{i+p}$, $T^py\in A_{i+1+p}$ with $A_{i+p}=A_i$, $K_{i+p}=K_i$, for all $i\in\overline{p}$, where T^0 is the identity mapping on $\bigcup_{i\in\overline{p}}A_i$. Now, define $\widehat{K}\left(y\right):=[\prod_{j=1}^{p}K_{p-j+1}(T^{p-j+1-i}y)]$ so that one gets

$$\begin{aligned}
&D_f\left(T^{np}x,T^{np}y\right)\\
&\le\widehat{K}^n\left(y\right)D_f\left(x,y\right)\\
&\quad+\sum_{k=1}^{np}\left(\prod_{j=k+1}^{np}\left[K_{np-j+i}\left(T^{np-j+1}\right)y\right]\right)\lambda\left(T^{k-1}y\right)\\
&\quad\times\left\langle T^{k-1}x-T^kx,f'\left(T^{k-1}y\right)-f'\left(T^ky\right)\right\rangle\\
&\le\widehat{K}^n\left(y\right)D_f\left(x,y\right)\\
&\quad+\sum_{k=1}^{(n-1)p}\left(\prod_{j=k+1}^{np}\left[K_{np-j+i}\left(T^{np-j+1}\right)y\right]\right)\lambda\left(T^{k-1}y\right)\\
&\quad\times\left\langle T^{k-1}x-T^kx,f'\left(T^{k-1}y\right)-f'\left(T^ky\right)\right\rangle\\
&\quad+\sum_{k=(n-1)p}^{np}\left(\prod_{j=k+1}^{np}\left[K_{np-j+i}\left(T^{np-j+1}\right)y\right]\right)\lambda\left(T^{k-1}y\right)\\
&\quad\times\left\langle T^{k-1}x-T^kx,f'\left(T^{k-1}y\right)-f'\left(T^ky\right)\right\rangle\\
&\le\widehat{K}^n\left(y\right)D_f\left(x,\ y\right)\\
&\quad+\left(\frac{1-\widehat{K}^{(n-1)p+1}\left(\widehat{y}\right)}{1-\widehat{K}\left(y\right)}+M_{np}\right)\\
&\quad\times\left(\max_{1\le j\le np}\left[\lambda\left(T^{j-1}y\right)\right.\right.\\
&\qquad\left.\left.\times\left\langle T^{j-1}x-T^jx,f'\left(T^{j-1}y\right)-f'\left(\ T^jy\right)\right\rangle\right]\right),
\end{aligned}\tag{17}$$

since $K\left(y\right)=[K_j(T^{j-i}y)]<1$, for all $y\in A_{i+1}$, since $T:\bigcup_{i\in\overline{p}}A_i\to\bigcup_{i\in\overline{p}}A_i$ is a generalized contractive point-dependent (K,λ)-hybrid $p(\ge2)$-cyclic self-mapping relative to D_f, implies that $\widehat{K}\left(y\right)<1$, for all $y\in\bigcup_{i\in\overline{p}}A_i$, since $K(y)=\prod_{j=i}^{i+p-1}[K_j(T^{j-i}y)]<1$ for any $y\in A_{i+1}$, for all $i\in\overline{p}$ (then for any $y\in\bigcup_{i\in\overline{p}}A_i$), where

$$\begin{aligned}
&M_{np}\ge M_{np}^0\\
&\quad:=\sum_{k=(n-1)p}^{np}\left(\prod_{j=k+1}^{np}\left[K_{np-j+i}\left(T^{np-j+1}\right)y\right]\right)\\
&\qquad\times\lambda\left(T^{k-1}y\right)\\
&\qquad\times\left\langle T^{k-1}x-T^kx,f'\left(T^{k-1}y\right)-f'\left(T^ky\right)\right\rangle,
\end{aligned}$$

$$\begin{aligned}
&D_f\ \left(T^{mnp}x,T^{mnp}y\right)\\
&\quad\le\widehat{K}^m\left(y\right)D_f\left(T^{np}x,T^{np}y\right)\\
&\qquad+\left(\frac{1-\widehat{K}^{(m-1)p+1}\left(\widehat{y}\right)}{1-\widehat{K}\left(y\right)}+M_{nmp}\right)\\
&\qquad\times\left(\max_{np+1\le j\le nmp}\left[\lambda\ \left(T^{j-1}y\right)\left\langle T^{j-1}x-T^jx,f'\left(T^{j-1}y\right)\right.\right.\right.\\
&\qquad\left.\left.\left.-f'\left(T^jy\right)\right\rangle\right]\right),
\end{aligned}\tag{18}$$

where $\widehat{K}^n(y):=\widehat{K}(y)\cdot\widehat{K}(T^py)\cdots\widehat{K}(T^{np}y)<1$, since $K(y)=\prod_{j=i}^{i+p-1}[K_j(T^{j-i}y)]<1$, for all $y\in\bigcup_{i\in\overline{p}}A_i$, so that

$$\begin{aligned}
&0\le\limsup_{n,m\to\infty}D_f\left(T^{mnp}x,T^{mnp}y\right)\\
&\quad\le\left(\frac{1}{1-\widehat{K}\left(y\right)}+\limsup_{n,m\to\infty}M_{nmp}\right)\\
&\qquad\times\limsup_{n,m\to\infty}\left(\max_{np+1\le j\le nmp}\right.\\
&\qquad\left.\times\left[\lambda\left(T^{j-1}y\right)\left\langle T^{j-1}x-T^jx,f'\left(T^{j-1}y\right)\right.\right.\right.\\
&\qquad\left.\left.\left.-f'\left(T^jy\right)\right\rangle\right]\right)=0,
\end{aligned}\tag{19}$$

since $\lambda : \bigcup_{i\in\overline{p}} A_i \to \Lambda \subset \mathbf{R}$ is bounded, $f : D \subset X \to (-\infty, \infty]$ is lower-semicontinuous then with all subgradients in any bounded subsets of int D being bounded, and $\{T^j x\}$ and $\{T^{j-1}x - T^j x\}$, for all $x \in \bigcup_{i\in\overline{p}} A_i$, for all $i \in \overline{p}$, converge so that they are Cauchy sequences being then bounded, for all $x \in \bigcup_{i\in\overline{p}} A_i$, for all $i \in \overline{p}$, where $z \in \bigcap_{i\in\overline{p}} A_i$, since $\bigcap_{i\in\overline{p}} A_i$ is nonempty and closed, is some fixed point of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$. As a result, $\exists \lim_{n\to\infty} D_f(T^{np}x, T^{np}y) = \lim_{n\to\infty} D_f(T^n x, T^n y) = 0$, for all $x \in A_i$, for all $y \in A_{i+1}$, for all $i \in \overline{p}$. From a basic property of Bregman distance, $T^n y \to z$, $T^n x \to z$ as $n \to \infty$, for all $x \in A_i$, for all $y \in A_{i+1}$, for all $i \in \overline{p}$, if $f : D\ (\equiv \operatorname{dom} f) \subset X \to (-\infty, \infty]$ is sequentially consistent. But, since $\bigcup_{i\in\overline{p}} A_i$ is closed, $f : D \mid \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is sequentially consistent if and only if it is totally convex [19]. Thus, $\{T^n y\}$ converges also to z for any $x \in A_i$ and $y \in A_{i+1}$, for all $i \in \overline{p}$, so that $z = Tz$ is a fixed point of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$. Assume not and proceed by contradiction so as then obtaining $D_f(T^n x, T^n z) \to 0$; $D_f(z, T^n z) \to 0$ as $n \to \infty$ from a basic property of Bregman distance. Thus, $[D_f(z, T^n z) - f(z) + f(T^n z)] \to 0$ as $n \to \infty$ since $\langle z - T^n z, f'(T^n z)\rangle \to 0$ as $n \to \infty$. As a result, $f(T^n z) \to f(z)$, $T^n z \to z = Tz$ as $n \to \infty$ from the continuity of $f : D\ (\equiv \operatorname{dom} f) \subset X \to (-\infty, \infty]$, and z is a fixed point of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$. Now, take any $y_1 \in A_j$ so that $y = T^{i+1-j} y_1 \in A_{i+1}$, then, $D_f(T^n x, T^{n+i+1-j} y) \to D_f(z, T^{i+1-j} z) = D_f(z, z) = 0$ since z is a fixed point of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ and $f : D\ (\equiv \operatorname{dom} f) \subset X \to (-\infty, \infty]$ is a proper strictly totally convex function. As a result, $\{T^n y\}$ converges to z, for all $y \in A_{i+1}$.

It is now proven that $z \in \bigcap_{i\in\overline{p}} A_i$ is the unique fixed point of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$. Assume not so that there is $z_1 (\neq z) = Tz_1 \in \bigcap_{i\in\overline{p}} A_i$. Then, $D_f(T^n x, T^n z_1) \to 0$ as $n \to \infty$ from (17) for $y = z_1$ since $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a generalized point-dependent (K, λ)-hybrid $p(\geq 2)$-cyclic self-mapping relative to D_f with $\lambda : \bigcup_{i\in\overline{p}} A_i \to \Lambda \subset \mathbf{R}$ and $K : \bigcup_{i\in\overline{p}} A_i \to (0, 1)$ so that $\{T^n x\} \to z_1 = T^n z_1$, since $D_f(x, y) > 0$ if $x, y (\neq x) \in \bigcap_{i\in\overline{p}} A_i = \operatorname{int}(\bigcap_{i\in\overline{p}} A_i)$ since $f : D (\equiv \operatorname{dom} f) \subset X \to (-\infty, \infty]$ is proper and totally strictly convex, and since $T^n x \to z$, and $z \in \bigcap_{i\in\overline{p}} A_i$. Since $\bigcap_{i\in\overline{p}} A_i$ is closed and convex, it turns out that z is the unique fixed point of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$.

Note that the result also holds for any for all $y_1 \in \bigcup_{i\in\overline{p}} A_i$ since $y \in \bigcup_{j(\neq i)\in\overline{p}} A_j$ maps to $y_1 = T^{k_i} y \in A_{i+1}$ for some nonnegative integer $k_i \leq p - 1$ through the self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ so that $D_f(T^n x, T^n y_1) = D_f(T^n x, T^{n+k_i} y) \to D_f(z, T^{k_i} z) = D_f(z, z) = 0$ as $n \to \infty$ since $z \in \bigcap_{i\in\overline{p}} A_i$ is the unique fixed point of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ and $\{T^n y\}$ converges to z for any $y \in \bigcup_{i\in\overline{p}} A_i$. □

The subsequent result directly extends Theorem 3 to the p-composite self-mappings $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$, for all $i \in \overline{p}$, defined as $T^p x = T^j (T^{p-j} x)$; for all $x \in \bigcup_{i\in\overline{p}} A_i$, subject to $i = p - j - k$, for all $i, j \in \overline{p}$. The subsets $A_i \subset X$, $i \in \overline{p}$ are not required to intersect since the restricted composite mappings as defined earlier are self-mappings on nonempty, closed, and convex sets.

Corollary 4. *Assume that*

(1) *$f_i : D(\equiv \operatorname{dom} f) \subset X \to (-\infty, \infty]$ is a proper strictly totally convex function which is lower-semicontinuous and Gâteaux differentiable in* int D*, and, furthermore, it is bounded on any bounded subsets of* int D*;*

(2) *$A_i (\neq \varnothing) \subseteq \operatorname{int} D \subset H$ is bounded and closed, for all $i \in \overline{p}$, $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a p-cyclic self-mapping so that $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ for some $i \in \overline{p}$ is a generalized point-dependent (K, λ_i)-hybrid $p(\geq 2)$-cyclic self-mapping relative to D_f for some given functions $\lambda_i : \bigcup_{i\in\overline{p}} A_i \to \Lambda \subset \mathbf{R}$ and $K : \bigcup_{i\in\overline{p}} A_i \to (0, 1)$ for some $i \in \overline{p}$ defined by $K(y) = \prod_{j=i}^{i+p-1} [K_j(T^{j-i} y)]$ for any $y \in A_{i+1}$, for all $i \in \overline{p}$, and $K_i : A_{i+1} \to (0, a_i]$ for some $a_i \in \mathbf{R}_+$, for all $i \in \overline{p}$, where $A := \bigcup_{i\in\overline{p}} A_i \subset H$, Λ being bounded and A_i being, furthermore, convex for the given $i \in \overline{p}$;*

(3) *there is a convergent sequence $\{T_i^{np} x\}$ to some $z_i \in A_i$ for some $x \in A_i$ and $i \in \overline{p}$.*

Then, $z_i = Tz_i$ is a unique fixed point of $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ to which all sequences $\{T_i^{np} x\}$ converge for any $x \in A_i$ for $i \in \overline{p}$.

Also, if conditions (1)–(3) are satisfied with all the subsets A_i, for all $i \in \overline{p}$, being nonempty, closed, and convex for some proper strictly convex function $f \equiv f_i : D(\equiv \operatorname{dom} f) \subset X \to (-\infty, \infty]$ which is Gâteaux differentiable in int D*, then $z_i = Tz_i$, for all $i \in \overline{p}$, is a unique fixed point of $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$, for all $i \in \overline{p}$, to which all sequences $\{T_i^{np} x\}$ converge for any $x \in A_i$, for all $i \in \overline{p}$. The p unique fixed points of each generalized point-dependent (K, λ_i)-hybrid 1-cyclic composite self-mappings $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$, for all $i \in \overline{p}$, fulfil the relations $z_{p-i} = T^j z_k$ for $i = p - j - k$, for all $i \in \overline{p-1}$, for all $j \in \overline{p}$.*

Outline of Proof. Note that $D \cap \bigcap_{i\in\overline{p}} (A_i) \neq \varnothing$. Equation (14) is now extended to $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ for the given $i \in \overline{p}$ leading to

$$D_f\left(T_i^p x, T_i^p y\right) \leq K_i(y) D_f(x, y) + \lambda_i(y) \left\langle x - T_i^p x, f'(y) - f'\left(T_i^p y\right)\right\rangle, \quad \forall x, y \in A_i,\ i \in \overline{p}, \tag{20}$$

since T_i^p is a trivial 1-cyclic self-mapping on A_i for $i \in \overline{p}$. The previous relation leads recursively to.

$$D_f\left(T_i^{np}x, T_i^{np}y\right) \le \widehat{K}^n(y) D_f(x,y) + \left(\frac{1}{1-\widehat{K}(y)} + \widehat{M}_{inp}\right) \times \left(\max_{1\le j\le n}\left[\lambda\left(T_i^{(j-1)p}y\right)\left\langle T_i^{(j-1)p}x - T_i^{jp}x, f'\left(T_i^{(j-1)p}y\right) - f'\left(T_i^{jp}y\right)\right\rangle\right]\right), \tag{21}$$

with $T_i^{np}x,\ T_i^{np}y \in A_i$, for all $y \in A_{i+1}$, for the given $i \in \overline{p}$ with $K(y) < 1$, where $K(y)$ is independent of the particular $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ for $i \in \overline{p}$. One gets by using very close arguments to those used in the proof of Theorem 3 that $\exists \lim_{n,m\to\infty} D_f(T_i^{nmp}x, T_i^{nmp}y) = 0$. Then, $\{T_i^{nmp}x\}$ converges to some $z_i \in A_i$ which is proven to be a unique fixed point in the nonempty, closed, and convex set A_i for $i \in \overline{p}$. The remaining of the proof is similar to that of Theorem 3. The last part of the result follows by applying its first part to each of the p generalized point-dependent (K,λ)-hybrid 1-cyclic composite self-mappings $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ relative to D_f, for all $i \in \overline{p}$.

Remark 5. If $f : D \mid \bigcup_{i\in\overline{p}} A_i \subset X \to (-\infty,\infty]$ is totally convex if it is a continuous strictly convex function which is Gâteaux differentiable in $\operatorname{int} D$, $\dim X < \infty$ and $D \mid \bigcup_{i\in\overline{p}} A_i$ is closed, [20]. In view of this result, Theorem 3 and Corollary 4 are still valid if the condition of its strict total convexity of $f : D \mid \bigcup_{i\in\overline{p}} A_i \to (-\infty,\infty]$ is replaced by its continuity and its strict convexity if the Banach space is finite dimensional. Since $v_f(x,t) \ge \delta_f(t) > 0$, for all $t \in \mathbf{R}_+$, it turns out that if $f : D \mid \bigcup_{i\in\overline{p}} A_i \to (-\infty,\infty]$ is uniformly convex, then it is totally convex. Therefore, Theorem 3 and Corollary 4 still hold if the condition of strict total convexity is replaced with the sufficient one of strict uniform convexity. Note that if a convex function f is totally convex then it is sequentially consistent in the sense that $D_f(x_n, y_n) \to 0$ as $n \to \infty$ if $\|x_n - y_n\| \to 0$ as $n \to \infty$ for any sequences $\{x_n\}$ and $\{y_n\}$ in D.

Some results on weak cluster points of average sequences built with the iterated sequences generated from hybrid cyclic self-mappings $\{T^n x\}$ relative to a Bregman distance D_f, for $x \in A_i$ and some $i \in \overline{p}$, are investigated in the following results related to the fixed points of $\{T^n x\}$.

Theorem 6. *Assume that*

(1) *X is a reflexive space and $f : D \subset X \to (-\infty,\infty]$ is a lower-semicontinuous strictly convex function, so that it is Gâteaux differentiable in* $\operatorname{int}(D)$*, and it is bounded on any bounded subsets of* $\operatorname{int} D$*;*

(2) *a p-cyclic self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is given defining a composite self-mapping $T^p : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ with $A_i(\neq\varnothing) \subseteq \operatorname{int} D \subset H$ being bounded, convex, and closed, for all $i \in \overline{p}$, so that its restricted composite mapping $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ to A_i, for some given $i \in \overline{p}$, is generalized point-dependent $(1,\lambda_i)$-hybrid relative to D_f for some $\lambda_i : A_i \to \mathbf{R}$ and the given $i \in \overline{p}$.*

Define the sequence $\{S_n^{(i)}x\} \equiv \{(1/n)\sum_{k=0}^{n-1} T^{kp}x\}$ for $x \in A_i$, where $T_i^0 \equiv T^{0p}$ is the identity mapping on A_i so that $T^{0p}x = x$, for all $x \in A_i$, and assume that $\{T^n x\}$ is bounded for $x \in A_i$. Then, the following properties hold.

(i) Every weak cluster point of $\{S_n^{(i)}x\}$ for $x \in A_i$ is a fixed point $v_i \in A_i$ of $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ of $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ for the given $i \in \overline{p}$. Under the conditions of Theorem 3, there is a unique fixed point of $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ which coincides with the unique cluster point of $\{S_n^{(i)}x\}$.

(ii) Define sequences $\{S_n^{(i,j)}x\} \equiv \{(1/n)\sum_{k=0}^{n-1} T^{kp+j}x\}$ for any integer $1 \le j \le p-1$ and $x \in A_i$ where A_i are bounded, closed, and convex, for all $i \in \overline{p}$. Thus, $\{S_n^{(i,j)}x\}$ converges weakly to $v_{i+j} = T^j v_i \in A_{i+j}$ for $x \in A_i$, where $v_i \in A_i$ is a fixed point of $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ and a weak cluster point of $\{S_n^{(i)}x\}$ for $x \in A_i$ and $v_{i+j} \in A_{i+j}$ $(1 \le j \le p-1)$ is both a fixed point of $T_{i+j}^p : \bigcup_{j\in\overline{p}} A_j \mid A_{i+j} \to A_{i+j}$ and a weak cluster point of $\{S_n^{(i,j)}x\}$ for $x \in A_i$. Furthermore, $v_{i+j} = T^j v_i$ if $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is continuous.

Proof. Using (14) with $T^p : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ being a generalized point-dependent (K,λ)-hybrid $p(\ge 2)$-cyclic self-mapping relative to D_f for $\lambda : \bigcup_{i\in\overline{p}} A_i \to \mathbf{R}$, with $K(y) = \prod_{j=i}^{i+p-1}[K_j(T^{j-i}y)] = 1$, for all $y \in \bigcup_{i\in\overline{p}} A_i$, yields

$$\begin{aligned} &D_f\left(T^{kp}x, y\right) - D_f\left(T^{(k+1)p}x, T^p y\right) \\ &\quad + \lambda(y)\left\langle T^{kp}x - T^{(k+1)p}x, f'(y) - f'\left(T^p y\right)\right\rangle \\ &= f\left(T^{kp}x\right) - f\left(T^{(k+1)p}x\right) + f\left(T^p y\right) \\ &\quad - f(y) - \left\langle T^{kp}x - y, f'(y)\right\rangle \\ &\quad + \left\langle T^{(k+1)p}x - T^p y, f'\left(T^p y\right)\right\rangle \\ &\quad + \lambda(y)\left\langle T^{kp}x - T^{(k+1)p}x, f'(y) - f'\left(T^p y\right)\right\rangle \\ &\ge 0, \quad \forall k \in \mathbf{N}_0 = \mathbf{N}\cup\{0\}. \end{aligned} \tag{22}$$

Summing up from $k = 0$ to $k = n-1$ and taking $n \to \infty$ yields

$$\frac{f(x) - f\left(T^{np}x\right)}{n} + f\left(T^p y\right) - f(y) + \lambda(y)\left\langle \frac{x - T^{np}x}{n}, f'(y) - f'\left(T^p y\right)\right\rangle$$

$$
\begin{aligned}
&-\left\langle \frac{1}{n}\sum_{k=0}^{n-1} T^{kp}x - y, f'(y)\right\rangle \\
&+\left\langle \frac{1}{n}\sum_{k=0}^{n-1} T^{(k+1)p}x - T^{p}y\ , f'\left(T^{p}y\right)\right\rangle \\
&= \frac{f(x)-f\left(T^{np}x\right)}{n} + f\left(T^{p}y\right) - f(y) \\
&+\lambda(y)\left\langle \frac{x-T^{np}x}{n}, f'(y) - f'\left(T^{p}y\right)\right\rangle \\
&-\left\langle \frac{1}{n}\sum_{k=0}^{n-1} T^{kp}x - y, f'(y)\right\rangle \\
&+\left\langle \frac{1}{n}\sum_{k=0}^{n-1} T^{kp}x + \frac{T^{np}x}{n} - T^{p}y, f'\left(T^{p}y\right)\right\rangle \\
&\longrightarrow f\left(T^{p}v_i\right) - f(v_i) + \langle v_i - T^{p}y, f'\left(T^{p}y\ \right)\rangle \\
&\quad + \left\langle y - v_i, f'(v_i)\right\rangle (\geq 0) \quad \text{as } n\longrightarrow\infty,\ \forall y\in A_i,
\end{aligned}
\tag{23}
$$

since $\{T^n x\}$ is bounded for $x \in A_i$, its subsequence $\{T^{np}x\}$ is then bounded for $x \in A_i$, and $\{f(T^{np}x)\}$ is also bounded on the bounded subset $\bigcup_{i\in\overline{p}} A_i$ of int D. Then, $\{(x - T^{np}x)/n\}$ converges to zero since $\{x - T^{np}x\}$ is bounded for $x \in A_i$. Since $f : D \subset X \to (-\infty, \infty]$ is lower-semicontinuous, the set of subgradients is bounded in all bounded subsets $A_i \subseteq \text{int}\, D$, for all $i \in \overline{p}$. As a result, $\{S_n^{(i)}x\} \equiv \{(1/n)\sum_{k=0}^{n-1} T^{kp}x\}$ is bounded for $x \in A_i$ and has a subsequence $\{S_{n_i}^{(i)}x\}$ being weakly convergent to some $v_i \in A_i$ for $x \in A_i$ since X is reflexive. One gets by taking $y = v_i$ that

$$
\begin{aligned}
-D_f\left(v_i, T_i^{p}v_i\right) &= f\left(T_i^{p}v_i\right) - f(v_i) \\
&\quad + \left\langle v_i - T_i^{p}v_i, f'\left(T_i^{p}v_i\right)\right\rangle \geq 0.
\end{aligned}
\tag{24}
$$

Hence, it follows from Lemma 1 that $v_i = T_i^{p}v_i$ is a fixed point $v_i \in A_i$ of $T_i^{p} : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ for the given $i \in \overline{p}$. Now, consider $\{S_{n_i}^{(i,j)}x\}$, $x \in A_i$, for any integers $1 \leq j \leq p-1$ so that $T^j x \in A_{i+j}$ for $x \in A_i$. Hence, Property (i) follows with the uniqueness and the coincidence of the weak cluster point of $\{S_n^{(i)}x\}$ and fixed point of $T_i^{p} : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ under Theorem 3. The previous reasoning remains valid so that $\{S_{n_i}^{(i,j)}x\}$ is weakly convergent to some $v_{i+j} \in A_{i+j}$ since $\{T^{n+j}x\} \in A_{i+j}$ is bounded for $x \in A_i$ (since $\{T^n x\}$ is bounded for $x \in A_i$ and j is finite), its subsequence $\{T^{np+k}x\}$ for $x \in A_i$ is also bounded so that $\{(T^j x - T^{np+j}x)/n\}$ converges to zero since $\{T^j x - T^{np+j}x\}$ in A_{i+j} is bounded for $x \in A_i$, $1 \leq j \leq p-1$. Now, take $y = v_{i+j}$ so that $D_f(v_{i+j}, T_{i+j}^{p}v_{i+j}) = 0$, and then $v_{i+j} = T_{i+j}^{p}v_{i+j}$, $1 \leq j \leq p-1$, is both a fixed point of $T_{i+j}^{p} : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_{i+j}$ and a weak cluster point of $\{S_{n_i}^{(i,j)}x\}$. Now, take $x \in A_i$ so that $T^j x \in A_{i+j}$. Then, $T^j(T^{np}x) = T^{np+j}x = T^{np}(T^j x) \to v_{i+j}$ as $n \to \infty$ for $x \in A_i$ and, in addition, $v_{i+j} = T^j v_i$ if $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is continuous. Hence, Property (ii) follows. □

3. Extensions for Generalized Point-Dependent Cyclic Hybrid Self-Mappings on Nonintersecting Subsets: Weak Convergence to Weak Cluster Points of a Class of Sequences

Some of the results of Section 2 are now generalized to the case when the subsets of the cyclic mapping do not intersect $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$, in general, by taking advantage of the fact that best proximity points of such a self-mapping are fixed points of the restricted composite mapping $T_i^{p} : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ for $i \in \overline{p}$. Weak convergence of averaging sequences to weak cluster points and their links with the best proximity points in the various subsets of the p-cyclic self-mappings is discussed. Firstly, the following result follows from a close proof to that of Theorem 6 which is omitted.

Theorem 7. *Let X be a reflexive space, and let $f : D \subset X \to (-\infty, \infty]$ be a lower-semicontinuous strictly convex function so that it is Gâteaux differentiable in* int(D) *and it is bounded on any bounded subsets of* int D. *Consider the generalized point-dependent $(p \geq 1)$-cyclic hybrid self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ being $(1, \lambda_i)$ relative to D_f for some $\lambda_i : A_i \to \mathbf{R}$ such that $A_i (\neq \varnothing) \subseteq \text{int}\, D \subset H$ are all bounded, convex, closed, and with nonempty intersection. Define the sequence $\{S_n x\} \equiv \{(1/n)\sum_{k=0}^{n-1} T^k x\}$ for $x \in \bigcup_{i\in\overline{p}} A_i$, where T^0 is the identity mapping on $\bigcup_{i\in\overline{p}} A_i$, and assume that $\{T^n x\}$ is bounded for $x \in \bigcup_{i\in\overline{p}} A_i$. Then, the following properties hold.*

(i) *Every weak cluster point of $\{S_n^{(i)}x\}$ for $x \in A_i$ is a fixed point $v_i \in A_i$ of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$.*

(ii) *Define the sequence $\{S_n x\} \equiv \{(1/n)\sum_{k=0}^{n-1} T^k x\}$ for $x \in \bigcup_{i\in\overline{p}} A_i$ which is bounded, closed, and convex, for all $i \in \overline{p}$ and any integer $1 \leq j \leq p-1$. Thus, $\{S_n x\}$ converges weakly to the fixed point $v = Tv \in \bigcap_{i\in\overline{p}} A_i$ of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ for $x \in \bigcup_{i\in\overline{p}} A_i$ which is also a weak cluster point of $\{S_n x\}$.*

Remark 8. The results of Theorems 6 and 7 are extendable without difficulty to the weak cluster points of other related sequences to the considered ones.

(1) Define sequences $\{S_n^{(j)}x\} \equiv \{(1/n)\sum_{k=0}^{n-1} T^{k+j}x\}$, $x \in \bigcup_{i\in\overline{p}} A_i$, for any given finite non-negative integer j under all the hypotheses of Theorem 7. With this notation, the sequence considered in such a corollary is $\{S_n x\} \equiv \{S_n^{(0)}x\}$. Direct calculation yields $(S_n^{(j)}x - S_n x) = (1/n)\sum_{k=0}^{j-1}(T^{k+n}x - T^k x) \to 0$ for $x \in \bigcup_{i\in\overline{p}} A_i$ as $n \to \infty$ since $\{T^{k+n}x - T^k x\}$, and then $\{\sum_{k=0}^{j-1}(T^{k+n}x - T^k x)\}$, is bounded. Then, $S_n^{(j)}x \to v$ weakly which is the same fixed point of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ in $\bigcap_{i\in\overline{p}} A_i$ which is a weak cluster point of $\{S_n^{(j)}x\}$ for $x \in \bigcup_{i\in\overline{p}} A_i$ for any finite non-negative integer j.

(2) Consider all the hypotheses of Theorem 7 and now define sequences $\{S_n^{[j]}x\} \equiv \{(1/n)\sum_{k=0}^{n+j-1} T^k x\}$, $x \in \bigcup_{i\in\overline{p}} A_i$, for any given finite non-negative integer j. With this notation, the sequence considered in the corollary is $\{S_n x\} \equiv \{S_n^{[0]}x\}$. Direct calculation yields $(S_n^{[j]}x - S_n x) = (1/n)\sum_{k=0}^{j-1}(T^{k+n}x) \to 0$ weakly for $x \in \bigcup_{i\in\overline{p}} A_i$ as $n \to \infty$ since $\{T^{k+n}x\}$, and then $\{\sum_{k=0}^{j-1}(T^{k+n}x)\}$, is bounded. Then, $S_n^{[j]}x \to \nu$ weakly which is the same fixed point of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ in $\bigcap_{i\in\overline{p}} A_i$ which is a weak cluster point of $\{S_n^{[j]}x\}$ for $x \in \bigcup_{i\in\overline{p}} A_i$ and for any finite non-negative integer j.

(3) Now consider the hypotheses of Theorem 6. It turns out that the sequence $\{S_n^{(i,j)}x\} \equiv \{(1/n)\sum_{k=0}^{n-1} T^{kp+j}x\}$ for $x \in A_i$ satisfies for any integer $1 \le j \le p-1$,

$$\begin{aligned}
S_n^{(i,j)}x &= \frac{1}{n+1}\frac{n+1}{n}\left(\sum_{k=0}^{n-1} T^{kp+j}x\right) \\
&\quad + \frac{T^j}{n}\left(\sum_{k=0}^{n-1} T^{kp}x - \sum_{k=0}^{n} T^{kp}x\right) \\
&= T^j\left(\frac{n+1}{n}S_{n+1}^{(i)}x - \frac{1}{n}T^{np}x\right), \\
S_n^{(i,j)}x &= \frac{1}{n+1}\frac{n+1}{n}\left(\sum_{k=0}^{n-1} T^{kp}\left(T^j x\right)\right) \\
&\quad + \frac{1}{n}\left(\sum_{k=0}^{n-1} T^{kp}\left(T^j x\right) - \sum_{k=0}^{n} T^{kp}x\right) \\
&= \left(\frac{n+1}{n}S_{n+1}^{(i+j)}x_{i+j} - \frac{1}{n}T^{np}x_{i+j}\right) \longrightarrow \nu_{i+j} \\
&\equiv T_{i+j}^p \nu_{i+j} \left(\in A_{i+k}\right)
\end{aligned} \tag{25}$$

weakly as $n \to \infty$, where $x_{i+j}(= T^j x) \in A_{i+j}$ since $x \in A_i$, $\{T^n x\}$ is bounded, and $\{S_{n+1}^{(i,j)}z\} \equiv (1/n)\sum_{k=0}^{n-1} T^{kp}z = (1/n)\sum_{k=0}^{n-1}\left(T_{i+j}^p\right)^k z$ for $z \in A_{i+j}$ and $1 \le j \le p-1$. Thus, ν_{i+j} is a fixed point of $T_{i+j}^p : \bigcup_{j\in\overline{p}} A_j \mid A_{i+j} \to A_{i+j}$ which is also a weak cluster point of the sequences $\{S_{n+1}^{(i,j)}z\}$ for $1 \le j \le p-1$. However, it is not guaranteed that $\nu_{i+j} = T^j\nu_i = T_i^p\nu_i = T_{i+j}^p\nu_{i+j}$ without additional hypotheses on $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ such as its continuity, or at least that of the composite mapping $T^j : A_i \to A_{i+j}$ allowing to equalize the function of the limit with the limit of the function at such a fixed point.

(4) Now, define $\{S_n^{[i,j]}x\} \equiv \{(1/n)\sum_{k=0}^{n+j-1} T^{kp}x\}$ for $x \in A_i$. Note that for $x \in A_i$, $\exists \nu_i \in A_i$,

$$\begin{aligned}
S_n^{[i,j]}x &= \frac{1}{n}\sum_{k=0}^{n-1} T^{kp}x \\
&\quad + \frac{1}{n}\sum_{k=0}^{j-1} T^{(n+k)p}x = S_n^{(i)}x \\
&\quad + \frac{1}{n}\sum_{k=0}^{j-1} T^{(n+k)p}x \longrightarrow \nu_i \left(= T_i^p\nu_i\right)
\end{aligned} \tag{26}$$

weakly as $n \to \infty$ since j is finite, which is a fixed point in A_i of the composite mapping T_i^p and a weak cluster point of $\{S_n^{[i,j]}x\}$ for finite j.

Note that Theorem 6 are supported by boundedness constraints for the sequences of iterates obtained through the cyclic self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ which is generalized point-dependent with respect to some convex function. The results of identification of weak cluster points of some average sequences with fixed points of the cyclic self-mapping or its composite mappings do not guarantee uniqueness of fixed points and weak cluster points because the cyclic self-mapping is not restricted to be contractive. By incorporating some background contractive-type conditions for the cyclic self-mapping, the previous results can be extended to include uniqueness of fixed points as follows.

Theorem 9. *Assume that.*

(1) *Assumption 1 of Theorem 6 holds with the restriction of* $(X, \|\|)$ *to be a uniformly convex Banach space;*

(2) *Assumption 2 of Theorem 6 holds, and, furthermore, all the p-cyclic composite mappings with restricted domain* $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$*; for all* $i \in \overline{p}$ *are either contractive or Meir-Keeler contractions.*

Then, the following properties hold.

(i) Theorem 6(i)-(ii) holds. Furthermore, each of the mappings $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ has a unique fixed point $\nu_i \in A_i$ which are also best proximity points of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ in A_i so that $\nu_{i+j} = T^j\nu_i$; for all $j \in \overline{p-i}$, for all $i \in \overline{p}$.

(ii) If, in addition, $\bigcap_{i\in\overline{p}} A_i \neq \varnothing$, then, there is a unique fixed point $\nu \in \bigcap_{i\in\overline{p}} A_i$ of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ and $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$, for all $i \in \overline{p}$.

Proof. Note that uniformly convex Banach spaces $(X, \|\|)$ are also reflexive spaces required by Theorem 6. Each mapping $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ has a unique fixed point $\nu_i \in A_i$, for all $i \in \overline{p}$, irrespective of $\bigcap_{i\in\overline{p}} A_i$ being empty or not if $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ is either a cyclic contraction or a Meir-Keeler contraction [1–3], since A_i $(i \in \overline{p})$ are non-empty, closed, and convex, and $(X, \|\|)$ is a uniformly convex Banach space so that each $\nu_i \in A_i$; for all $i \in \overline{p}$ is a best proximity point in A_i of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$. It follows from the hypothesis that there is a unique weak cluster point of $\{S_n^{(i)}x\}$ for $x \in A_i$ which is the unique fixed point of $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$, for all $i \in \overline{p}$, and also the unique best proximity point of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ in A_i for $i \in \overline{p}$.

It is now proven that if $\bigcap_{i\in\overline{p}} A_i \neq \varnothing$ then $(\bigcap_{i\in\overline{p}} A_i) \ni \nu = \nu_i \in A_i$, for all $i \in \overline{p}$. Take some $x \in A_i \cap A_j$ for some $i, j(\neq i) \in \overline{p}$. Thus, $T^{pn}x \to \nu_i(\in A_i)$ and $T^{pn}x \to \nu_j$ as $n \to \infty$ since ν_i is the unique fixed point of $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ and $\nu_j(\in A_j)$ is the unique fixed point of $T_j^p : \bigcup_{i\in\overline{p}} A_i \mid A_j \to A_j$. Then, $\nu = \nu_i = T^k\nu$, for all $i \in \overline{p}$; for all $k \in \mathbf{N}$,

$S_n^{(i)}x \to v$ weakly as $n \to \infty$, for all $i \in \overline{p}$, and $v \in \bigcap_{i\in\overline{p}} A_i$ is the unique weak cluster point of $\{S_n^{(i)}x\}$, for all $i \in \overline{p}$. □

Theorem 9 can be also extended "mutatis-mutandis" to the convergence of weak cluster points of the alternative sequences discussed in Remark 8. It is now proven that the sets of fixed points of the restricted composite mapping $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$, some $i \in \overline{p}$, are convex if such mappings are quasi-nonexpansive with respect to D_f in the sense that it has (at least) a fixed point in A_i and $D_f(v, T_i^p x) \le D_f(v, x)$, for all $x \in A_i$, and $f : D \subset X \to (-\infty, \infty]$ is a proper strictly convex function, [16]. The concept of quasi-nonexpansive mapping is addressed in the subsequent result to discuss the topology of fixed points and best proximity points of composite mappings of cyclic self-mappings.

Theorem 10. *Let $f : D \subset X \to (-\infty, \infty]$ be a proper strictly convex function on the Banach space $(X, \|\|)$ so that it is Gâteaux differentiable in* int D*, and consider the restricted composite mapping $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ for some given $i \in \overline{p}$ built from the p-cyclic self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ so that A_i is nonempty, convex, and closed. Assume that $A_j \subseteq$* int D*, for all $j \in \overline{p}$, and that the composite mapping $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ is quasi-nonexpansive with respect to D_f for the given $i \in \overline{p}$.*

Then, the following properties hold.

(i) The set of fixed points $F(T_i^p)$ of $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ is a closed and convex subset of A_i for the given $i \in \overline{p}$.

(ii) Assume, in addition, that A_i, for all $i \in \overline{p}$, are nonempty convex closed subsets of H subject to $\bigcup_{i\in\overline{p}} A_i \subseteq$ int D, and assume also that $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ are quasi-nonexpansive with respect to D_f, for all $i \in \overline{p}$. Then, the set of best proximity points in A_i of the p-cyclic self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ coincides with $F(T_i^p)$, and it is then a closed and convex subset of A_i, for all $i \in \overline{p}$. Furthermore, if $\bigcap_{i\in\overline{p}} A_i \neq \varnothing$, then $F(T) = \text{cl}\, F(T) \subseteq \bigcap_{i\in\overline{p}} F(T_i^p) \subseteq \bigcap_{i\in\overline{p}} A_i$ which is then nonempty, closed, and convex.

Proof. Take $x \in \text{cl}\, F(T_i^p) \subseteq A_i \subseteq \bigcup_{i\in\overline{p}} A_i \subseteq$ int D and $\{x_n\} \subseteq \text{cl}\, F(T_i^p)$ so that $\{x_n\} \to x$ as $n \to \infty$. Note that $F(T_i^p)$ and $\text{cl}\, F(T_i^p)$ are nonempty sets since $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ is quasi-nonexpansive with respect to D_f then possessing at least a fixed point. By the continuity of $D_f(\cdot, T_i^p x)$ and that of $D_f(x, \cdot)$, the strict convexity of $f : D \subset X \to (-\infty, \infty]$, and the assumption $D_f(v, T_i^p x) \le D_f(v, x)$, one has

$$\begin{aligned} D_f\left(x, T_i^p x\right) &= \lim_{n\to\infty} D_f\left(x_n, T_i^p x\right) \\ &\le \lim_{n\to\infty} D_f\left(x_n, x\right) \\ &\le D_f(x, x) = 0 \end{aligned} \tag{27}$$

and $x = T_i^p x$, from the strict convexity of $f : D \subset X \to (-\infty, \infty]$ and Lemma 1, which is in $F(T_i^p)$ which is then a closed subset of A_i as a result. Now, it is proven that is convex. Following the steps of a parallel result proven in [16] for noncyclic self-mappings, take $x, y(\neq x) \in F(T_i^p)$ and consider for some arbitrary real constant $\alpha \in [0, 1]$ a point $z = \alpha x + (1-\alpha)y$ which is in A_i since such a set is convex. Since $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ is quasi-nonexpansive with respect to D_f leading to

$$\begin{aligned} D_f\left(x, T_i^p z\right) &\le D_f(x, z) \\ &= f(x) - f(z) - \left\langle x - z f'(z)\right\rangle, \\ D_f\left(y, T_i^p z\right) &\le D_f(y, z) \\ &= f(y) - f(z) - \left\langle y - z f'(z)\right\rangle, \end{aligned} \tag{28}$$

and, since $q = \alpha q + (1-\alpha)q$ and $f(q) = \alpha f(q) + (1-\alpha) f(q)$ for any $q \in A_i$, that

$$\begin{aligned} &D_f\left(z, T_i^p z\right) \\ &= f(z) - f\left(T_i^p z\right) - \left\langle z - T_i^p z, f'\left(T_i^p z\right)\right\rangle \\ &= f(z) + \left[\alpha f(x) - f\left(T_i^p z\right) - \langle x - T_i^p z, f'\left(T_i^p z\right)\rangle\right] \\ &\quad + (1-\alpha)\left[f(y) - f\left(T_i^p z\right) - \langle y - T_i^p z, f'\left(T_i^p z\right)\rangle\right] \\ &\quad - \left[\alpha f(x) + (1-\alpha) f(y)\right] \\ &= f(z) + \alpha D_f\left(x, T_i^p\right) + (1-\alpha) D_f\left(y, z\right) \\ &\quad - \left[\alpha f(x) + (1-\alpha) f(y)\right] \\ &\le f(z) + \alpha D_f(x, z) + (1-\alpha) D_f\left(y, z\right) \\ &\quad - \left[\alpha f(x) + (1-\alpha) f(y)\right] \\ &= -\left\langle \alpha x + (1-\alpha) y - (\alpha z + (1-\alpha)) z, f'(z)\right\rangle \\ &= -\langle 0, f'(z)\rangle = 0, \end{aligned} \tag{29}$$

which implies from Lemma 1 that $Tz = z = \alpha x + (1-\alpha)y$ for any $x, y(\neq x) \in F(T_i^p)$ since $f : D \subset X \to (-\infty, \infty]$ is strictly convex and $F(T_i^p) = \text{cl}\, F(T_i^p) \subseteq (\text{int}\, D \cap A_i)$. Thus, $F(T_i^p)$ is a convex subset of A_i. Hence, Property (ii) follows. The first part of property (ii) is a direct consequence of property (i) if the p composite self-mappings on all the sets A_i are quasi-nonexpansive with respect to D_f since the respective sets of fixed points are the best proximity points of the p-cyclic self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ in each of the sets A_i, for all $i \in \overline{p}$. If, furthermore, the sets A_i, for all $i \in \overline{p}$, have a nonempty intersection, then its set of fixed points coincides with the intersection of the sets of best proximity points of the composite mappings $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ which are all identical for $i \in \overline{p}$. The proof is trivial. Take any $x \in \bigcap_{i\in\overline{p}} A_i (\neq \varnothing)$. Then, the sequence of iterates

obtained through the composite $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ converges to some $z \in \bigcap_{i\in\overline{p}}(A_i \cap F(T_i^p))$. This implies that $\bigcap_{i\in\overline{p}} F(T_i^p) \neq \varnothing$, closed and convex from property (i). Thus, $F(T)(\neq \varnothing) \subseteq \bigcap_{i\in\overline{p}} F(T_i^p)$. Then, the set of fixed points of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is $F(T) \equiv \operatorname{cl} F(T) \subseteq \bigcap_{i\in\overline{p}}(A_i \cap F(T_i^p))$. Property (ii) has been proven. □

Concerning that Theorem 10(ii), note that the set inclusion $F(T) = \bigcap_{i\in\overline{p}} F(T_i^p)$ does not guarantee, in general, that the identity $F(T) \subseteq \bigcap_{i\in\overline{p}} F(T_i^p)$ is not guaranteed for the case when $\bigcap_{i\in\overline{p}} A_i \neq \varnothing$ except for cases under extra conditions such as the contractiveness of the composite mappings built from the p-cyclic one leading, for instance, to the uniqueness of the fixed point of the cyclic self-mapping. See, for instance, Theorem 3 and Corollary 4.

It is direct to give sufficient conditions for the restricted composite mappings of the p-cyclic self-mapping to be quasi-nonexpansive under the relevant conditions of Theorem 3, Corollary 4, Theorem 6, and Theorem 7 (see Proposition 3.5 of [16]) for noncyclic self-mappings, as follows.

Theorem 11. *Assume that.*

(1) *X is a reflexive space and $f : D \subset X \to (-\infty, \infty]$ is a proper strictly convex function, so that it is Gâteaux differentiable in* $\operatorname{int}(D)$*, and it is bounded on any bounded subsets of* $\operatorname{int} D$.

(2) *A p-cyclic self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is given defining a composite self-mapping $T^p : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ with the subsets $A_i(\neq \varnothing) \subseteq \operatorname{int} D \subset H$ being bounded, closed, and convex, for all $i \in \overline{p}$.*

(3) *The restricted composite mapping to A_i for some given $i \in \overline{p}$, that is, $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$, is generalized point-dependent $(1, \lambda_i)$-hybrid relative to D_f for some function $\lambda_i : A_i \to \mathbf{R}$ and the given $i \in \overline{p}$ which possesses a bounded sequence $\{T^{pn}x\} \subset A_i$ for some point $x \in A_i$.*

Then, the restricted composite mapping $T_i^p : \bigcup_{j\in\overline{p}} A_j \mid A_i \to A_i$ is quasi-nonexpansive with respect to D_f so that $F(T_i^p)$ is a nonempty closed convex subset of A_i.

Remark 12. The well-known concepts of nonexpansive, nonspreading, hybrid, and contractive cyclic self-mappings [12–16, 19] are useful in the context of particular cases of interest of (14) within the given framework for generalized nonexpansive p-cyclic self-mappings relative to D_f.

(1) If (14) holds $0 \le K_i(y) \le 1$ and $\lambda_i(y) = 0$, for all $y \in A_{i+1}$, for each $x \in A_i$, for all $i \in \overline{p}$, then $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is said to be a generalized nonexpansive p-cyclic self-mapping relative to D_f.

(2) If (14) holds $0 \le K_i(y) \le 1$ and $\lambda_i(y) = 2$, for all $y \in A_{i+1}$, for each $x \in A_i$, for all $i \in \overline{p}$, then $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is said to be a generalized nonspreading p-cyclic self-mapping relative to D_f.

(3) If (14) holds, for all $x \in A_i$, for all $y \in A_{i+1}$, for all $i \in \overline{p}$ with $0 \le K_i(y) \le 1$ and $\lambda_i(y) = 1$, for all $i \in \overline{p}$, then $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is said to be a generalized nonexpansive $(1,1)$-hybrid p-cyclic self-mapping relative to D_f.

(4) If (14) holds, for all $x \in A_i$, for all $y \in A_{i+1}$, for all $i \in \overline{p}$ with $0 \le K_i(y) \le 1$ and $\lambda_i(y) \neq 1$ for some $y \in A_{i+1}$, for all $i \in \overline{p}$, then $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is said to be a generalized nonexpansive (point-dependent if some $\lambda_i(y)$ for some $i \in \overline{p}$ is not constant) $(1, \lambda)$-hybrid p-cyclic self-mapping relative to D_f.

(5) If (14) holds, for all $x \in A_i$, for all $y \in A_{i+1}$, for all $i \in \overline{p}$ with $0 \le K_i(y) \le 1$ and $\lambda_i(y) \neq 0$ for some $y \in A_{i+1}$; for all $i \in \overline{p}$ then $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is said to be a generalized nonexpansive (point-dependent if some $\lambda_i(y)$ for some $i \in \overline{p}$ is not constant) p-cyclic self-mapping relative to D_f;

(6) If (14) holds, for all $x \in A_i$, for all $y \in A_{i+1}$, for all $i \in \overline{p}$ with $0 \le K_i(y) \le K < 1$ and $\lambda_i(y) \neq 1$ for some $y \in A_{i+1}$, for all $i \in \overline{p}$, then $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is said to be a generalized contractive (point-dependent if some $\lambda_i(y)$ for some $i \in \overline{p}$ is not constant) (K, λ)-hybrid p-cyclic self-mapping relative to D_f.

(7) If (14) holds $0 \le K_i(y) \le K < 1$ and $\lambda_i(y) = 0$, for all $y \in A_{i+1}$ for each $x \in A_i$, for all $i \in \overline{p}$, then $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is said to be a generalized K-contractive p-cyclic self-mapping relative to D_f.

The various given results can be easily focused on these particular cases.

4. Examples

Dynamic systems are a very important tool to describe and design control systems in applications. Fixed point theory has been found useful to study their controllability and stability properties. See, for instance, [21–26] and references there in. Two examples are now given related to discrete dynamic systems in order to illustrate the theoretical aspects of this paper.

Example 1. Consider the scalar discrete dynamic system

$$x_{k+1} = Tx_k := a_k\, x_k + \eta_k, \quad \forall k \in \mathbf{N}, \tag{30}$$

for given initial condition x_0 with $\eta_k = \eta_k(\overline{x}_k), \overline{x}_k := \{x_j : j = 0, 1, \ldots, k\}$ being a state disturbance which can include combined effects of parametrical disturbances and unmodeled dynamics (roughly speaking, the neglected dynamic effects of describing a higher-order difference equation by the previous first-order one). The solution sequence is defined by the self-mapping $T : \operatorname{cl}\mathbf{R} \to \operatorname{cl}\mathbf{R}$ ($\operatorname{cl}\mathbf{R} = [-\infty, +\infty]$ being the extended real line including the infinity points) given by $x_{k+1} = (\prod_{i=0}^{k}[a_i])x_0 + \sum_{i=0}^{k}(\prod_{j=i+1}^{k}[a_j])\eta_i$, for all $k \in \mathbf{N}$. It

is assumed that a fixed point exists for some $x_0 \in \mathbf{R}$; that is, a sequence $\{Tx_k\}$ for some initial point x_0 is bounded; That is,

$$\exists \lim_{k\to\infty} \left[\left(\prod_{i=0}^{k}[a_i]\right)x_0 + \sum_{i=0}^{k}\left(\prod_{j=i+1}^{k}[a_j]\right)\eta_i\right] = x^* = x^*(x_0), \tag{31}$$

with $|x^*| < +\infty$. In particular, this holds for the unperturbed system with $|a_k| < 1$ and $\eta_k \equiv 0$ which possess a unique globally asymptotically stable equilibrium $x^* = 0$ which is also a unique fixed point of the solution. If $a_k = 1$, then there is a stable constant solution $x_k = x_0$ for each initial condition which is also a (nonunique) fixed point. In both cases, the mapping $T : \mathrm{cl}\,\mathbf{R} \to \mathrm{cl}\,\mathbf{R}$ is trivially nonexpansive and, in the first case, it is also contractive. Note that the previous mapping is also a trivial cyclic self-mapping for $p = 1$. Under a cyclic repetition of the sequence $\{a_k\}$ with

$$a = \prod_{i=qk}^{q(k+1)-1}[a_i], \qquad b = \sum_{i=0}^{q-1}\left(\prod_{j=i+1}^{q-1}\left[a_{qk+j}\right]\right)\eta_{qk+i}, \quad \forall k \in \mathbf{N}_0, \tag{32}$$

for some $q \in \mathbf{N}$ being constant with $|a| < 1$ and $|b| < +\infty$. In this case, we can describe the given difference equation equivalently as

$$x_{(k+1)q} = T^q x_k := a x_{kq} + b_q \tag{33}$$

for the same initial condition. Then, the composite self-mapping $T^q x_{qk} = x_{(k+1)q}$ generating the subsequence $\{x_{qk}\} \subset \{x_k\}$ of the solution has a unique fixed point $x_q^* = b/(1-a)$ for any given x_0. If, furthermore, there are finite limits $a_{qk+i} \to a_i^*$, $\eta_{qk+i} \to \eta_i^* = (1-a_i^*)\,x_q^* = b(1-a_i^*)/(1-a)$ as $k \to \infty$, for all $i \in \overline{q-1}$, then $Tx_k \to x_q^*$ as $k \to \infty$ which is a fixed point of $T : \mathrm{cl}\,\mathbf{R} \to \mathrm{cl}\,\mathbf{R}$. If the second set of limits exists being all finite, but arbitrary, that is, the identities $\eta_i^* = (1-a_{qi})x_q^*$ do not all hold for $i \in \overline{q-1}$, then the solution sequence converges to a cycle $\{x_q^* = b/(1-a), x_{q+1}^* = a_1^* b/(1-a) + \eta_1^*, \ldots, x_{2q-1}^* = a_{q-1}^* b/(1-a) + \eta_{q-1}^*\}$. We now retake the example under the point if view of a point-dependent λ-hybrid map where $f(x) = \alpha x^2 + \beta$ ($\alpha > 0, \beta \geq 0$). The considered Banach space is $(R, \|\,\|)$ which is a Hilbert space for the inner product being the Euclidean scalar product and the norm is the Euclidean norm. Then,

$$\begin{aligned} D_f(y_k, x_k) &= f(y_k) - f(x_k) - 2\alpha(y_k - x_k)x_k \\ &= \alpha\left[\left(y_k^2 - x_k^2\right) - 2(y_k - x_k)x_k\right] \\ &= \alpha\,(y_k - x_k)^2, \end{aligned}$$

$$\begin{aligned} D_f(y_{k+1}, x_{k+1}) &= \alpha(y_{k+1} - x_{k+1})^2 = \alpha\left[a_k^2(y_k - x_k)^2 + \left(\eta_{yk} - \eta_{xk}\right)^2\right. \\ &\quad \left. + 2a_k(y_k - x_k)\left(\eta_{yk} - \eta_{xk}\right)\right]. \end{aligned} \tag{34}$$

Condition (14) for $T : \mathrm{cl}\,\mathbf{R} \to \mathrm{cl}\,\mathbf{R}$ to be point-dependent 1-cyclic λ-hybrid becomes in particular for some real functions $\lambda(y)$ and $K(y) \in [0,1]$ for $y \in \mathbf{R}$

$$\begin{aligned} &\alpha\left[a_k^2(y_k - x_k)^2 + \left(\eta_{yk} - \eta_{xk}\right)^2\right. \\ &\quad \left. + 2a_k(y_k - x_k)\left(\eta_{yk} - \eta_{xk}\right)\right] \\ &\leq \alpha\left[K(y_k)(y_k - x_k)^2\right. \\ &\quad \left. + 2\lambda(y_k)(x_k - x_{k+1})(y_k - y_{k+1})\right] \\ &= \alpha\left[K(y_k)(y_k - x_k)^2 + 2\lambda(y_k)\left((1-a_k)x_k - \eta_{xk}\right)\right. \\ &\quad \left. \times\left((1-a_k)y_k - \eta_{yk}\right)\right] \\ &= \alpha\left[K(y_k)(y_k - x_k)^2 + 2\lambda(y_k)\right. \\ &\quad \times\left((1-a_k)^2 x_k y_k + \eta_{xk}\eta_{yk}\right. \\ &\quad \left.\left. - (1-a_k)\left(y_k\eta_{xk} + x_k\eta_{yk}\right)\right)\right], \quad \forall k \in \mathbf{N}_0, \end{aligned} \tag{35}$$

and, equivalently,

$$\begin{aligned} &\left(a_k^2 - K(y_k)\right)(y_k - x_k)^2 \\ &\quad \leq \left(\eta_{yk} - \eta_{xk}\right)\left[2a_k(y_k - x_k) - \left(\eta_{yk} - \eta_{xk}\right)\right] \\ &\qquad + 2\lambda(y_k)\left((1-a_k)^2 x_k y_k + \eta_{xk}\eta_{yk}\right. \\ &\qquad\quad \left. - (1-a_k)\left(y_k\eta_{xk} + x_k\eta_{yk}\right)\right), \\ &\qquad\qquad \forall k \in \mathbf{N}_0. \end{aligned} \tag{36}$$

Note that if $x_k = x_{k+1}$ or if $y_k = y_{k+1}$, equivalently, if $\eta_{xk} = (1-a_k)x_k$ or if $\eta_{yk} = (1-a_k)y_k$, then the previous equivalent constraints (35)-(36) cannot be satisfied by a choice of some finite value of $\lambda(y_k)$ unless

$$\begin{aligned} &\left|a_k(y_k - x_k) + (1-a_k)y_k - \eta_{xk}\right| \\ &\quad \leq \sqrt{K(y_k)}\,|y_k - x_k| \quad \text{if } y_k = y_{k+1}, \\ &\left|a_k(y_k - x_k) + (1-a_k)x_k - \eta_{yk}\right| \\ &\quad \leq \sqrt{K(y_k)}\,|y_k - x_k| \quad \text{if } x_k = x_{k+1}, \end{aligned} \tag{37}$$

so that any arbitrary value of $\lambda(y_k)$ would satisfy the inequalities. Note that both inequalities hold directly for any fixed points.

If the previous constraint (36) holds, subject to (37), for some real sequence $\{\lambda(y_k)\}$, then any weak cluster point of $\{S_n^{(i)}x\} \equiv \{(1/n)\sum_{k=0}^{n-1}T^kx\}$ is a fixed point of $T : \mathrm{cl}\,\mathbf{R} \to \mathrm{cl}\,\mathbf{R}$ according to Theorem 6. If there is a unique fixed point according to Theorem 3, then the unique fixed point of $T : \mathrm{cl}\,\mathbf{R} \to \mathrm{cl}\,\mathbf{R}$ and weak cluster point of $\{S_n^{(i)}x\} \equiv \{(1/n)\sum_{k=0}^{n-1}T^kx\}$ coincide. The same property holds for the weak cluster point of the average sequences referred to in Remark 8. Note in particular the following.

(1) If $\eta_{xk} = \eta_{yk} \equiv 0$ and $|a_k| \le \sqrt{K(y_k)} \le 1$, then the previous constraint leads to $0 \le 2\lambda(y_k)(1-a_k)^2x_ky_k$ which guarantees that Theorem 6 holds for any real sequence $\{\lambda(y_k)\}$ satisfying $\lambda(y_k) = \mathrm{sign}(x_ky_k)$ if $|1-a_k||y_k - x_k|x_ky_k \ne 0$ and taking any arbitrary real value, otherwise for each $k \in \mathbf{N}_0$. Since $\eta_{xk} = \eta_{yk} \equiv 0$, a choice of $\lambda = \lambda(y_k)$ independent of x_k is as follows:

$$\begin{aligned}\lambda = \lambda(y_k) = \lambda_0 \ge 0 \quad &\text{if } \min(x_0, y_0) \ge 0,\ \forall k \in \mathbf{N}_0\\ \lambda = \lambda(y_k) = \lambda_0 < 0 \quad &\text{if } \min(x_0, y_0) < 0\\ &\text{or if } \mathrm{sgn}(x_0) = -\mathrm{sgn}(y_0),\ \forall k \in \mathbf{N}_0.\end{aligned} \tag{38}$$

Thus, any cluster point of $\{S_n^{(i)}x\}$ is a fixed point of $T : \mathrm{cl}\,\mathbf{R} \to \mathrm{cl}\,\mathbf{R}$. There is a unique such fixed point $x_q^* = b/(1-a)$ if $|a| < 1$ and $|b| < +\infty$, which is also a globally asymptotically stable equilibrium point of the solution, and there are finite limits $a_{qk+i} \to a_i^*, \eta_{qk+i} \to \eta_i^* = (1-a_i^*)\,x_q^* = b(1-a_i^*)/(1-a)$ as $k \to \infty$ for some given $q \in \mathbf{N}$ and such a fixed point is also a fixed point of the composite self-mapping $T^q : \mathrm{cl}\,\mathbf{R} \to \mathrm{cl}\,\mathbf{R}$. If $|a_k| > 1$, then the constraint of point-dependent λ-hybrid self-mapping is satisfied for $\lambda(y_k) \ge (a_k^2 - K(y_k))\,(y_k - x_k)^2/(1-a_k)^2x_ky_k$ if $x_ky_k \ne 0$, for all $k \in \mathbf{N}_0$. If $x_ky_k \ge 0$ (i.e., both x_k and y_k have the same sign or one of them is zero) then $T : \mathrm{cl}\,\mathbf{R} \to \mathrm{cl}\,\mathbf{R}$ is λ-hybrid for $\lambda = \lambda(y_k) = \lambda_0 \ge 0$, for all $k \in \mathbf{N}_0$. However, Theorem 6 is not applicable for cluster fixed points of the averaging sequence since there is no fixed point of the difference equation, in general.

(2) If $\eta_k = \eta_{xk} = \eta_{yk}$ is not identically zero, then the constraint is satisfied if $|a_k| \le \sqrt{K(y_k)} \le 1$ and

$$\begin{aligned}\lambda(y_k) = \mathrm{sign}\big[&(1-a_k)^2x_ky_k\\ &+ [\eta_k - (1-a_k)(y_k + x_k)]\,\eta_k\big] = \mathrm{sign}\end{aligned} \tag{39}$$

$$\begin{aligned}\Bigg\{&(1-a_k)^2\left(\prod_{i=0}^{k-1}[a_i]\right)x_0y_k + \sum_{i=0}^{k-1}\left(\prod_{j=i+1}^{k-1}[a_j]\right)\eta_iy_k\\ &+\left[\eta_k - (1-a_k)\left(y_k + \left(\prod_{i=0}^{k-1}[a_i]\right)x_0\right.\right.\\ &\left.\left.+\sum_{i=0}^{k-1}\left(\prod_{j=i+1}^{k-1}[a_j]\right)\eta_i\right)\right]\eta_k\Bigg\}\end{aligned} \tag{40}$$

if $[(1-a_k)^2x_ky_k + [\eta_k - (1-a_k)(y_k+x_k)]\eta_k]|x_k - y_k| \ne 0$ and taking any arbitrary real value, otherwise.

(3) In the general case, the constraint is satisfied if $|a_k| \le \sqrt{K(y_k)} \le 1$ with

$$\begin{aligned}&\lambda(y_k)\\ &\ge \frac{1}{2}\Big(\big(\eta_{xk} - \eta_{yk}\big)\big[2a_k(y_k - x_k) - \big(\eta_{yk} - \eta_{xk}\big)\big]\\ &\quad + \big(a_k^2 - K(y_k)\big)(y_k - x_k)^2\Big)\\ &\quad\times\Big(\big(1-a_k\big)^2x_ky_k + \eta_{xk}\eta_{yk} - \big(1-a_k\big)\big(y_k\eta_{xk} + x_k\eta_{yk}\big)\Big)^{-1},\\ &\qquad\qquad \forall k \in \mathbf{N}_0,\end{aligned} \tag{41}$$

provided that the denominator of (41) is nonzero or if so (37) hold so that $\lambda(y_k)$ may take an arbitrary value.

(4) Assume that $a_k \in [0,1]$, for all $k \in \mathbf{N}_0$, $a_k \to 1$ as $k \to \infty$ with $\alpha_k = 1 - a_k \ge 0$ and η_k satisfying $\sum_{k=0}^{\infty}\alpha_k = +\infty$, $\eta_k \ge 0$ and $\sum_{k=0}^{\infty}\eta_k < +\infty$ and that $x_0 \ge 0$. Thus, the difference equation can be described equivalently by $x_{k+1} - x_k = -\alpha_kx_k + \eta_k$, and summing up both sides from $k = 0$ to $k = n-1$ yields since the solution sequence is non-negative for nonnegative initial conditions and since $\sum_{k=0}^{\infty}\eta_k < +\infty$:

$$0 \le x_n = x_0 - \sum_{k=0}^{n-1}\alpha_kx_k + \sum_{k=0}^{n-1}\eta_k \le x_0 - \sum_{k=0}^{n-1}\alpha_kx_k + C \tag{42}$$

for some real constant $C \ge 0$. Thus, $0 \le \sum_{k=0}^{n-1}\alpha_kx_k \le x_0 + C - \sum_{k=0}^{n-1}\eta_k < +\infty$ which implies that $\liminf_{k\to+\infty}x_k = 0$, $\limsup_{k\to+\infty}x_k < +\infty$, and $\sup_{k\to+\infty}x_k < +\infty$. But if $0 < \bar{x} = \limsup_{k\to+\infty}x_k < +\infty$, then $0 < \bar{x} = \limsup_{n\to\infty}x_n \le x_0 + C + \limsup_{n\to\infty}(-\sum_{k=0}^{n-1}\alpha_kx_k) = -\infty$ which is a contradiction. Then, $\{x_n\}$ is bounded and $\exists\lim_{n\to\infty}x_n = 0$. Thus, $x^* = 0$ is a fixed point of $T : \mathrm{cl}\,\mathbf{R} \to \mathrm{cl}\,\mathbf{R}$ and the previous particular cases (1)–(3) can be applied for weak cluster points of the average sequences.

(5) Now, consider the $r(\ge 2)$-dimensional dynamic system $x_{k+1} = Tx_k := A_k\,x_k + \eta_k$, for all $k \in \mathbf{N}$, where $A_k \in \mathbf{R}^{r\times r}$, $\eta_k \in \mathbf{R}^r$, for all $k \in \mathbf{N}_0$, and the convex function $f(x) = (1/2)x^TQx$ with $Q = Q^T \succ 0$ (i.e., a positive definite square r-matrix with the superscript T standing for transposes). The Bregman distance becomes $D_f(y_k, x_k) = (1/2)(y_k^TQy_k - x_k^TQx_k) - Q(y_k - x_k)^Tx_k$ resulting in the point-dependent λ-hybrid constraint:

$$\begin{aligned}&(y_k - x_k)^T\big(A_k^TA_k - K(y_k)\,I_r\big)(y_k - x_k)\\ &\le \big(\eta_{yk} - \eta_{xk}\big)^T\big[2A_k(y_k - x_k) - \big(\eta_{yk} - \eta_{xk}\big)\big]\\ &\quad + 2\lambda(y_k)\big(x_k^T(I_r - A_k)^T(I_r - A_k)\,y_k + \eta_{xk}^T\eta_{yk}\\ &\quad - \big[y_k^T(I_r - A_k)\,\eta_{xk} + x_k^T(I_r - A_k)\,\eta_{yk}\big]\big),\end{aligned} \tag{43}$$

where I_r is the rth identity matrix. A finite, in general nonunique, real sequence $\{\lambda(y_k)\}$ exists satisfying the previous constraint if for any $k \in \mathbf{N}_0$,

$$\begin{aligned}
&\left(x_k^T(I_r - A_k)^T(I_r - A_k)\,y_k + \eta_{xk}^T\eta_{yk}\right.\\
&\quad\left. - \left[y_k^T(I_r - A_k)\,\eta_{xk} + x_k^T(I_r - A_k)\,\eta_{yk}\right]\right) = 0\\
&\quad\Longrightarrow \left\{(y_k - x_k)^T\left(A_k^TA_k - K(y_k)\,I_r\right)(y_k - x_k)\right.\\
&\qquad\left. -\left(\eta_{yk} - \eta_{xk}\right)^T\left[2A_k(y_k - x_k) - \left(\eta_{yk} - \eta_{xk}\right)\right]\right\}\\
&\quad\le 0,
\end{aligned} \tag{44}$$

which is a generalization of (37) to the n-the dimensional case. Thus,

$$\begin{aligned}
&\lambda(y_k)\\
&\ge \left((y_k - x_k)^T\left(A_k^TA_k - K(y_k)\,I_r\right)(y_k - x_k)\right.\\
&\qquad\left. -\left(\eta_{yk} - \eta_{xk}\right)^T\left[2A_k(y_k - x_k) - \left(\eta_{yk} - \eta_{xk}\right)\right]\right) \quad (45a)\\
&\times\left(2\left(x_k^T(I_r - A_k)^T(I_r - A_k)\,y_k + \eta_{xk}^T\eta_{yk}\right.\right.\\
&\qquad\left.\left. -\left[y_k^T(I_r - A_k)\,\eta_{xk} + x_k^T(I_r - A_k)\,\eta_{yk}\right]\right)\right)^{-1}\\
&\text{if}\left(x_k^T(I_r - A_k)^T(I_r - A_k)\,y_k + \eta_{xk}^T\eta_{yk}\right.\\
&\qquad\left. - \left[y_k^T(I_r - A_k)\,\eta_{xk} + x_k^T(I_r - A_k)\,\eta_{yk}\right]\right) \ne 0 \quad (45b)\\
&\qquad\lambda(y_k) = \lambda_0, \quad \text{otherwise.}
\end{aligned}$$

The previous discussion for the particular scalar difference equation may be generalized for this case with the replacement $a_k^2 \to \lambda_{\max}(A_k^TA_k) = \|A_k\|_2^2 \le K(y_k) \le 1$, for all $k \in \mathbf{N}_0$, where $\lambda_{\max}(\cdot)$ stands for the maximum (real) eigenvalue of the symmetric matrix $(\cdot)$ leading to the results for point-hybrid mappings to hold if there is a sequence $\{\lambda(y_k)\}$ satisfying the previous constraint (45a) and (45b).

Example 2. It is direct to extend Example 1 to a 2-cyclic self-mapping as follows. For instance, consider a scalar difference equation of the form

$$x_{k+1} = Tx_k := a_kx_k + u_k + \eta_k, \quad \forall k \in \mathbf{N}, \tag{46}$$

for a given initial condition $x_0 \ge 0$ where $\{u_k\}$ is a control sequence. Recursive computation for two consecutive samples yields

$$\begin{aligned}
x_{k+2} = T^2x_k &:= \widehat{a}_kx_k + \widehat{\eta}_k\\
&= \widehat{a}_kx_k + \widehat{\eta}_k^0 + u_{k+1}, \quad \forall k \in \mathbf{N},
\end{aligned} \tag{47}$$

where $x_0 \ge 0$, $\widehat{a}_k = a_{k+1}a_k$; $\widehat{\eta}_k = \widehat{\eta}_k^0 + u_{k+1} = a_{k+1}(\eta_k + u_k) + \eta_{k+1} + u_{k+1}$. Define the sets A_i $(i = 1, 2)$ as $A_1 = \mathbf{R}_{0+} = \{z \in \text{cl}\,\mathbf{R} : z \ge 0\} = -A_2$ so that $A_1 \cap A_2 = \{0\}$. If the control sequence $\{u_k\}$ is chosen as $u_k = Me^{-k}(-1)^{k+1}$, for all $k \in \mathbf{N}_0$, for some constant $M > 0$, then $x_{k+1} = Me^{-(k+1)}(-1)^k$, $\text{sgn}\,x_{k+1} = -\,\text{sgn}\,x_k$, for all $k \in \mathbf{N}_0$, $x_k \to 0$ as $k \to \infty$, and the sequences $\{x_{2k}\} \subset A_1$, $\{x_{2k+1}\} \subset A_2$ both converge to the unique fixed point $x^* = 0$ of both $T : \text{cl}\,\mathbf{R} \to \text{cl}\,\mathbf{R}$ and $T^2 : \text{cl}\,\mathbf{R} \to \text{cl}\,\mathbf{R}$. Now, suppose that the control sequence is changed to $u_k = \max(Me^{-k}(-1)^{k+1}, \varepsilon\,\text{sgn}((-1)^{k+1}))$, for all $k \in \mathbf{N}_0$ for some positive real constant ε, then $\{x_{2k}\} \subset A_1$, $\{x_{2k+1}\} \subset A_2$ and $x_{2k} \to \varepsilon$, $x_{2k+1} \to -\varepsilon$ as $k \to \infty$ with A_i $(i = 1, 2)$ being redefined as $A_1 = \mathbf{R}_{0+} = \{z \in \text{cl}\,\mathbf{R} : \infty \ge z \ge \varepsilon\} = -A_2$. In this case, $A_1 \cap A_2 = \varnothing$ and $\pm\varepsilon$ are the best proximity points of $T : \text{cl}\,\mathbf{R} \to \text{cl}\,\mathbf{R}$ in A_1 and A_2, respectively, while ε and $(-\varepsilon)$ are also fixed points of $T^2 : A_1 \to A_1$ and $T^2 : A_2 \to A_2$, respectively.

Now, note that $T^{2k}x_0 \in A_1$ and $T^{2k}x_1 (= T^{2k+1}x_0) \in A_2$ if $x_0 \in A_1$ and $T^{2k}x_0 \in A_2$ and $T^{2k}x_1 (= T^{2k+1}x_0) \in A_1$ if $x_0 \in A_2$. Thus, one gets

$$\begin{aligned}
x_{2k} &= \left(\prod_{i=0}^{k-1}\left[\widehat{a}_{2i}\right]\right)x_0 + \sum_{i=0}^{k-1}\left(\prod_{j=i+1}^{2k-2}\left[\widehat{a}_{2j}\right]\right)\widehat{\eta}_{2i}\\
&= \left(\prod_{i=0}^{k-1}\left[\widehat{a}_{2i}\right]\right)x_0 + \sum_{i=0}^{k-1}\left(\prod_{j=i+1}^{2k-2}\left[\widehat{a}_{2j}\right]\right)\\
&\qquad\times\left(a_{2i+1}\left(\eta_{2i} + u_{2i}\right) + \eta_{2i+1} + u_{2i+1}\right),\\
x_{2k+1} &= a_{2k}x_{2k} + u_{2k} + \eta_{2k}, \quad \forall k \in \mathbf{N}_0,
\end{aligned} \tag{48}$$

with $x_{2k} \in A_1$ and $x_{2k+1} \in A_2$ if $x_0 \in A_1$ and $x_{2k} \in A_2$ and $x_{2k+1} \in A_1$ if $x_0 \in A_2$. The Bregman constraint for the composite self-mapping $T^2 : A_1 \to A_1$ to be $\lambda(y)$-hybrid relative to D_f holds in a similar way as (36), subject to (37), by replacing the subscripts $k \to 2k$ and the sequences $\{a_k\} \to \{\widehat{a}_{2k}\}$, $\{x_k\} \to \{x_{2k}\} \subset A_1$, $\{y_k\} \to \{y_{2k}\} \subset A_1$, $\{\eta_{xk}\} \to \{\widehat{\eta}_{x(2k)}\}$, $\{\eta_{yk}\} \to \{\widehat{\eta}_{y(2k)}\}$ and "mutatis-mutandis" performed replacements for subscripts $k \to 2k + 1$ for the composite self-mapping $T^2 : A_2 \to A_2$ for $x_0 \in A_1$. If $x_0 \in A_2$, then the modifications in the Bregman constraint (36), subject to (37), are referred to $\{x_k\} \to \{x_{2k}\} \subset A_2$, $\{y_k\} \to \{y_{2k}\} \subset A_2$.

Then, we have the following.

(a) If $A_1 = \mathbf{R}_{0+} = \{z \in \text{cl}\,\mathbf{R} : z \ge 0\} = -A_2$ and the control sequence $\{u_k\}$ is chosen as $u_k = Me^{-k}(-1)^{k+1}$; for all $k \in \mathbf{N}_0$, then $\{\widehat{S}_n^{(i)}x\} \equiv \{(1/n)\sum_{k=0}^{n-1}T^{2k}x\}$ has a unique weak cluster point $\{0\}$ for any real x which is the unique fixed point and best proximity point of $T^{2k} : A_1 \to A_1$, $T^{2k} : A_2 \to A_2$ and a fixed point of $T : \text{cl}\,\mathbf{R} \to \text{cl}\,\mathbf{R}$ provided that the previous modified Bregman constraint (36), subject to (37), holds.

(b) Take a control $u_k = \max(Me^{-k}(-1)^{k+1}, \varepsilon\,\text{sgn}((-1)^{k+1}))$, for all $k \in \mathbf{N}_0$, and A_i $(i = 1, 2)$ are redefined as $A_1 = \mathbf{R}_{0+} = \{z \in \text{cl}\,\mathbf{R} : \infty \ge z \ge \varepsilon\} = -A_2$ for some positive real constant ε, then $\{x_{2k}\} \subset A_1$, $\{x_{2k+1}\} \subset A_2$, and $x_{2k} \to \varepsilon$, $x_{2k+1} \to -\varepsilon$ as $k \to \infty$ if $x_0 \in A_1$. Then, $\{\widehat{S}_n^{(i)}x\} \equiv \{(1/n)\sum_{k=0}^{n-1}T^{2k}x\}$ has a unique weak cluster point $\{\varepsilon\}$ which is the unique fixed point of

$T^{2k}: A_1 \to A_1$ and the unique best proximity point in A_1 of $T: \operatorname{cl}\mathbf{R} \to \operatorname{cl}\mathbf{R}$ for any $x_0 = x \in A_1$. Also, $\{\widehat{S}_n^{(i)}x\} \equiv \{(1/n)\sum_{k=0}^{n-1} T^{2k}x\}$ has a unique weak cluster point $\{-\varepsilon\}$ which is the unique fixed point of $T^{2k}: A_2 \to A_2$ and the unique best proximity point in A_2 of $T: \operatorname{cl}\mathbf{R} \to \operatorname{cl}\mathbf{R}$ for any $x_0 = x \in A_2$ provided that the mentioned modified Bregman constraint (36), subject to (37), holds.

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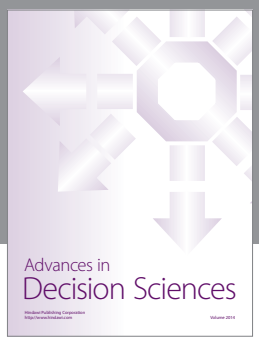

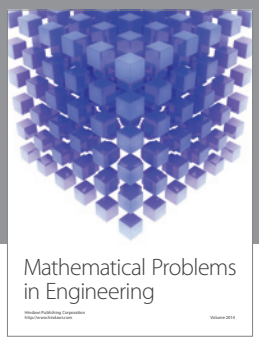

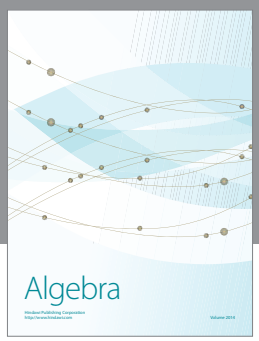

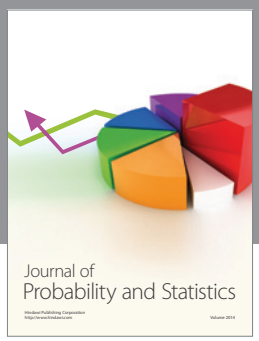

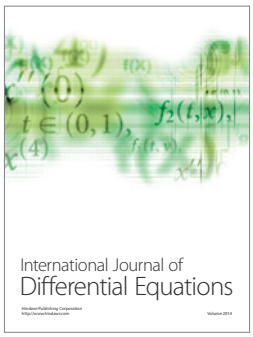

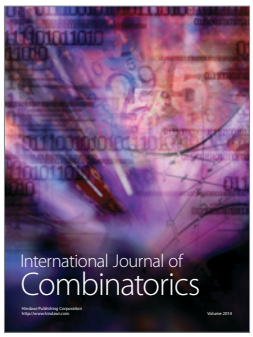

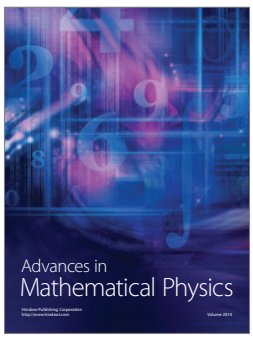

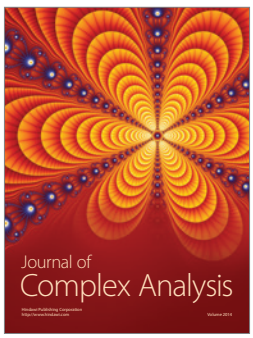

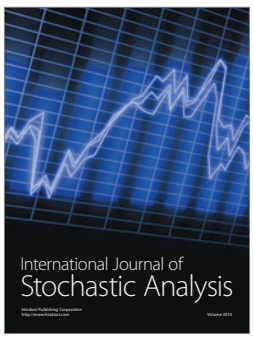

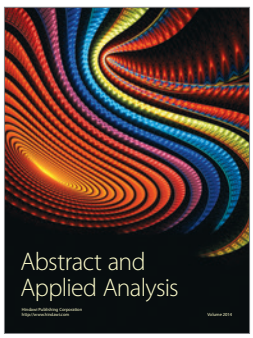

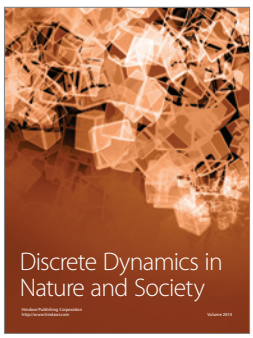

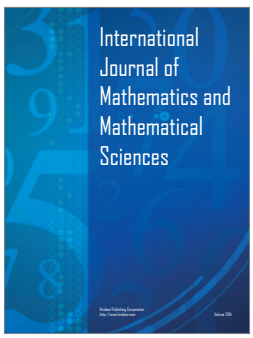

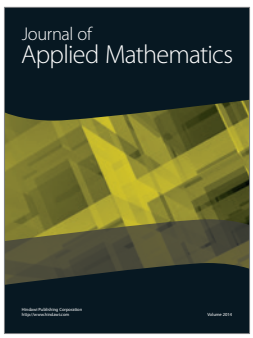

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