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Research Article

On Strong Convergence for Weighted Sums of a Class of Random Variables

Aiting Shen

School of Mathematical Science, Anhui University, Hefei 230601, China

Correspondence should be addressed to Aiting Shen; empress201010@126.com

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Let $\{X_n, n \geq 1\}$ be a sequence of random variables satisfying the Rosenthal-type maximal inequality. Complete convergence is studied for linear statistics that are weighted sums of identically distributed random variables under a suitable moment condition. As an application, the Marcinkiewicz-Zygmund-type strong law of large numbers is obtained. Our result generalizes the corresponding one of Zhou et al. (2011) and improves the corresponding one of Wang et al. (2011, 2012).

1. Introduction

Throughout the paper, let $I(A)$ be the indicator function of the set A. C denotes a positive constant which may be different in various places, and $a_n = O(b_n)$ stands for $a_n \leq Cb_n$. Denote $\log x = \ln \max(x, e)$.

Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ an array of constants. The strong convergence results for weighted sums $\sum_{i=1}^{n} a_{ni} X_i$ have been studied by many authors; see, for example, Choi and Sung [1], Cuzick [2], Wu [3], Bai and Cheng [4], Chen and Gan [5], Cai [6], Sung [7, 8], Shen [9], Wang et al. [10–14], Zhou et al. [15], Wu [16–18], Xu and Tang [19], and so forth. Many useful linear statistics are these weighted sums. Examples include least squares estimators, nonparametric regression function estimators, and jackknife estimates among others. Bai and Cheng [4] proved the strong law of large numbers for weighted sums:

$$\frac{1}{b_n} \sum_{i=1}^{n} a_{ni} X_i \longrightarrow 0, \quad \text{a.s.}, \tag{1}$$

when $\{X, X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables with $EX = 0$ and $E \exp(h|X|^{\gamma}) < \infty$ for some $h > 0$ and $\gamma > 0$, and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of constants satisfying

$$A_{\alpha} \doteq \limsup_{n \to \infty} A_{\alpha,n} < \infty, \qquad A_{\alpha,n}^{\alpha} \doteq \frac{1}{n} \sum_{i=1}^{n} |a_{ni}|^{\alpha}, \tag{2}$$

for some $1 < \alpha < 2$, where $b_n = n^{1/\alpha} (\log n)^{1/\gamma + \gamma(\alpha-1)/\alpha(1+\gamma)}$.

Cai [6] generalized the result of Bai and Cheng [4] to the case of negatively associated (NA, in short) random variables and obtained the following complete convergence result for weighted sums of identically distributed NA random variables.

Theorem 1. *Let $\{X, X_n, n \geq 1\}$ be a sequence of NA random variables with identical distributions. And let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants satisfying $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$ for $0 < \alpha \leq 2$. Let $b_n = n^{1/\alpha} \log^{1/\gamma} n$ for some $\gamma > 0$. Furthermore, assume that $EX = 0$ when $1 < \alpha \leq 2$. If $E \exp(h|X|^{\gamma}) < \infty$ for some $h > 0$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > \varepsilon b_n \right) < \infty, \quad \forall \varepsilon > 0. \tag{3}$$

Recently, Wang et al. [14] extended the result of Cai [6] for sequences of NA random variables to the case of arrays of rowwise negatively orthant-dependent (NOD, in short) random variables. Sung [8] improved the result of Cai [6] for NA random variables under much weaker conditions. Zhou et al. [15] generalized the result of Sung [8] to the case of ρ^*-mixing random variables when $\alpha > \gamma$. The technique used in Sung [8] is the result of Chen et al. [20] for NA random variables, which is not proved for ρ^*-mixing random variables. The main purpose of the paper is to further study the strong convergence for a class of random variables satisfying the Rosenthal-type maximal inequality by using a different method from that of Sung [8]. We not only generalize the result of Zhou et al. [15] for ρ^*-mixing random variables to the case of a sequence of random variables satisfying the Rosenthal-type maximal inequality, but also consider the case of $\alpha \le \gamma$. In addition, our main result improves the corresponding one of Wang et al. [11, 14], since the exponential moment condition is weakened to moment condition.

2. Main Results

In this section, we will study the strong convergence for a class of random variables satisfying the Rosenthal-type maximal inequality by using a different method from that of Sung [8]. As an application, the Marcinkiewicz-Zygmund-type strong law of large numbers is obtained.

Theorem 2. *Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed random variables. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ for some $0 < \alpha \le 2$. $EX_n = 0$ when $1 < \alpha \le 2$. Let $b_n = n^{1/\alpha}\log^{1/\gamma} n$ for some $\gamma > 0$. Assume that for any $q \ge 2$, there exists a positive constant C_q depending only on q such that*

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\right|^q\right) \le C_q\left[\sum_{i=1}^{n}E\left|Y_{ni}\right|^q+\left(\sum_{i=1}^{n}EY_{ni}^2\right)^{q/2}\right], \tag{4}$$

where $Y_{ni} = -b_n I(a_{ni}X_i < -b_n) + a_{ni}X_i I(|a_{ni}X_i| \le b_n) + b_n I(a_{ni}X_i > b_n)$ or $Y_{ni} = a_{ni}X_i I(|a_{ni}X_i| \le b_n)$. Furthermore, suppose that

$$\sum_{n=1}^{\infty} n^{-1}\left[\sum_{i=1}^{n}P\left(\left|a_{ni}X_i\right| > b_n\right)\right]^{q/2} < \infty \tag{5}$$

for $Y_{ni} = -b_n I(a_{ni}X_i < -b_n) + a_{ni}X_i I(|a_{ni}X_i| \le b_n) + b_n I(a_{ni}X_i > b_n)$. If

$$\begin{aligned} &E\left|X_1\right|^\alpha < \infty, \quad \text{for } \alpha > \gamma,\\ &E\left|X_1\right|^\alpha \log\left|X_1\right| < \infty, \quad \text{for } \alpha = \gamma,\\ &E\left|X_1\right|^\gamma < \infty, \quad \text{for } \alpha < \gamma, \end{aligned} \tag{6}$$

then (3) holds.

Proof. We only need to prove that (3) holds for $Y_{ni} = -b_n I(a_{ni}X_i < -b_n) + a_{ni}X_i I(|a_{ni}X_i| \le b_n) + b_n I(a_{ni}X_i > b_n)$. The proof for $Y_{ni} = a_{ni}X_i I(|a_{ni}X_i| \le b_n)$ is analogous.

Without loss of generality, we may assume that $\sum_{i=1}^{n} |a_{ni}|^\alpha \le n$. It is easy to check that for any $\varepsilon > 0$,

$$\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right| > \varepsilon b_n\right) \subset \left(\max_{1\le i\le n}\left|a_{ni}X_i\right| > b_n\right)\bigcup\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}Y_{ni}\right| > \varepsilon b_n\right), \tag{7}$$

which implies that

$$\begin{aligned} P&\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right| > \varepsilon b_n\right)\\ &\le P\left(\max_{1\le i\le n}\left|a_{ni}X_i\right| > b_n\right) + P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}Y_{ni}\right| > \varepsilon b_n\right)\\ &\le \sum_{i=1}^{n}P\left(\left|a_{ni}X_i\right| > b_n\right)\\ &\quad + P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\right| > \varepsilon b_n - \max_{1\le j\le n}\left|\sum_{i=1}^{j}EY_{ni}\right|\right). \end{aligned} \tag{8}$$

Firstly, we will show that

$$b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}EY_{ni}\right| \longrightarrow 0, \quad \text{as } n\longrightarrow\infty. \tag{9}$$

When $1 < \alpha \le 2$, we have by $EX_n = 0$, Markov's inequality and (6) that

$$\begin{aligned} b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}EY_{ni}\right| &\le \sum_{i=1}^{n}P\left(\left|a_{ni}X_i\right| > b_n\right)\\ &\quad + b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}EX_i I\left(\left|a_{ni}X_i\right| > b_n\right)\right|\\ &\le \sum_{i=1}^{n}P\left(\left|a_{ni}X_1\right| > b_n\right)\\ &\quad + b_n^{-1}\sum_{i=1}^{n}E\left|a_{ni}X_1\right| I\left(\left|a_{ni}X_1\right| > b_n\right)\\ &\le b_n^{-\alpha}\sum_{i=1}^{n}\left|a_{ni}\right|^\alpha E\left|X_1\right|^\alpha\\ &\quad + b_n^{-\alpha}\sum_{i=1}^{n}\left|a_{ni}\right|^\alpha E\left|X_1\right|^\alpha\\ &\le 2E\left|X_1\right|^\alpha(\log n)^{-\alpha/\gamma} \longrightarrow 0, \quad \text{as } n\longrightarrow\infty. \end{aligned} \tag{10}$$

When $0 < \alpha \leq 1$, we have by Markov's inequality and (6) again that

$$\begin{aligned} & b_n^{-1} \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} EY_{ni} \right| \\ & \leq \sum_{i=1}^{n} P\left(\left|a_{ni}X_i\right| > b_n\right) + b_n^{-1} \sum_{i=1}^{n} E\left|a_{ni}X_i\right| I\left(\left|a_{ni}X_i\right| \leq b_n\right) \\ & = \sum_{i=1}^{n} P\left(\left|a_{ni}X_1\right| > b_n\right) + b_n^{-1} \sum_{i=1}^{n} E\left|a_{ni}X_1\right| I\left(\left|a_{ni}X_1\right| \leq b_n\right) \\ & \leq b_n^{-\alpha} \sum_{i=1}^{n} \left|a_{ni}\right|^{\alpha} E\left|X_1\right|^{\alpha} + b_n^{-\alpha} \sum_{i=1}^{n} E\left|a_{ni}X_1\right|^{\alpha} I\left(\left|a_{ni}X_1\right| \leq b_n\right) \\ & \leq b_n^{-\alpha} n E\left|X_1\right|^{\alpha} + b_n^{-\alpha} n E\left|X_1\right|^{\alpha} \\ & = 2E\left|X_1\right|^{\alpha} (\log n)^{-\alpha/\gamma} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{11}$$

By (10) and (11), we can get (9) immediately. Hence, for n large enough,

$$\begin{aligned} & P\left(\max_{1\leq j\leq n} \left| \sum_{i=1}^{j} a_{ni}X_i \right| > \varepsilon b_n \right) \\ & \leq \sum_{i=1}^{n} P\left(\left|a_{ni}X_i\right| > b_n\right) + P\left(\max_{1\leq j\leq n} \left| \sum_{i=1}^{j} \left(Y_{ni} - EY_{ni}\right) \right| > \frac{\varepsilon}{2} b_n \right). \end{aligned} \tag{12}$$

To prove (3), we only need to show that

$$I \doteq \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P\left(\left|a_{ni}X_i\right| > b_n\right) < \infty, \tag{13}$$

$$J \doteq \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq j\leq n} \left| \sum_{i=1}^{j} \left(Y_{ni} - EY_{ni}\right) \right| > \frac{\varepsilon}{2} b_n \right) < \infty. \tag{14}$$

Firstly, we will prove (13). By $\sum_{i=1}^{n} |a_{ni}|^{\alpha} \leq n$ and (6), we can get that

$$\begin{aligned} I & \leq \sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} E\left|a_{ni}X_1\right|^{\alpha} I\left(\left|a_{ni}X_1\right| > b_n\right) \\ & \leq \sum_{n=1}^{\infty} n^{-2} (\log n)^{-\alpha/\gamma} \\ & \quad \times \sum_{i=1}^{n} E\left|a_{ni}X_1\right|^{\alpha} I\left(\sum_{i=1}^{n} \left|a_{ni}X_1\right|^{\alpha} > n (\log n)^{\alpha/\gamma} \right) \\ & \leq \sum_{n=1}^{\infty} n^{-2} (\log n)^{-\alpha/\gamma} \\ & \quad \times \sum_{i=1}^{n} E\left|a_{ni}X_1\right|^{\alpha} I\left(\left|X_1\right|^{\alpha} > (\log n)^{\alpha/\gamma} \right) \\ & \leq \sum_{n=1}^{\infty} n^{-1} (\log n)^{-\alpha/\gamma} E\left|X_1\right|^{\alpha} I\left(\left|X_1\right|^{\gamma} > \log n \right) \\ & = \sum_{n=1}^{\infty} n^{-1} (\log n)^{-\alpha/\gamma} \\ & \quad \times \sum_{m=n}^{\infty} E\left|X_1\right|^{\alpha} I\left(\log m < \left|X_1\right|^{\gamma} \leq \log(m+1) \right) \\ & = \sum_{m=1}^{\infty} E\left|X_1\right|^{\alpha} I\left(\log m < \left|X_1\right|^{\gamma} \leq \log(m+1) \right) \\ & \quad \times \sum_{n=1}^{m} n^{-1} (\log n)^{-\alpha/\gamma} \\ & \leq \begin{cases} C \sum_{m=1}^{\infty} E\left|X_1\right|^{\alpha} I\left(\log m < \left|X_1\right|^{\gamma} \leq \log(m+1) \right), & \text{for } \alpha > \gamma, \\ C \sum_{m=1}^{\infty} E\left|X_1\right|^{\alpha} I\left(\log m < \left|X_1\right|^{\gamma} \leq \log(m+1) \right) \times \log\log m, & \text{for } \alpha = \gamma, \\ C \sum_{m=1}^{\infty} E\left|X_1\right|^{\alpha} I\left(\log m < \left|X_1\right|^{\gamma} \leq \log(m+1) \right) \times (\log m)^{1-\alpha/\gamma}, & \text{for } \alpha < \gamma \end{cases} \\ & \leq \begin{cases} CE\left|X_1\right|^{\alpha}, & \text{for } \alpha > \gamma, \\ CE\left|X_1\right|^{\alpha} \log\left|X_1\right|, & \text{for } \alpha = \gamma, \\ CE\left|X_1\right|^{\gamma}, & \text{for } \alpha < \gamma \end{cases} \\ & < \infty, \end{aligned} \tag{15}$$

which implies (13).

In the following, we will prove (14). Let $q > \max\{2, \alpha, \gamma, (2\gamma/\alpha)\}$. By Markov's inequality and condition (4), we have

$$\begin{aligned} J & \leq C \sum_{n=1}^{\infty} n^{-1} b_n^{-q} E\left(\max_{1\leq j\leq n} \left| \sum_{i=1}^{j} \left(Y_{ni} - EY_{ni}\right) \right|^{q} \right) \\ & \leq C \sum_{n=1}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} E\left|Y_{ni}\right|^{q} \\ & \quad + C \sum_{n=1}^{\infty} n^{-1} b_n^{-q} \left(\sum_{i=1}^{n} EY_{ni}^{2} \right)^{q/2} \\ & \doteq J_1 + J_2. \end{aligned} \tag{16}$$

To prove (14), it suffices to show that $J_1 < \infty$ and $J_2 < \infty$.

For $j \geq 1$ and $n \geq 2$, denote

$$I_{nj} = \left\{1 \leq i \leq n : \frac{n}{j+1} < |a_{ni}|^{\alpha} \leq \frac{n}{j}\right\}. \quad (17)$$

In view of $\sum_{i=1}^{n} |a_{ni}|^{\alpha} \leq n$, it is easy to see that $\{I_{nj}, j \geq 1\}$ are disjoint and $\bigcup_{j=1}^{\infty} I_{nj} = \{1 \leq i \leq n : a_{ni} \neq 0\}$. Hence, we have for all $m \geq 1$ that

$$\begin{aligned} n &\geq \sum_{i=1}^{n} |a_{ni}|^{\alpha} = \sum_{\{1 \leq i \leq n : a_{ni} \neq 0\}} |a_{ni}|^{\alpha} = \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} |a_{ni}|^{\alpha} \\ &\geq n \sum_{j=1}^{\infty} (j+1)^{-1} \sharp I_{nj} \\ &\geq n \sum_{j=m}^{\infty} (j+1)^{-q/\alpha} (j+1)^{q/\alpha - 1} \sharp I_{nj} \\ &\geq n \sum_{j=m}^{\infty} (j+1)^{-q/\alpha} (m+1)^{q/\alpha - 1} \sharp I_{nj}, \end{aligned} \quad (18)$$

which implies that for all $m \geq 1$,

$$\sum_{j=m}^{\infty} (j+1)^{-q/\alpha} \sharp I_{nj} \leq C m^{1-q/\alpha}, \quad n \geq 2. \quad (19)$$

By C_r's inequality, (13) and (17), we can get that

$$\begin{aligned} J_1 &\leq C \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P(|a_{ni} X_i| > b_n) \\ &\quad + C \sum_{n=1}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} E|a_{ni} X_i|^q I(|a_{ni} X_i| \leq b_n) \\ &\leq C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} E|a_{ni} X_1|^q I(|a_{ni} X_1| \leq b_n) \\ &\leq C \sum_{n=2}^{\infty} n^{-1-q/\alpha} (\log n)^{-q/\gamma} \\ &\quad \times \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} E|a_{ni} X_1|^q I\left(|X_1| \leq (j+1)^{1/\alpha} (\log n)^{1/\gamma}\right) \\ &\leq C \sum_{n=2}^{\infty} n^{-1-q/\alpha} (\log n)^{-q/\gamma} \\ &\quad \times \sum_{j=1}^{\infty} n^{q/\alpha} j^{-q/\alpha} E|X_1|^q I\left(|X_1| \leq (j+1)^{1/\alpha} (\log n)^{1/\gamma}\right) \sharp I_{nj} \\ &\leq C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-q/\gamma} \\ &\quad \times \sum_{j=1}^{\infty} j^{-q/\alpha} E|X_1|^q I\left(|X_1| \leq (\log n)^{1/\gamma}\right) \sharp I_{nj} \\ &\quad + C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-q/\gamma} \\ &\quad \times \sum_{j=1}^{\infty} j^{-q/\alpha} \sum_{k=1}^{j} E|X_1|^q I\left(k^{1/\alpha} (\log n)^{1/\gamma} < |X_1| \right. \\ &\qquad\qquad \left. \leq (k+1)^{1/\alpha} (\log n)^{1/\gamma}\right) \sharp I_{nj} \\ &\doteq J_{11} + J_{12}. \end{aligned} \quad (20)$$

If $\alpha > \gamma$, we have by (19) and $E|X_1|^{\alpha} < \infty$ that

$$\begin{aligned} J_{11} &\leq C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-q/\gamma} E|X_1|^q I\left(|X_1| \leq (\log n)^{1/\gamma}\right) \\ &\leq C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-\alpha/\gamma} E|X_1|^{\alpha} I\left(|X_1| \leq (\log n)^{1/\gamma}\right) \\ &\leq C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-\alpha/\gamma} < \infty. \end{aligned} \quad (21)$$

If $\alpha \leq \gamma$, we have by (6) and (19) that

$$\begin{aligned} J_{11} &\leq C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-q/\gamma} E|X_1|^q I\left(|X_1| \leq (\log n)^{1/\gamma}\right) \\ &\leq C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-q/\gamma} \\ &\quad \times \sum_{m=2}^{n} E|X_1|^q I\left((\log(m-1))^{1/\gamma} < |X_1| \leq (\log m)^{1/\gamma}\right) \\ &= C \sum_{m=2}^{\infty} E|X_1|^q I\left((\log(m-1))^{1/\gamma} < |X_1| \leq (\log m)^{1/\gamma}\right) \\ &\quad \times \sum_{n=m}^{\infty} n^{-1} (\log n)^{-q/\gamma} \end{aligned}$$

$$\begin{aligned}
&\le C\sum_{m=2}^{\infty}(\log m)^{1-q/\gamma}E|X_1|^q I\\
&\qquad\times\left((\log(m-1))^{1/\gamma}<|X_1|\le(\log m)^{1/\gamma}\right)\\
&\le C\sum_{m=2}^{\infty}E|X_1|^{\gamma} I\left((\log(m-1))^{1/\gamma}<|X_1|\le(\log m)^{1/\gamma}\right)\\
&\le CE|X_1|^{\gamma}<\infty.
\end{aligned}\tag{22}$$

By (21) and (22), we can get that $J_{11}<\infty$. Next, we will prove that $J_{12}<\infty$.

It follows by (6) and (19) again that

$$\begin{aligned}
J_{12}&=C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-q/\gamma}\\
&\quad\times\sum_{k=1}^{\infty}E|X_1|^q I\left(k^{1/\alpha}(\log n)^{1/\gamma}<|X_1|\right.\\
&\qquad\qquad\left.\le(k+1)^{1/\alpha}(\log n)^{1/\gamma}\right)\\
&\quad\times\sum_{j=k}^{\infty}j^{-q/\alpha}\sharp I_{nj}\\
&\le C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-q/\gamma}\\
&\quad\times\sum_{k=1}^{\infty}k^{1-q/\alpha}E|X_1|^q I\\
&\qquad\times\left(k^{1/\alpha}(\log n)^{1/\gamma}<|X_1|\le(k+1)^{1/\alpha}(\log n)^{1/\gamma}\right)\\
&\le C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-\alpha/\gamma}\\
&\quad\times\sum_{k=1}^{\infty}E|X_1|^{\alpha} I\left(k^{1/\alpha}(\log n)^{1/\gamma}<|X_1|\right.\\
&\qquad\qquad\left.\le(k+1)^{1/\alpha}(\log n)^{1/\gamma}\right)\\
&=C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-\alpha/\gamma}E|X_1|^{\alpha}I\left(|X_1|>(\log n)^{1/\gamma}\right)\\
&=C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-\alpha/\gamma}\\
&\quad\times\sum_{m=n}^{\infty}E|X_1|^{\alpha}I\left((\log m)^{1/\gamma}<|X_1|\le(\log(m+1))^{1/\gamma}\right)\\
&=C\sum_{m=2}^{\infty}E|X_1|^{\alpha}I\left((\log m)^{1/\gamma}<|X_1|\le(\log(m+1))^{1/\gamma}\right)\\
&\quad\times\sum_{n=2}^{m}n^{-1}(\log n)^{-\alpha/\gamma}\\
&\le\begin{cases}
C\sum_{m=1}^{\infty}E|X_1|^{\alpha}I\left(\log m<|X_1|^{\gamma}\le\log(m+1)\right), & \text{for } \alpha>\gamma,\\
C\sum_{m=1}^{\infty}E|X_1|^{\alpha}I\left(\log m<|X_1|^{\gamma}\le\log(m+1)\right)\times\log\log m, & \text{for } \alpha=\gamma,\\
C\sum_{m=1}^{\infty}E|X_1|^{\alpha}I\left(\log m<|X_1|^{\gamma}\le\log(m+1)\right)\times(\log m)^{1-\alpha/\gamma}, & \text{for } \alpha<\gamma
\end{cases}\\
&\le\begin{cases}
CE|X_1|^{\alpha}, & \text{for } \alpha>\gamma,\\
CE|X_1|^{\alpha}\log|X_1|, & \text{for } \alpha=\gamma,\\
CE|X_1|^{\gamma}, & \text{for } \alpha<\gamma
\end{cases}\\
&<\infty.
\end{aligned}\tag{23}$$

By $J_{11}<\infty$ and $J_{12}<\infty$, we can get that $J_1<\infty$.

To prove (14), it suffices to show that $J_2<\infty$. By C_r's inequality, conditions (5) and (6), we can get that

$$\begin{aligned}
J_2&\le C\sum_{n=1}^{\infty}n^{-1}\left[\sum_{i=1}^{n}P\left(|a_{ni}X_i|>b_n\right)\right]^{q/2}\\
&\quad+C\sum_{n=1}^{\infty}n^{-1}b_n^{-q}\left[\sum_{i=1}^{n}E|a_{ni}X_i|^2 I\left(|a_{ni}X_i|\le b_n\right)\right]^{q/2}\\
&\le C\sum_{n=1}^{\infty}n^{-1}\left[\sum_{i=1}^{n}b_n^{-\alpha}E|a_{ni}X_1|^{\alpha}I\left(|a_{ni}X_1|\le b_n\right)\right]^{q/2}\\
&\le C\left(E|X_1|^{\alpha}\right)^{q/2}\sum_{n=1}^{\infty}n^{-1}(\log n)^{-\alpha q/2\gamma}<\infty.
\end{aligned}\tag{24}$$

Therefore, (14) follows from (16) and $J_1<\infty$, $J_2<\infty$ immediately. This completes the proof of the theorem. □

The following result provides the Marcinkiewicz-Zygmund-type strong law of large numbers for weighted sums $\sum_{i=1}^{n}a_iX_i$ of a class of random variables satisfying the Rosenthal-type maximal inequality.

Theorem 3. *Let $\{X_n, n\ge 1\}$ be a sequence of identically distributed random variables. Let $\{a_n, n\ge 1\}$ be a sequence of constants satisfying $\sum_{i=1}^{n}|a_i|^{\alpha}=O(n)$ for some $0<\alpha\le 2$. $EX_n=0$ when $1<\alpha\le 2$. Let $b_n=n^{1/\alpha}\log^{1/\gamma}n$ for some $\gamma>0$. Assume that for any $q\ge 2$, there exists a positive constant C_q depending only on q such that (4) holds, where*

$Y_{ni} = -b_n I(a_i X_i < -b_n) + a_i X_i I(|a_i X_i| \le b_n) + b_n I(a_i X_i > b_n)$ *or* $Y_{ni} = a_i X_i I(|a_i X_i| \le b_n)$. *Furthermore, suppose that*

$$\sum_{n=1}^{\infty} n^{-1} \left[\sum_{i=1}^{n} P\left(|a_i X_i| > b_n\right) \right]^{q/2} < \infty \tag{25}$$

for $Y_{ni} = -b_n I(a_i X_i < -b_n) + a_i X_i I(|a_i X_i| \le b_n) + b_n I(a_i X_i > b_n)$. *If* (6) *holds, then*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\le j\le n} \left| \sum_{i=1}^{j} a_i X_i \right| > \varepsilon b_n \right) < \infty, \quad \forall \varepsilon > 0, \tag{26}$$

$$\frac{1}{b_n} \sum_{i=1}^{n} a_i X_i \longrightarrow 0 \text{ a.s.}, \quad \text{as } n \longrightarrow \infty. \tag{27}$$

Proof. The proof of (26) is the same as that of Theorem 2. So the details are omitted. It suffices to show (27). Denote $S_n = \sum_{i=1}^{n} a_i X_i$ for each $n \ge 1$. It follows by (26) that

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n} |S_j| > \varepsilon b_n \right) \\ &= \sum_{i=0}^{\infty} \sum_{n=2^i}^{2^{i+1}-1} n^{-1} P\left(\max_{1\le j\le n} |S_j| > \varepsilon n^{1/\alpha} (\log n)^{1/\gamma} \right) \\ &\ge \frac{1}{2} \sum_{i=1}^{\infty} P\left(\max_{1\le j\le 2^i} |S_j| > \varepsilon 2^{(i+1)/\alpha} \left(\log 2^{i+1}\right)^{1/\gamma} \right). \end{aligned} \tag{28}$$

By Borel-Cantelli lemma, we obtain that

$$\lim_{i\to\infty} \frac{\max_{1\le j\le 2^i} |S_j|}{2^{(i+1)/\alpha} \left(\log 2^{i+1}\right)^{1/\gamma}} = 0 \text{ a.s.} \tag{29}$$

For all positive integers n, there exists a positive integer i_0 such that $2^{i_0-1} \le n < 2^{i_0}$. We have by (29) that

$$\begin{aligned} \frac{|S_n|}{b_n} &\le \max_{2^{i_0-1}\le n<2^{i_0}} \frac{|S_n|}{b_n} \\ &\le \frac{2^{2/\alpha} \max_{1\le j\le 2^{i_0}} |S_j|}{2^{(i_0+1)/\alpha} \left(\log 2^{i_0+1}\right)^{1/\gamma}} \left(\frac{i_0+1}{i_0-1} \right)^{1/\gamma} \longrightarrow 0 \text{ a.s.}, \\ &\qquad \text{as } i_0 \longrightarrow \infty, \end{aligned} \tag{30}$$

which implies (27). This completes the proof of the theorem. □

If the Rosenthal type inequality for the maximal partial sum is replaced by the partial sum, then we can get the following complete convergence result for a class of random variables. The proof is similar to that of Theorem 2. So the details are omitted.

Theorem 4. *Let* $\{X_n, n \ge 1\}$ *be a sequence of identically distributed random variables. Let* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *be an array of constants satisfying* $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ *for some* $0 < \alpha \le 2$. $EX_n = 0$ *when* $1 < \alpha \le 2$. *Let* $b_n = n^{1/\alpha} \log^{1/\gamma} n$ *for some* $\gamma > 0$. *Assume that for any* $q \ge 2$, *there exists a positive constant* C_q *depending only on* q *such that*

$$\begin{aligned} E\left(\left| \sum_{i=1}^{n} (Y_{ni} - EY_{ni}) \right|^q \right) \le C_q \left[\sum_{i=1}^{n} E|Y_{ni}|^q + \left(\sum_{i=1}^{n} EY_{ni}^2 \right)^{q/2} \right], \end{aligned} \tag{31}$$

where $Y_{ni} = -b_n I(a_{ni} X_i < -b_n) + a_{ni} X_i I(|a_{ni} X_i| \le b_n) + b_n I(a_{ni} X_i > b_n)$ *or* $Y_{ni} = a_{ni} X_i I(|a_{ni} X_i| \le b_n)$. *Furthermore, suppose that* (5) *holds for* $Y_{ni} = -b_n I(a_{ni} X_i < -b_n) + a_{ni} X_i I(|a_{ni} X_i| \le b_n) + b_n I(a_{ni} X_i > b_n)$. *If* (6) *satisfies, then*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\left| \sum_{i=1}^{n} a_{ni} X_i \right| > \varepsilon b_n \right) < \infty, \quad \forall \varepsilon > 0. \tag{32}$$

Remark 5. There are many sequences of dependent random variables satisfying (4) for all $q \ge 2$. Examples include sequences of NA random variables (see Shao [21]), ρ^*-mixing random variables (see Utev and Peligrad [22]), φ-mixing random variables with the mixing coefficients satisfying certain conditions (see Wang et al. [23]), ρ^--mixing random variables with the mixing coefficients satisfying certain conditions (see Wang and Lu [24]), and asymptotically almost negatively associated random variables (see Yuan and An [25]). There are also many sequences of dependent random variables satisfying (31) for all $q \ge 2$. Examples not only include the sequences of above, but also include sequences of NOD random variables (see Asadian et al. [26]) and extended negatively dependent random variables (see Shen [27]).

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