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Research Article

Note on the Lower Bound of Least Common Multiple

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Consider a sequence of positive integers in arithmetic progression $u_k = u_0 + kr$ with $(u_0, r) = 1$. Denote the least common multiple of $u_0, \ldots, u_n$ by L_n. We show that if $n \geqslant r^2 + r$, then $L_n \geqslant u_0 r^{r+1}(r + 1)$, and we obtain optimum result on n in some cases for such estimate. Besides, for quadratic sequences $m^2 + c, (m + 1)^2 + c, \ldots, n^2 + c$, we also show that the least common multiple is at least 2^n when $m \leqslant \lceil n/2 \rceil$, which sharpens a recent result of Farhi.

1. Introduction

Integer sequences in arithmetic progressions constitute a recurrent theme in number theory. The most notable result in this new century is perhaps the existence of arbitrary long sequences of primes in arithmetic progressions due to Green and Tao [1].

The bounds of the least common multiple for the finite sequences in arithmetic progressions also attract some attention. The prime number theorem assures that the least common multiple of the first n positive integers is asymptotically upper bounded by $(e + \varepsilon)^n$ and lower bounded by $(e - \varepsilon)^n$ for any prefixed ε. As for effective uniform estimate, Hanson [2] obtained the upper bound 3^n about forty years ago by considering Sylvester series of one. Nair [3] gave 2^n as alower bound in a simple proof ten years later in view of obtaining a Chebyshev-type estimate on the number of prime numbers as in a tauberian theorem due to Shapiro [4].

Recently, some results concerning the lower bound of the least common multiple of positive integers in finite arithmetic progressions were obtained by Farhi [5]. Some other results about the least common multiple of consecutive integers and consecutive arithmetic progression terms are given by Farhi and Kane [6] and by Hong and Qian [7], respectively. If $a_0, \ldots, a_n$ are integers, we denote their least common multiple by $[a_0, \ldots, a_n]$. Consider two coprime positive integers u_0 and r, and put $u_k = u_0 + kr$, $L_k = [u_0, \ldots, u_k]$. A recent result of Hong et al. (cf. [8, 9]) shows that for any positive integers a, α, r, and n such that $a \geqslant 2$, $\alpha, r \geqslant a$, $n \geqslant 2\alpha r$, we have $L_n \geqslant u_0 r^{\alpha+a-2}(r + 1)^n$.

Recently, Wu et al. [10] improved the Hong-Kominers lower bound. A special case of Hong-Kominers result tells us that if $n \geqslant r(r + 3)$ (or $n \geqslant r(r + 4)$) if r is odd (or even), then $L_n \geqslant u_0 r^{r+1}(r + 1)^n$. In this note, we find that the Hong-Kominers lower bound is still valid if $n \in [r(r + 1), r(r + r_0))$ with $r_0 = 3$ or 4 if r is odd or even. That is, we have the following result.

Theorem 1. *Let $n, u_0, r \in \mathbb{N}$ with $(u_0, r) = 1$. One puts for any $k \in [\![0, n]\!]$, $u_k = u_0 + kr$ and $L_n = [u_0, \ldots, u_n]$. Then, for any $n \geqslant r(r + 1)$,*

$$L_n \geqslant u_0 r^{r+1}(r + 1)^n. \tag{1}$$

Furthermore, if $u_0 > r$ or $u_0 < \min\{3, r\}$, the same estimate holds when $n = r^2 + r - 1$.

In 2007, Farhi [11] showed that $[1^2 + 1, 2^2 + 1, \ldots, n^2 + 1] \geqslant 0.32(1.442)^n$. Note that Qian et al. [12] obtained some results on the least common multiple of consecutive terms in a quadratic progression. We can now state the second result of this paper.

Theorem 2. *Let $c, m, n \in \mathbb{N}$ be such that $0 < m < n$. Suppose that $m \leqslant \lceil n/2 \rceil$. Then, one has*

$$\left[m^2 + c, (m + 1)^2 + c, \ldots, n^2 + c\right] \geqslant 2^n. \tag{2}$$

This theorem improves the result in [11].

2. Proof of the First Theorem

Let $x, y \in \mathbb{R}$ with $y \neq 0$. We say that x is a multiple of y if there is an integer z such that $x = yz$. As usual, $\lfloor x \rfloor$ denotes the largest integer not larger than x, and $\lceil x \rceil$, denotes the smallest integer not smaller than x.

We will introduce the two following results. The first is a known result which tells us that L_n is a multiple of $(u_0 \cdots u_n/n!)$. This is can be proved by considering a suitable partial fraction expansion (cf. [11]) or by considering the integral $\int_0^1 x^{u_0/r-1}(1-x)^n dx$ (cf. [13]).

For $\ell \in [\![0, n]\!]$, with a slight modification of notation as in [11], we put $L_{n,\ell} = [u_{n-\ell}, \ldots, u_n]$ and

$$B_{n,\ell} := \frac{u_{n-\ell} \cdots u_n}{\ell!}. \tag{3}$$

Clearly, we have $L_n = L_{n,n}$ and for any $\ell \in [\![0, n]\!]$, $L_n \geqslant L_{n,\ell}$. This result can be restated as

Lemma 3. *For any $\ell \in [0, n]$, one can find a positive integer $A_{n,\ell}$ such that $L_{n,\ell} = A_{n,\ell} B_{n,\ell}$.*

Our modification aims to emphasize the estimate of the terms $A_{n,\ell}$ and $B_{n,\ell}$ that will give us some improvement. If $n \geqslant u_0$, then we see that the first term u_0 does not play an important role as it will be a factor of the term $u_0 + u_0 r = u_0(1+r)$. Hence, the behaviour should be different when n is large. It will be more interesting to give a control over the last ℓ terms.

Note that $B_{n,\ell+1} = (u_{n-\ell-1}/(\ell+1)) B_{n,\ell}$, thus

$$B_{n,\ell+1} \leqslant B_{n,\ell} \Longleftrightarrow u_0 + (n - (\ell+1))\, r \leqslant \ell + 1 \Longleftrightarrow \ell + 1 \geqslant \frac{nr + u_0}{r+1}. \tag{4}$$

We will put $\ell_n = \min\{\lfloor u_n/(r+1) \rfloor, n\}$. The following lemma tells us that keeping the first smaller terms can increase at least the power n in the estimate of lower bound in such a way (cf. also [13]).

Lemma 4. *One has $B_{n,\ell_n} \geqslant u_0(r+1)^n$.*

Proof. We can just proceed by mathematical induction.

If $n = 0$, then $\ell_n = 0$ and $B_{0,0} = u_0$, and it holds. In fact, if $n \leqslant u_0$, then we have $\ell_n = n$ and

$$B_{n,n} = u_0 \left(\frac{u_0}{1} + r\right) \cdots \left(\frac{u_0}{n} + r\right) \geqslant u_0 (1+r)^n. \tag{5}$$

When $n \geqslant u_0$, we have $(nr+u_0)/(r+1) \leqslant (nr+n)/(r+1) \leqslant n$ and $\ell_n = \lfloor (nr+u_0)/(r+1) \rfloor$. It remains to derive the result for the case $n+1$ from that of $n \geqslant u_0$.

It is obvious that $\ell_n \leqslant \ell_{n+1} \leqslant \ell_n + 1$ and $u_n/(r+1) - 1 \leqslant \ell_n \leqslant u_n/(r+1)$.

If $\ell_n = \ell_{n+1}$, then $(n+1)r + u_0 \leqslant (\ell_n + 1)(r+1)$ and $u_{n-\ell_n} = u_0 + (n - \ell_n) r \leqslant \ell_n + 1$.

Thus,

$$\frac{u_{n+1}}{u_{n-\ell_n}} = \frac{u_{n-\ell_n} + (\ell_n + 1)\, r}{u_{n-\ell_n}} = 1 + \frac{\ell_n + 1}{u_{n-\ell_n}} r \geqslant 1 + r. \tag{6}$$

Hence,

$$B_{n+1,\ell_{n+1}} = \frac{u_{n+1}}{u_{n-\ell_n}} \cdot B_{n,\ell_n} \geqslant u_0 (r+1)^{n+1}. \tag{7}$$

If $\ell_{n+1} = \ell_n + 1$, then

$$B_{n+1,\ell_{n+1}} = \frac{u_{n+1}}{\ell_n + 1} \cdot B_{n,\ell_n} \geqslant u_0 (r+1)^{n+1}. \tag{8}$$

In either case, the principle of mathematical induction assures the result. □

We can complete our proof now. Suppose that $n \geqslant r(r+1)$. Then,

$$\ell_n = \left\lfloor \frac{u_n}{r+1} \right\rfloor \geqslant \left\lfloor r^2 + \frac{u_0}{r+1} \right\rfloor \geqslant r^2. \tag{9}$$

By considering the first r multiples of r, we have $r^{r+1} \mid \ell_n!$. Since $(r, u_0) = 1$, we deduce that for all $k \in [\![0, n]\!]$, $(r, u_k) = 1$. Writing the result of Lemma 3 for $\ell = \ell_n$ as

$$\ell_n! \cdot L_{n,\ell_n} = A_{n,\ell_n} \cdot u_{n-\ell_n} \cdots u_n, \tag{10}$$

we conclude that $r^{r+1} \mid A_{n,\ell_n}$.

Using the Lemma 4, we obtain

$$L_n \geqslant L_{n,\ell_n} = A_{n,\ell_n} B_{n,\ell_n} \geqslant u_0 r^{r+1} (r+1)^n, \tag{11}$$

which is our conclusion.

Consider the case $n = r^2 + r - 1$. If $u_0 > r$, then we still have $\ell_n \geqslant r^2$.

Supposing now that $r \geqslant 2$ and $u_0 \leqslant \min\{2, r-1\}$, we shall prove that, for $n_0 = r^2 + r - 1$, it is still possible to choose $\ell'_{n_0} = r^2$ and $n_0 - \ell'_{n_0} = r - 1$ so that $B_{n_0,\ell'_{n_0}} \geqslant u_0 (r+1)^{n_0}$.

On the one hand, when $n = r^2 - r - 1$, we have

$$\begin{aligned} \ell_n &= \left\lfloor \frac{u_0 + r\left(r^2 - r - 1\right)}{r+1} \right\rfloor \\ &= \left\lfloor \frac{r^2 (r+1) - 2r (r+1) + r + u_0}{r+1} \right\rfloor \\ &= r^2 - 2r + \left\lfloor \frac{r + u_0}{r+1} \right\rfloor \\ &= r^2 - 2r + 1. \end{aligned} \tag{12}$$

Thus, $n - \ell_n = r - 2$ and

$$\frac{u_{r-2} \cdots u_{r^2-r-1}}{(r^2 - 2r + 1)!} \geqslant u_0 (r+1)^{r^2-r-1}. \tag{13}$$

On the other hand, we write

$$\begin{aligned} B_{n_0,\ell'_{n_0}} &= \frac{u_{r-1} \cdots u_{r^2+r-1}}{r^2!} \\ &= \frac{u_{r-2} \cdots u_{r^2-r-1}}{(r^2 - 2r + 1)!} \cdot \frac{u_{r^2-r} u_{r^2}}{u_{r-2} \left(r^2 - r + 1\right)} \\ &\quad \cdot \prod_{s=1}^{r-1} \frac{u_{r^2-r+s} u_{r^2+r-s}}{\left(r^2 - 2r + s + 1\right)\left(r^2 - (s-1)\right)}. \end{aligned} \tag{14}$$

It suffices to show that for any $s \in [\![1, r-1]\!]$,

$$\frac{u_{r^2-r+s}u_{r^2+r-s}}{(r^2-2r+s+1)(r^2-(s-1))} \geqslant (r+1)^2, \quad (15)$$

$$\frac{u_{r^2-r}u_{r^2}}{u_{r-2}(r^2-r+1)} \geqslant (r+1)^2. \quad (16)$$

Equation (15) is equivalent to

$$u_{r^2-r+s}u_{r^2+r-s} \geqslant \left(r^2-2r+s+1\right)\left(r^2-(s-1)\right)(r+1)^2. \quad (17)$$

By expanding, it remains to verify for any integer $r \geqslant s+1 \geqslant 2$ that

$$f_s(r) := 2u_0r^3 - (4s-1)r^2 + 2s(s-1)r + s^2 + u_0^2 - 1 \geqslant 0. \quad (18)$$

With simple computation, we obtain for any real $r, s > 0$ with $r \geqslant s+1$,

$$\begin{aligned} f_s(s+1) &= 2(u_0-1)s^3 + 6(u_0-1)s^2 \\ &\quad + 2(3u_0-2)s + u_0^2 + 2u_0 \geqslant 0, \\ f_s'(r) &\geqslant 6r^2 - 2(4s-1)r + 2s(s-1) \\ &\geqslant 2(r-s)(3r-s+1) \geqslant 0, \end{aligned} \quad (19)$$

which allow to establish (15).

Equation (16) is equivalent to

$$\begin{aligned} &\left(u_0 + \left(r^2-r\right)r\right)\left(u_0+r^3\right) \\ &\quad \geqslant \left(u_0+(r-2)r\right)\left(r^2-r+1\right)(r+1)^2 \end{aligned} \quad (20)$$

or

$$\begin{aligned} &(2-u_0)r^4 + (u_0-1)r^3 + (1-u_0)r^2 \\ &\quad + (2-u_0)r + u_0^2 - u_0 \geqslant 0. \end{aligned} \quad (21)$$

Such inequality holds for any positive integer r when $u_0 = 1$ or 2.

Finally, we still have

$$B_{n_0,\ell'_{n_0}} \geqslant u_0(r+1)^{n_0}. \quad (22)$$

and the condition $\ell'_{n_0} \geqslant r^2$ allows us to conclude.

3. Proof of the Second Theorem

We shall start by proving the following lemma.

Lemma 5. *Let $c, m, n \in \mathbb{N}$ be such that $0 < m < n$. Put*

$$L'_{m,n} := \left[m^2+c, (m+1)^2+c, \ldots, n^2+c\right]. \quad (23)$$

Then,

$$L'_{m,n} \geqslant \frac{\prod_{k=m}^{n}\sqrt{k^2+c}}{(n-m)!}. \quad (24)$$

Proof. We shall denote $x^{\imath} = \cos(\log x) + \imath\sin(\log x)$.

Consider the integral of complex-valued function of a real variable

$$\int_0^1 x^{m-1+\sqrt{c}\imath}(1-x)^{n-m}dx. \quad (25)$$

Firstly, by integrating by parts $(n-m)$ times,

$$\begin{aligned} &\int_0^1 x^{m-1+\sqrt{c}\imath}(1-x)^{n-m}dx \\ &= \left.\frac{x^{m+\sqrt{c}\imath}(1-x)^{n-m}}{m+\sqrt{c}\imath}\right|_0^1 \\ &\quad + \frac{n-m}{m+\sqrt{c}\imath}\int_0^1 x^{m+\sqrt{c}\imath}(1-x)^{n-m-1}dx \\ &= \frac{(n-m)!}{\prod_{k=m}^{n-1}(k+\sqrt{c}\imath)}\int_0^1 x^{n-1+\sqrt{c}\imath}dx \\ &= \frac{(n-m)!}{\prod_{k=m}^{n}(k+\sqrt{c}\imath)}. \end{aligned} \quad (26)$$

Secondly, by expanding

$$\begin{aligned} &\int_0^1 x^{m-1+\sqrt{c}\imath}(1-x)^{n-m}dx \\ &= \int_0^1 \sum_{k=0}^{n-m}(-1)^k\binom{n-m}{k}x^{m-1+k+\sqrt{c}\imath}dx \\ &= \sum_{k=0}^{n-m}\frac{(-1)^k\binom{n-m}{k}}{m+k+\sqrt{c}\imath}. \end{aligned} \quad (27)$$

On one hand, put such complex number in Cartesian form, and after multiplying it by $L'_{m,n}$, we get a linear combination of $x_k + y_k\sqrt{c}\imath$ with integer coefficients, where $0 \leqslant k \leqslant n-m$ and $x_k, y_k \in \mathbb{Z}$. On the other hand, it is easy to see that the number obtained by integrating by parts $(n-m)$ times is not zero. So, its modulus is not smaller than 1, and we conclude that

$$L'_{m,n} \geqslant \frac{\prod_{k=m}^{n}\sqrt{k^2+c}}{(n-m)!}. \quad (28)$$

□

Now, we write furthermore

$$L'_{m,n} \geqslant \frac{n!}{(m-1)!(n-m)!} = m\binom{n}{m}. \quad (29)$$

Suppose now that $m \leqslant m' := \lceil n/2 \rceil$, then $L'_{m,n} \geqslant L'_{m',n} \geqslant m'\binom{n}{m'}$ and $n = 2m'$ or $2m'-1$.

The following Stirling estimate

$$\left(\frac{n}{e}\right)^n\sqrt{2\pi n} \leqslant n! \leqslant \left(\frac{n}{e}\right)^n\sqrt{2\pi n}e^{1/(12n)} \quad (30)$$

allows us to obtain

$$\binom{2n}{n} \geqslant \frac{2^{2n}}{\sqrt{n\pi}} e^{-1/(6n)}. \tag{31}$$

If $n = 2m'$, then

$$m' \binom{2m'}{m'} \geqslant 2^{2m'} \sqrt{\frac{m'}{\pi}} e^{-1/(6m')}. \tag{32}$$

As the function $\sqrt{(x/\pi)} e^{-1/(6x)}$ is increasing and its value at $x = 4$ is $\geqslant 1.08$, we conclude that for any integer $m' \geqslant 4$

$$m' \binom{2m'}{m'} \geqslant 2^{2m'}. \tag{33}$$

If $n = 2m' - 1$, then we can conclude similarly for $m' \geqslant 5$

$$\begin{aligned} m' \binom{2m'-1}{m'} &\geqslant \left(2m'-1\right) \binom{2\left(m'-1\right)}{m'-1} \\ &\geqslant 2^{2m'-1} \frac{m'-1/2}{\sqrt{m'\pi}} e^{-1/(6m')} \\ &\geqslant 2^{2m'-1}. \end{aligned} \tag{34}$$

In brief, for any integer m, n with $n \geqslant 8$ and $m \leqslant \lceil n/2 \rceil$, we just establish that

$$L'_{m,n} \geqslant 2^n. \tag{35}$$

The case $n = 7$ can be checked directly as $4\binom{7}{4} \geqslant 2^7$. For the case $n = 6$, we have

$$\begin{aligned} [16+c, 25+c, 36+c] &\geqslant [25+c, 36+c] \\ &= \frac{(25+c)(36+c)}{\gcd(25+c, 36+c)} \\ &\geqslant \frac{25 \times 36}{11} \\ &\geqslant 64 = 2^6. \end{aligned} \tag{36}$$

For the case $n = 5$, we have

$$\begin{aligned} [9+c, 16+c, 25+c] &\geqslant [16+c, 25+c] \\ &\geqslant \frac{16 \times 25}{25-16} \geqslant 2^5. \end{aligned} \tag{37}$$

It is easy to check the cases $n = 2, 3$, and 4 as it involves only two terms, and we can make our final conclusion.

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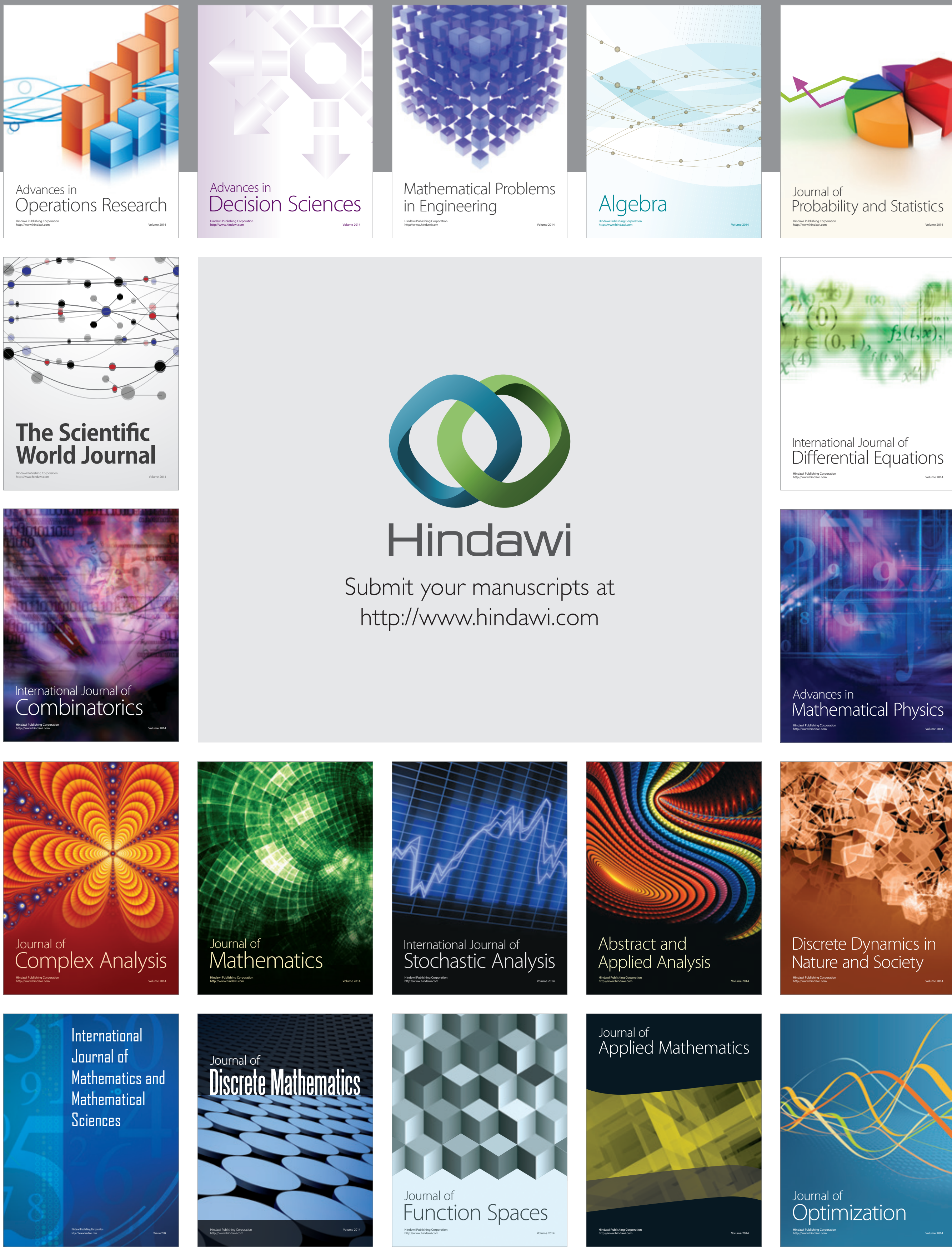

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