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Research Article

Antiperiodic Solutions for a Generalized High-Order (p,q)-Laplacian Neutral Differential System with Delays in the Critical Case

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By applying the method of coincidence degree, some criteria are established for the existence of antiperiodic solutions for a generalized high-order (p,q)-Laplacian neutral differential system with delays $(\varphi_p((x(t)-cx(t-\tau))^{(k)}))^{(m-k)} = F(t, x_{\theta_0(t)}, x'_{\theta_1(t)}, \ldots, x^{(k)}_{\theta_k(t)}, y_{\vartheta_0(t)}, y'_{\vartheta_1(t)}, \ldots, y^{(l)}_{\vartheta_l(t)})$, $(\varphi_q((y(t)-dy(t-\sigma))^{(l)}))^{(n-l)} = G(t, y_{\mu_0(t)}, y'_{\mu_1(t)}, \ldots, y^{(l)}_{\mu_l(t)}, x_{\nu_0(t)}, x'_{\nu_1(t)}, \ldots, x^{(k)}_{\nu_k(t)})$ in the critical case $|c| = |d| = 1$. The results of this paper are completely new. Finally, an example is employed to illustrate our results.

1. Introduction

During the last twenty years, there have been quite a few results on the existence of periodic solutions for delay differential equations and neutral functional differential equations. We can see [1–7]. For example, the authors of [8–11] investigated the existence of periodic solutions for the following types of neutral functional differential equations:

$$
\begin{gathered}
(x(t)-cx(t-\tau))' = g_1(t, x(t)) + g_2(t, x(t-\delta)) + e(t), \\
(x(t)-cx(t-\tau))'' + g(x(t-\delta)) = e(t), \\
\left(\varphi_p(x(t)-cx(t-\tau))'\right)' + g(t, x(t-\eta(t))) = e(t), \\
\left(\varphi_p\left(x'(t)-cx'(t-\tau)\right)\right)' = f(x(t))x'(t) + \xi(t)g\left(x(t-\eta(t))\right) + e(t),
\end{gathered}
\tag{1}
$$

respectively. But the condition of constant $|c| \neq 1$ is required. For example, under the assumption $|c| \neq 1$, they obtain that $A : C_{2\pi} := \{x : x \in C(\mathbb{R},\mathbb{R}), x(t+2\pi) \equiv x(t)\} \to C_{2\pi}$, $(Ax)(t) = x(t) - cx(t-\tau)$ has a unique inverse $A^{-1} : C_{2\pi} \to C_{2\pi}$ defined by

$$
\left(A^{-1}f\right)(t) = \begin{cases} \sum_{j\geq 0} c^j f(t-j\tau), & |c| < 1, \\ -\sum_{j\geq 1} c^{-j} f(t+j\tau), & |c| > 1, \end{cases} \tag{2}
$$

and then

$$
\int_0^{2\pi} \left|\left(A^{-1}f\right)(s)\right| ds \leq \frac{1}{|1-|c||} \int_0^{2\pi} |f(s)| ds, \quad \forall f \in C_{2\pi}, \tag{3}
$$

which was crucial to obtaining estimation of a priori bounds of periodic solutions in the noncritical case $|c| \neq 1$.

Under the critical case $|c| = 1$, the authors of [12–15] studied a first-order neutral differential equation

$$
(x(t)-cx(t-\tau))' = g(t, x(t-\delta(t))) + e(t), \tag{4}
$$

a Duffing differential equation of neutral type

$$
(x(t)-cx(t-\tau))'' = g(t, x(t-\delta(t))) + e(t), \tag{5}
$$

a Rayleigh differential equation of neutral type

$$(x(t) - cx(t-\tau))'' = f(x(t))x'(t) + g(t, x(t-\delta(t))) + e(t), \quad (6)$$

and a p-Laplacian differential equation of neutral type

$$\left(\varphi_p(x(t) - cx(t-\tau))'\right)' = f(x(t))x'(t) + g(t, x(t-\delta(t))) + e(t), \quad (7)$$

respectively.

In the past thirty years, there has been a great deal of work on the problem of the periodic solutions of high-order nonlinear differential equations, especially for the third-order and fourth-order differential equations which have been used to describe nonlinear oscillations [16–20], and fluid mechanical and nonlinear elastic mechanical phenomena [21–27]. In [28], Jin and Lu discussed the existence of periodic solutions of third-order p-Laplacian equation with a deviating argument

$$\left(\varphi_p\left(x''(t)\right)\right)' + f\left(t, x'(t), x''(t)\right) + g\left(t, x\left(t-\eta(t)\right)\right) = e(t). \quad (8)$$

Before continuing, by applying Mawhin's continuation theorem of coincidence degree theory, the authors of [29] studied the existence of periodic solutions for a fourth-order p-Laplacian equation with a deviating argument

$$\left(\varphi_p\left(x''(t)\right)\right)'' + f\left(x''(t)\right) + g\left(x\left(t-\eta(t)\right)\right) = e(t). \quad (9)$$

Arising from problems in applied sciences, it is well known that the existence of antiperiodic solutions plays a key role in characterizing the behavior of nonlinear differential equations as a special periodic solution and has been extensively studied by many authors during the past twenty years; see [30–44] and references therein. For example, antiperiodic trigonometric polynomials are important in the study of interpolation problems [45, 46], and antiperiodic wavelets are discussed in [47].

However, to the best of our knowledge, due to the neutral term and p-Laplace operator term, the existence of antiperiodic solutions for (4)–(7) is very difficult to obtain by applying traditional researching methods. Therefore, to date, there are few papers to investigate the existence of antiperiodic solutions for (4)–(7).

Motivated by above statements, in this paper, we will apply the method of coincidence degree to study the existence of antiperiodic solutions for a generalized high-order (p,q)-Laplacian neutral differential system with delays in the critical case

$$\begin{aligned}
&\left(\varphi_p\left((x(t) - cx(t-\tau))^{(k)}\right)\right)^{(m-k)} \\
&\quad = F\left(t, x_{\theta_0(t)}, x'_{\theta_1(t)}, \ldots, x^{(k)}_{\theta_k(t)}, y_{\vartheta_0(t)}, y'_{\vartheta_1(t)}, \ldots, y^{(l)}_{\vartheta_l(t)}\right), \\
&\left(\varphi_q\left((y(t) - dy(t-\sigma))^{(l)}\right)\right)^{(n-l)} \\
&\quad = G\left(t, y_{\mu_0(t)}, y'_{\mu_1(t)}, \ldots, y^{(l)}_{\mu_l(t)}, x_{\nu_0(t)}, x'_{\nu_1(t)}, \ldots, x^{(k)}_{\nu_k(t)}\right),
\end{aligned} \quad (10)$$

where $|c| = |d| = 1$, $\varphi_p(s) = |s|^{p-2}s$, $\varphi_q(s) = |s|^{q-2}s$, and $s \in \mathbb{R}$, $p, q \geq 2$; $\theta_i(t)$ $(0 \leq i \leq k)$, $\vartheta_i(t)$ $(0 \leq i \leq l)$, $\mu_i(t)$ $(0 \leq i \leq l)$, and $\nu_i(t)$ $(0 \leq i \leq k)$ $\in C(\mathbb{R}, \mathbb{R})$ are π-periodic functions; for any $\rho(t) \in C(\mathbb{R}, \mathbb{R})$, $x_{\rho(t)}$ is defined by $x_{\rho(t)} = x(t - \rho(t))$; $F, G \in C(\mathbb{R}^{k+l+3}, \mathbb{R})$ are 2π-periodic in their first arguments; τ, σ are constants; m, n, k, and l are nonnegative integers, $k < m$, $l < n$.

Throughout this paper, we will denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{N}_1$ the set of odd positive integers.

Let $p = q$, $k = l$, $m = n$, $x = y$, $c = d$, $\tau = \sigma$, and $F = G$ in system (10), then system (10) is reformulated as

$$\begin{aligned}
&\left(\varphi_p\left((x(t) - cx(t-\tau))^{(k)}\right)\right)^{(m-k)} \\
&\quad = F\left(t, x_{\theta_0(t)}, x'_{\theta_1(t)}, \ldots, x^{(k)}_{\theta_k(t)}, x_{\vartheta_0(t)}, x'_{\vartheta_1(t)}, \ldots, x^{(l)}_{\vartheta_l(t)}\right).
\end{aligned} \quad (11)$$

Furthermore, one can easily obtain the following.

(a) If $p = q = 2$, $x = y$, $|c| = |d| = 1$, $k = l = 0$, $m = n = 1$, $\theta_0(t) = \delta(t)$,

$$F = F\left(t, x_{\theta_0(t)}\right) = g(t, x(t-\delta(t))) + e(t), \quad (12)$$

then system (10) reduces to (4).

(b) If $p = q = 2$, $x = y$, $|c| = |d| = 1$, $k = l = 0$, $m = n = 2$, $\theta_0(t) = \delta(t)$,

$$F = F\left(t, x_{\theta_0(t)}\right) = g(t, x(t-\delta(t))) + e(t), \quad (13)$$

then system (10) reduces to (5).

(c) If $p = q = 2$, $x = y$, $|c| = |d| = 1$, $k = l = 0$, $m = n = 2$, $\theta_0(t) = \theta_1(t) \equiv 0$, $\vartheta_0(t) = \delta(t)$,

$$\begin{aligned}
F &= F\left(t, x_{\theta_0(t)}, x'_{\theta_1(t)}, x_{\vartheta_0(t)}\right) \\
&= f(x(t))x'(t) + g(t, x(t-\delta(t))) + e(t),
\end{aligned} \quad (14)$$

then system (10) reduces to (6).

(d) If $p = q$, $x = y$, $|c| = |d| = 1$, $k = l = 1$, $m = n = 2$, $\theta_0(t) = \theta_1(t) \equiv 0$, $\vartheta_0(t) = \delta(t)$,

$$\begin{aligned} F &= F\left(t, x_{\theta_0(t)}, x'_{\theta_1(t)}, x_{\vartheta_0(t)}\right) \\ &= f(x(t))\, x'(t) + g(t, x(t-\delta(t))) + e(t), \end{aligned} \quad (15)$$

then system (10) reduces to (7).

The main purpose of this paper is to establish sufficient conditions for the existence of π-antiperiodic solutions to system (10) by using the method of coincidence degree.

The organization of this paper is as follows. In Section 2, we make some preparations. In Section 3, by using the method of coincidence degree, we establish sufficient conditions for the existence of π-antiperiodic solutions to system (10). An illustrative example is given in Section 4.

2. Preliminaries

The following continuation theorem of coincidence degree is crucial in the arguments of our main results.

Lemma 1 (see [48]). *Let $\mathbb{X}, \mathbb{Y}$ be two Banach spaces; let $\Omega \subset \mathbb{X}$ be open bounded and symmetric with $0 \in \Omega$. Suppose that $L : D(L) \subset \mathbb{X} \to \mathbb{Y}$ is a linear Fredholm operator of index zero with $D(L) \cap \overline{\Omega} \neq \emptyset$ and $N : \overline{\Omega} \to \mathbb{Y}$ is L-compact. Further, one also assumes that*

(H) $Lx - Nx \neq \lambda(-Lx - N(-x))$, for all $x \in D(L) \cap \partial\Omega$, $\lambda \in (0, 1]$.

Then equation $Lx = Nx$ has at least one solution on $D(L) \cap \overline{\Omega}$.

Definition 2. Let $u(t) : \mathbb{R} \to \mathbb{R}$ be continuous. $u(t)$ is said to be $T/2$-antiperiodic on $\mathbb{R}$, if

$$u(t+T) = u(t), \quad u\left(t + \frac{T}{2}\right) = -u(t), \quad \forall t \in \mathbb{R}. \quad (16)$$

We will adopt the following notations:

$$\begin{aligned} C_{2\pi}^k &:= \{u \in C(\mathbb{R}, \mathbb{R}) : u \text{ is } 2\pi\text{-periodic}\}, \\ & k \in \mathbb{N}, \quad |u|_\infty = \max_{t \in [0, 2\pi]} |u(t)|, \end{aligned} \quad (17)$$

where u is a 2π-periodic function.

For the sake of convenience, we introduce the following assumptions.

(H_1) There exist nonnegative constants $\alpha_1, \alpha_2, \ldots, \alpha_{k+l+2}$, $\beta_1, \beta_2, \ldots, \beta_{k+l+2}$, such that

$$\begin{aligned} &\left|F(t, s_1, s_2, \ldots, s_{k+l+2}) - F(t, z_1, z_2, \ldots, z_{k+l+2})\right| \\ &\quad \leq \sum_{i=1}^{k+l+2} \alpha_i |s_i - z_i|, \\ &\left|G(t, s_1, s_2, \ldots, s_{k+l+2}) - G(t, z_1, z_2, \ldots, z_{k+l+2})\right| \\ &\quad \leq \sum_{i=1}^{k+l+2} \beta_i |s_i - z_i| \end{aligned} \quad (18)$$

for any $(t, s_1, s_2, \ldots, s_{k+l+2})$, $(t, z_1, z_2, \ldots, z_{k+l+2}) \in \mathbb{R}^{k+l+3}$.

(H_2) For all $(t, s_1, s_2, \ldots, s_{k+l+2}) \in \mathbb{R}^{k+l+3}$,

$$\begin{aligned} &F(t+\pi, -s_1, -s_2, \ldots, -s_{k+l+2}) \\ &\quad = -F(t, s_1, s_2, \ldots, s_{k+l+2}), \\ &G(t+\pi, -s_1, -s_2, \ldots, -s_{k+l+2}) \\ &\quad = -G(t, s_1, s_2, \ldots, s_{k+l+2}). \end{aligned} \quad (19)$$

In order to apply Lemma 1 to study the existence of antiperiodic solutions for system (10), we set

$$\begin{aligned} \mathbb{X} = \Big\{ & x = (x_1(t), x_2(t), y_1(t), y_2(t))^T \in C_{2\pi}^k \times C_{2\pi}^{m-k-1} \\ & \times C_{2\pi}^l \times C_{2\pi}^{n-l-1} : x(t+\pi) = -x(t) \Big\}, \\ \mathbb{Y} = \Big\{ & x = (x_1(t), x_2(t), y_1(t), y_2(t))^T \in C_{2\pi}^0 \times C_{2\pi}^0 \\ & \times C_{2\pi}^0 \times C_{2\pi}^0 : x(t+\pi) = -x(t) \Big\} \end{aligned} \quad (20)$$

are two Banach spaces with the norms

$$\begin{aligned} \|x\|_{\mathbb{X}} &= \sum_{i=0}^{k} \left|x_1^{(i)}\right|_\infty + \sum_{i=0}^{m-k-1} \left|x_2^{(i)}\right|_\infty + \sum_{i=0}^{l} \left|y_1^{(i)}\right|_\infty \\ &\quad + \sum_{i=0}^{n-l-1} \left|y_2^{(i)}\right|_\infty, \\ \|x\|_{\mathbb{Y}} &= \sum_{j=1}^{2} \left(\left|x_j\right|_\infty + \left|y_j\right|_\infty\right), \end{aligned} \quad (21)$$

respectively. Define

$$\begin{aligned} \mathbb{D} = \Big\{ & x = (x_1(t), x_2(t), y_1(t), y_2(t))^T \in C_{2\pi}^k \times C_{2\pi}^{m-k} \\ & \times C_{2\pi}^l \times C_{2\pi}^{n-l} : x(t+\pi) = -x(t) \Big\} \end{aligned} \quad (22)$$

and two difference operators A and B as follows:

$$\begin{aligned} A : \mathbb{Y} \longrightarrow \mathbb{Y}, \qquad (Ax)(t) = x(t) - cx(t-\tau), \\ B : \mathbb{Y} \longrightarrow \mathbb{Y}, \qquad (By)(t) = y(t) - dy(t-\sigma). \end{aligned} \quad (23)$$

Then system (10) reduces to

$$\begin{aligned}
&\left(Ax_1\right)^{(k)}(t)=\varphi_{p'}\left(x_2(t)\right),\\
&x_2^{(m-k)}(t)\\
&\quad=F\left(t,x_{1\theta_0(t)},x'_{1\theta_1(t)},\dots,x^{(k)}_{1\theta_k(t)},y_{1\vartheta_0(t)},y'_{1\vartheta_1(t)},\dots,y^{(l)}_{\vartheta_{1l}(t)}\right),\\
&\left(By_1\right)^{(l)}(t)=\varphi_{q'}\left(y_2(t)\right),\\
&y_2^{(n-l)}(t)\\
&\quad=G\left(t,y_{1\mu_0(t)},y'_{1\mu_1(t)},\dots,y^{(l)}_{1\mu_l(t)},x_{1\nu_0(t)},x'_{1\nu_1(t)},\dots,x^{(k)}_{1\nu_k(t)}\right),
\end{aligned}\tag{24}$$

where $1/p+1/p'=1$, $1/q+1/q'=1$, $1<p',\ q'\leq 2$. Obviously, the existence of antiperiodic solutions to system (10) is equivalent to that of antiperiodic solutions to system (24). Thus, the problem of finding a π-antiperiodic solution for system (10) reduces to finding one for system (24).

Define a linear operator $L: D(L)\equiv\mathbb{D}\subset\mathbb{X}\to\mathbb{Y}$ by setting

$$Lx=L\begin{pmatrix}x_1\\x_2\\y_1\\y_2\end{pmatrix}=\begin{pmatrix}\left(Ax_1\right)^{(k)}\\x_2^{(m-k)}\\\left(By_1\right)^{(l)}\\y_2^{(n-l)}\end{pmatrix},\quad\forall x\in D(L)\tag{25}$$

and $N:\mathbb{X}\to\mathbb{Y}$ by setting

$$Nx=N\begin{pmatrix}x_1\\x_2\\y_1\\y_2\end{pmatrix}=\begin{pmatrix}\varphi_{p'}\left(x_2(t)\right)\\F\left(t,x_{1\theta_0(t)},x'_{1\theta_1(t)},\dots,x^{(k)}_{1\theta_k(t)},y_{1\vartheta_0(t)},y'_{1\vartheta_1(t)},\dots,y^{(l)}_{1\vartheta_l(t)}\right)\\\varphi_{q'}\left(y_2(t)\right)\\G\left(t,y_{1\mu_0(t)},y'_{1\mu_1(t)},\dots,y^{(l)}_{1\mu_l(t)},x_{1\nu_0(t)},x'_{1\nu_1(t)},\dots,x^{(k)}_{1\nu_k(t)}\right)\end{pmatrix}.\tag{26}$$

It is easy to see that

$$\begin{aligned}
&\operatorname{Ker}L=\{0\},\\
&\operatorname{Im}L=\left\{x\in\mathbb{Y}:\int_0^{2\pi}\begin{pmatrix}x_1(s)\\x_2(s)\\y_1(s)\\y_2(s)\end{pmatrix}ds=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right\}\equiv\mathbb{Y}.
\end{aligned}\tag{27}$$

Thus, $\dim\operatorname{Ker}L=0=\operatorname{codim}\operatorname{Im}L$, and L is a linear Fredholm operator of index zero.

Define the continuous projector $P:\mathbb{X}\to\operatorname{Ker}L$ and the averaging projector $Q:\mathbb{Y}\to\mathbb{Y}$ by

$$P\begin{pmatrix}x_1\\x_2\\y_1\\y_2\end{pmatrix}=Q\begin{pmatrix}x_1\\x_2\\y_1\\y_2\end{pmatrix}=\frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix}x_1(s)\\x_2(s)\\y_1(s)\\y_2(s)\end{pmatrix}ds\equiv\begin{pmatrix}0\\0\\0\\0\end{pmatrix}.\tag{28}$$

Hence $\operatorname{Im}P=\operatorname{Ker}L$ and $\operatorname{Ker}Q=\operatorname{Im}L$. Denoting by $L_P^{-1}:\operatorname{Im}L\to D(L)\cap\operatorname{Ker}P$ the inverse of $L|_{D(L)\cap\operatorname{Ker}}P$, we have

$$\left(L_P^{-1}\begin{pmatrix}x_1\\x_2\\y_1\\y_2\end{pmatrix}\right)(t)=\begin{pmatrix}\left(A^{-1}Kx_1\right)(t)\\\left(Kx_2\right)(t)\\\left(B^{-1}Ky_1\right)(t)\\\left(Ky_2\right)(t)\end{pmatrix},\tag{29}$$

where

$$\begin{aligned}
\left(Kx_1\right)(t)&=\sum_{i=0}^{k-1}\frac{1}{i!}\left(Ah_1\right)^{(i)}(0)\,t^i\\
&\quad+\frac{1}{(k-1)!}\int_0^t(t-s)^{k-1}x_1(s)\,ds,\\
\left(Kx_2\right)(t)&=\sum_{i=0}^{m-k-1}\frac{1}{i!}h_2^{(i)}(0)\,t^i\\
&\quad+\frac{1}{(m-k-1)!}\int_0^t(t-s)^{m-k-1}x_2(s)\,ds,\\
\left(Ky_1\right)(t)&=\sum_{i=0}^{l-1}\frac{1}{i!}\left(Bh_3\right)^{(i)}(0)\,t^i\\
&\quad+\frac{1}{(l-1)!}\int_0^t(t-s)^{l-1}y_1(s)\,ds,\\
\left(Ky_2\right)(t)&=\sum_{i=0}^{n-l-1}\frac{1}{i!}h_4^{(i)}(0)\,t^i\\
&\quad+\frac{1}{(n-l-1)!}\int_0^t(t-s)^{n-l-1}y_2(s)\,ds,
\end{aligned}\tag{30}$$

in which $(Ah_1)^{(i)}(0)$ $(i=0,1,\dots,k-1)$ are decided by $E_1Z_1=B_1$, where

$$E_1=\begin{pmatrix}2&0&0&\cdots&0&0\\c_1&2&0&\cdots&0&0\\c_2&c_1&2&\cdots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\c_{k-2}&c_{k-3}&c_{k-4}&\cdots&2&0\\c_{k-1}&c_{k-2}&c_{k-3}&\cdots&c_1&2\end{pmatrix}_{k\times k},\qquad Z_1=\begin{pmatrix}\left(Ah_1\right)^{(k-1)}(0)\\\left(Ah_1\right)^{(k-2)}(0)\\\left(Ah_1\right)^{(k-3)}(0)\\\vdots\\\left(Ah_1\right)'(0)\\\left(Ah_1\right)(0)\end{pmatrix}_{k\times 1},\tag{31}$$

$B_1 = (b_{11}, b_{12}, \ldots, b_{1k})^T$, $b_{1i} = -(1/i!)\int_0^{2\pi}(\pi - s)^i x_1(s)ds$, $c_j = (\pi)^j/j!$, and $j = 1, 2, \ldots, k-1$; $h_2^{(i)}(0)$ $(i = 0, 1, \ldots, m-k-1)$ are decided by $E_2 Z_2 = B_2$, where

$$E_2 = \begin{pmatrix} 2 & 0 & \cdots & 0 & 0 \\ c_1 & 2 & \cdots & 0 & 0 \\ c_2 & c_1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ c_{m-k-2} & c_{m-k-3} & \cdots & 2 & 0 \\ c_{m-k-1} & c_{m-k-2} & \cdots & c_1 & 2 \end{pmatrix}_{(m-k)\times(m-k)},$$

$$Z_2 = \begin{pmatrix} h_2^{(m-k-1)}(0) \\ h_2^{(m-k-2)}(0) \\ h_2^{(m-k-3)}(0) \\ \vdots \\ h_2'(0) \\ h_2(0) \end{pmatrix}_{(m-k)\times 1}, \tag{32}$$

$B_2 = (b_{21}, b_{22}, \ldots, b_{2(m-k)})^T$, $b_{2i} = -(1/i!)\int_0^{2\pi}(\pi - s)^i x_2(s)ds$, $c_j = (\pi)^j/j!$, and $j = 1, 2, \ldots, m-k-1$; $(Bh_3)^{(i)}(0)$ $(i = 0, 1, \ldots, l-1)$ are decided by $E_3 Z_3 = B_3$, where

$$E_3 = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & 2 & 0 & \cdots & 0 & 0 \\ c_2 & c_1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_{l-2} & c_{l-3} & c_{l-4} & \cdots & 2 & 0 \\ c_{l-1} & c_{l-2} & c_{l-3} & \cdots & c_1 & 2 \end{pmatrix}_{l\times l},$$

$$Z_3 = \begin{pmatrix} (Bh_3)^{(l-1)}(0) \\ (Bh_3)^{(l-2)}(0) \\ (Bh_3)^{(l-3)}(0) \\ \vdots \\ (Bh_3)'(0) \\ (Bh_3)(0) \end{pmatrix}_{l\times 1}, \tag{33}$$

$B_3 = (b_{31}, b_{32}, \ldots, b_{3l})^T$, $b_{3i} = -(1/i!)\int_0^{2\pi}(\pi - s)^i y_1(s)ds$, $c_j = (\pi)^j/j!$, and $j = 1, 2, \ldots, l-1$; $h_4^{(i)}(0)$ $(i = 0, 1, \ldots, n-l-1)$ are decided by $E_4 Z_4 = B_4$, where

$$E_4 = \begin{pmatrix} 2 & 0 & \cdots & 0 & 0 \\ c_1 & 2 & \cdots & 0 & 0 \\ c_2 & c_1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ c_{n-l-2} & c_{n-l-3} & \cdots & 2 & 0 \\ c_{n-l-1} & c_{n-l-2} & \cdots & c_1 & 2 \end{pmatrix}_{(n-l)\times(n-l)},$$

$$Z_4 = \begin{pmatrix} h_4^{(n-l-1)}(0) \\ h_4^{(n-l-2)}(0) \\ h_4^{(n-l-3)}(0) \\ \vdots \\ h_4'(0) \\ h_4(0) \end{pmatrix}_{(n-l)\times 1}, \tag{34}$$

$B_4 = (b_{41}, b_{42}, \ldots, b_{4(n-l)})^T$, $b_{4i} = -(1/i!)\int_0^{2\pi}(\pi - s)^i y_2(s)ds$, $c_j = (\pi)^j/j!$, and $j = 1, 2, \ldots, n-l-1$.

Clearly, QN and $L_P^{-1}(I - Q)N$ are continuous. Using the Arzela-Ascoli theorem, it is not difficult to show that $QN(\overline{\Omega})$, $L_P^{-1}(I-Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset \mathbb{X}$. Therefore, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset \mathbb{X}$.

Lemma 3 (see [12]). *Let $|\tau| = (b/a)\pi$, where a and b are coprime positive integers. Then*

(1) *if $c = -1$, b is odd and a is even, then*

$$\omega_1 := \inf_{k\in\mathbb{N}_1} \left|1 - ce^{-ki\tau}\right| = \inf_{k\in\mathbb{N}_1} [2(1 + \cos k\tau)]^{1/2} > 0; \tag{35}$$

(2) *if $c = 1$ and b is odd, then*

$$\omega_2 := \inf_{k\in\mathbb{N}_1} \left|1 - ce^{-ki\tau}\right| = \inf_{k\in\mathbb{N}_1} [2(1 - \cos k\tau)]^{1/2} > 0; \tag{36}$$

(3) *if $c = 1$ and $a = b = 1$, then*

$$\omega_3 := \inf_{k\in\mathbb{N}_1} \left|1 - ce^{-ki\tau}\right| = \inf_{k\in\mathbb{N}_1} [2(1 - \cos k\tau)]^{1/2} = 2 > 0. \tag{37}$$

Lemma 4 (see [13]). *Suppose $A : \mathbb{Y} \to \mathbb{Y}$, $(Ax)(t) = x(t) - cx(t-\tau)$ for all $t \in [0, 2\pi]$. Then the following propositions are true.*

(1) *If $c = -1$, $|\tau| = (b/a)\pi$, where a and b are coprime positive integers with b odd and a even, then A has a unique inverse $A^{-1} : \mathbb{Y} \to \mathbb{Y}$ satisfying $\|A^{-1}\| \le (1/\omega_1)$.*

(2) *If $c = 1$, $|\tau| = (b/a)\pi$, where a and b are coprime positive integers with b odd, then A has a unique inverse $A^{-1} : \mathbb{Y} \to \mathbb{Y}$ satisfying $\|A^{-1}\| \le (1/\omega_2)$.*

(3) *If $c = 1$, $|\tau| = \pi$, then A has a unique inverse $A^{-1} : \mathbb{Y} \to \mathbb{Y}$ satisfying $\|A^{-1}\| \le (1/\omega_3)$.*

Lemma 5 (see [12]). *Let $|\sigma| = (\nu/\mu)\pi$, where μ and ν are coprime positive integers. Then*

(1) *if $d = -1$, ν is odd and μ is even, then*

$$\omega_1 := \inf_{k\in\mathbb{N}_1} \left|1 - de^{-ki\tau}\right| = \inf_{k\in\mathbb{N}_1} [2(1 + \cos k\tau)]^{1/2} > 0; \tag{38}$$

(2) *if* $d = 1$ *and* ν *is odd, then*

$$\omega_2 := \inf_{k\in\mathbb{N}_1} \left|1 - de^{-ki\tau}\right| = \inf_{k\in\mathbb{N}_1} [2(1-\cos k\tau)]^{1/2} > 0; \quad (39)$$

(3) *if* $d = 1$ *and* $\mu = \nu = 1$, *then*

$$\omega_3 := \inf_{k\in\mathbb{N}_1} \left|1 - de^{-ki\tau}\right| = \inf_{k\in\mathbb{N}_1} [2(1-\cos k\tau)]^{1/2} = 2 > 0. \quad (40)$$

Lemma 6 (see [13]). *Suppose* $B : \mathbb{Y} \to \mathbb{Y}$, $(Bx)(t) = x(t) - dx(t-\sigma)$ *for all* $t \in [0, 2\pi]$. *Then the following propositions are true.*

(1) *If* $d = -1$, $|\sigma| = (\nu/\mu)\pi$, *where* μ *and* ν *are coprime positive integers with* ν *odd and* μ *even, then* B *has a unique inverse* $B^{-1} : \mathbb{Y} \to \mathbb{Y}$ *satisfying* $\|B^{-1}\| \le (1/\omega_1)$.

(2) *If* $d = 1$, $|\sigma| = (\nu/\mu)\pi$, *where* μ *and* ν *are coprime positive integers with* ν *odd, then* B *has a unique inverse* $B^{-1} : \mathbb{Y} \to \mathbb{Y}$ *satisfying* $\|B^{-1}\| \le (1/\omega_2)$.

(3) *If* $d = 1$, $|\sigma| = \pi$, *then* B *has a unique inverse* $B^{-1} : \mathbb{Y} \to \mathbb{Y}$ *satisfying* $\|B^{-1}\| \le (1/\omega_3)$.

3. Main Result

In this section, we will study the existence of π-antiperiodic solutions for system (10) in the critical case $|c| = |d| = 1$.

Theorem 7. *Assume that* (H_1)-(H_2) *hold. Suppose further that* $c = -1$, $|\tau| = (b/a)\pi$, *where* a *and* b *are coprime positive integers with* b *odd and* a *even, then system* (10) *has at least one* π*-antiperiodic solution, if one of the following conditions holds.*

(1) $d = -1$, $|\sigma| = (\nu/\mu)\pi$, *where* μ *and* ν *are coprime positive integers with* ν *odd and* μ *even, and* $W_1 + W_2 < 1$, *where*

$$W_1 = \max\left\{\sum_{i=1}^{k+1} \frac{\pi^{m-i+1}}{\omega_1}\alpha_i, \sum_{i=1}^{k+1} \frac{\pi^{n-l+k-i+1}}{\omega_1}\beta_{l+i+1}\right\},$$
$$W_2 = \max\left\{\sum_{i=1}^{l+1} \frac{\pi^{m-k+l-i+1}}{\omega_1}\alpha_{k+i+1}, \sum_{i=1}^{l+1} \frac{\pi^{n-i+1}}{\omega_1}\beta_i\right\}. \quad (41)$$

(2) $d = 1$, $|\sigma| = (\nu/\mu)\pi$, *where* μ *and* ν *are coprime positive integers with* ν *odd, and* $W_1 + W_3 < 1$, *where*

$$W_3 = \max\left\{\sum_{i=1}^{l+1} \frac{\pi^{m-k+l-i+1}}{\omega_2}\alpha_{k+i+1}, \sum_{i=1}^{l+1} \frac{\pi^{n-i+1}}{\omega_2}\beta_i\right\}. \quad (42)$$

(3) $d = 1$, $|\sigma| = \pi$, *and* $W_1 + W_4 < 1$, *where*

$$W_4 = \max\left\{\sum_{i=1}^{l+1} \frac{\pi^{m-k+l-i+1}}{\omega_3}\alpha_{k+i+1}, \sum_{i=1}^{l+1} \frac{\pi^{n-i+1}}{\omega_3}\beta_i\right\}, \quad (43)$$

in which ω_1, ω_2, *and* ω_3 *are constants defined by Lemma 3 or Lemma 5.*

Proof. As the proof of other cases works almost exactly as the proof of case (1), we will prove case (1) only. Consider the operator equation

$$Lx - Nx = \lambda(-Lx - N(-x)), \quad \lambda \in (0, 1]. \quad (44)$$

Then we have

$$\begin{aligned}
(Ax_1)^{(k)}(t) &= \frac{1}{1+\lambda}\varphi_{p'}(x_2(t)) - \frac{\lambda}{1+\lambda}\varphi_{p'}(-x_2(t)),\\
x_2^{(m-k)}(t) &= \frac{1}{1+\lambda}F(t,x,y) - \frac{\lambda}{1+\lambda}F(t,-x,-y),\\
(By_1)^{(l)}(t) &= \frac{1}{1+\lambda}\varphi_{q'}(y_2(t)) - \frac{\lambda}{1+\lambda}\varphi_{q'}(-y_2(t)),\\
y_2^{(n-l)}(t) &= \frac{1}{1+\lambda}G(t,y,x) - \frac{\lambda}{1+\lambda}G(t,-y,-x),
\end{aligned} \quad (45)$$

where

$$\begin{aligned}
&F(t,x_1,y_1)\\
&\quad = F\left(t, x_{1\theta_0(t)}, x'_{1\theta_1(t)}, \ldots, x^{(k)}_{1\theta_k(t)}, y_{1\vartheta_0(t)}, y'_{1\vartheta_1(t)}, \ldots, y^{(l)}_{1\vartheta_l(t)}\right),\\
&G(t,y_1,x_1)\\
&\quad = G\left(t, y_{1\mu_0(t)}, y'_{1\mu_1(t)}, \ldots, y^{(l)}_{1\mu_l(t)}, x_{1\nu_0(t)}, x'_{1\nu_1(t)}, \ldots, x^{(k)}_{1\nu_k(t)}\right),\\
&F(t,-x_1,-y_1)\\
&\quad = F\left(t, -x_{1\theta_0(t)}, -x'_{1\theta_1(t)}, \ldots, -x^{(k)}_{1\theta_k(t)}, -y_{1\vartheta_0(t)}, -y'_{1\vartheta_1(t)},\right.\\
&\qquad\left.\ldots, -y^{(l)}_{1\vartheta_l(t)}\right),\\
&G(t,-y_1,-x_1)\\
&\quad = G\left(t, -y_{1\mu_0(t)}, -y'_{1\mu_1(t)}, \ldots, -y^{(l)}_{1\mu_l(t)}, -x_{1\nu_0(t)}, -x'_{1\nu_1(t)},\right.\\
&\qquad\left.\ldots, -x^{(k)}_{1\nu_k(t)}\right).
\end{aligned} \quad (46)$$

Suppose that $x(t) = (x_1(t), x_2(t), y_1(t), y_2(t))^T \in \mathbb{D}$ is an arbitrary π-antiperiodic solution of system (45). Because $x_1(t) \in C^k_{2\pi}$ is $T/2$-antiperiodic, hence, we have

$$\begin{aligned}
\int_0^{2\pi} x_1(s)\,ds &= \int_0^{\pi} x_1(s)\,ds + \int_{\pi}^{2\pi} x_1(s)\,ds\\
&= \int_0^{\pi} x_1(s)\,ds + \int_0^{\pi} x_1(s+\pi)\,ds = 0.
\end{aligned} \quad (47)$$

Then there exists a constant $\xi \in [0, 2\pi]$ such that

$$x_1(\xi) = 0. \tag{48}$$

Therefore, we have

$$\begin{aligned} |x_1(t)| &= \left|x_1(\xi) + \int_\xi^t x_1'(s)\,ds\right| \le \int_\xi^t |x_1'(s)|\,ds, \\ |x_1(t)| &= |x_1(t-2\pi)| = \left|x_1(\xi) - \int_{t-2\pi}^{\xi} x_1'(s)\,ds\right| \\ &\le \int_{t-2\pi}^{\xi} |x_1'(s)|\,ds \end{aligned} \tag{49}$$

for all $t \in [\xi, \xi + 2\pi]$. Combining the above two inequalities, we can get

$$\begin{aligned} |x_1|_\infty &= \max_{t\in[0,2\pi]} |x_1(t)| \\ &= \max_{t\in[\xi,\xi+2\pi]} |x_1(t)| \\ &\le \max_{t\in[\xi,\xi+2\pi]} \left\{\frac{1}{2}\left(\int_\xi^t |x_1'(s)|\,ds + \int_{t-2\pi}^{\xi} |x_1'(s)|\,ds\right)\right\} \\ &\le \frac{1}{2}\int_0^{2\pi} |x_1'(s)|\,ds. \end{aligned} \tag{50}$$

Similar to (50), one can easily get

$$\begin{gathered} |x_1'|_\infty \le \frac{1}{2}\int_0^{2\pi} |x_1''(s)|\,ds, \\ |x_1''|_\infty \le \frac{1}{2}\int_0^{2\pi} |x_1'''(s)|\,ds, \dots, |x_1^{(k-1)}|_\infty \le \frac{1}{2}\int_0^{2\pi} |x_1^{(k)}(s)|\,ds, \end{gathered} \tag{51}$$

which yield

$$|x_1^{(i)}|_\infty \le \pi^{k-i}|x_1^{(k)}|_\infty \quad \text{for } i = 0, 1, \dots, k-1. \tag{52}$$

By a parallel argument to (47)–(52), we can also obtain

$$|y_1^{(i)}|_\infty \le \pi^{l-i}|y_1^{(l)}|_\infty, \quad \text{for } i = 0, 1, \dots, l-1, \tag{53}$$

$$|x_2^{(j)}|_\infty \le \pi^{m-k-j-1}|x_2^{(m-k-1)}|_\infty, \quad \text{for } j = 0, 1, \dots, m-k-2, \tag{54}$$

$$|y_2^{(j)}|_\infty \le \pi^{n-l-j-1}|y_2^{(n-l-1)}|_\infty, \quad \text{for } j = 0, 1, \dots, n-l-2. \tag{55}$$

Since $x_2(t) \in C_{2\pi}^{m-k}$ is π-antiperiodic, similar to (47), there exists a constant ζ such that $x_2^{(m-k-1)}(\zeta) = 0$. By a parallel argument to (50), we can obtain from (45), (H_1), (52), and (53) that

$$\begin{aligned} 2|x_2^{(m-k-1)}|_\infty &\le 2|x_2^{(m-k-1)}(\zeta)| + \int_0^{2\pi} |x_2^{(m-k)}(s)|\,ds \\ &= \int_0^{2\pi}\left|\frac{1}{1+\lambda}F(s, x_1, y_1) - \frac{\lambda}{1+\lambda}F(s, -x_1, -y_1)\right|ds \\ &\le \frac{1}{1+\lambda}\int_0^{2\pi} |F(s, x_1, y_1)|\,ds \\ &\quad + \frac{\lambda}{1+\lambda}\int_0^{2\pi} |F(s, -x_1, -y_1)|\,ds \\ &\le \frac{1}{1+\lambda}\int_0^{2\pi} |F(s, x_1, y_1) - F(s, 0, 0)|\,ds \\ &\quad + \frac{\lambda}{1+\lambda}\int_0^{2\pi} |F(s, -x_1, -y_1) - F(s, 0, 0)|\,ds \\ &\quad + \frac{1}{1+\lambda}\int_0^{2\pi} |F(s, 0, 0)|\,ds + \frac{\lambda}{1+\lambda}\int_0^{2\pi} |F(s, 0, 0)|\,ds \\ &\le \frac{1}{1+\lambda}\int_0^{2\pi}\Big(\alpha_1|x_1(s-\theta_0(s))| + \alpha_2|x_1'(s-\theta_1(s))| + \cdots \\ &\qquad + \alpha_{k+1}|x_1^{(k)}(s-\theta_k(s))| \\ &\qquad + \alpha_{k+2}|y_1(s-\vartheta_0(s))| \\ &\qquad + \alpha_{k+3}|y_1'(s-\vartheta_1(s))| + \cdots \\ &\qquad + \alpha_{k+l+2}|y_1^{(l)}(s-\vartheta_l(s))|\Big)\,ds \\ &\quad + \frac{\lambda}{1+\lambda}\int_0^{2\pi}\Big(\alpha_1|x_1(s-\theta_0(s))| + \cdots \\ &\qquad + \alpha_{k+1}|x_1^{(k)}(s-\theta_k(s))| \\ &\qquad + \alpha_{k+2}|y_1(s-\vartheta_0(s))| + \cdots \\ &\qquad + \alpha_{k+l+2}|y_1^{(l)}(s-\vartheta_l(s))|\Big)\,ds \\ &\quad + \int_0^{2\pi} |F(s, 0, 0)|\,ds \\ &\le \int_0^{2\pi}\Big(\alpha_1|x_1(s-\theta_0(s))| + \alpha_2|x_1'(s-\theta_1(s))| + \cdots \\ &\qquad + \alpha_{k+1}|x_1^{(k)}(s-\theta_k(s))| + \alpha_{k+2}|y_1(s-\vartheta_0(s))| \\ &\qquad + \alpha_{k+3}|y_1'(s-\vartheta_1(s))| + \cdots \\ &\qquad + \alpha_{k+l+2}|y_1^{(l)}(s-\vartheta_l(s))|\Big)\,ds + \int_0^{2\pi} |F(s, 0, 0)|\,ds \end{aligned}$$

$$\le \int_0^{2\pi} \Big(\alpha_1 |x_1|_\infty + \alpha_2 |x_1'|_\infty + \cdots + \alpha_k |x_1^{(k-1)}|_\infty + \alpha_{k+1} |x_1^{(k)}|_\infty + \alpha_{k+2} |y_1|_\infty + \alpha_{k+3} |y_1'|_\infty + \cdots + \alpha_{k+l+1} |y_1^{(l-1)}|_\infty + \alpha_{k+l+2} |y_1^{(l)}|_\infty \Big) \mathrm{d}s + \int_0^{2\pi} |F(s,0,0)| \, \mathrm{d}s$$
$$\le 2\pi\alpha_1 |x_1|_\infty + 2\pi\alpha_2 |x_1'|_\infty + \cdots + 2\pi\alpha_k |x_1^{(k-1)}|_\infty + 2\pi\alpha_{k+1} |x_1^{(k)}|_\infty + 2\pi\alpha_{k+2} |y_1|_\infty + 2\pi\alpha_{k+3} |y_1'|_\infty + \cdots + 2\pi\alpha_{k+l+1} |y_1^{(l-1)}|_\infty + 2\pi\alpha_{k+l+2} |y_1^{(l)}|_\infty + 2\pi \max_{s\in[0,2\pi]} |F(s,0,0)|$$
$$\le 2\pi\alpha_1 \frac{(2\pi)^k}{2^k} |x_1^{(k)}|_\infty + 2\pi\alpha_2 \frac{(2\pi)^{k-1}}{2^{k-1}} |x_1^{(k)}|_\infty + \cdots + 2\pi\alpha_k \frac{2\pi}{2} |x_1^{(k)}|_\infty + 2\pi\alpha_{k+1} |x_1^{(k)}|_\infty + 2\pi\alpha_{k+2} \frac{(2\pi)^l}{2^l} |y_1^{(l)}|_\infty + 2\pi\alpha_{k+3} \frac{(2\pi)^{l-1}}{2^{l-1}} |y_1^{(l)}|_\infty + \cdots + 2\pi\alpha_{k+l+1} \frac{2\pi}{2} |y_1^{(l)}|_\infty + 2\pi\alpha_{k+l+2} |y_1^{(l)}|_\infty + 2\pi \max_{s\in[0,2\pi]} |F(s,0,0)|$$
$$= 2\sum_{i=1}^{k+1} \pi^{k-i+2} \alpha_i |x_1^{(k)}|_\infty + 2\sum_{i=1}^{l+1} \pi^{l-i+2} \alpha_{k+i+1} |y_1^{(l)}|_\infty + 2\pi \max_{s\in[0,2\pi]} |F(s,0,0)|. \quad (56)$$

Namely,

$$|x_2^{(m-k-1)}|_\infty \le \sum_{i=1}^{k+1} \pi^{k-i+2} \alpha_i |x_1^{(k)}|_\infty + \sum_{i=1}^{l+1} \pi^{l-i+2} \alpha_{k+i+1} |y_1^{(l)}|_\infty + \pi \max_{s\in[0,2\pi]} |F(s,0,0)|. \quad (57)$$

Since $y_2(t) \in C_{2\pi}^{n-l}$ is π-antiperiodic, similar to (50), there exists a constant η such that $y_2^{(n-l-1)}(\eta) = 0$. By a parallel argument to (50), we can obtain from (45), (H_1), (52), and (53) that

$$2|y_2^{(n-l-1)}|_\infty \le 2\left|y_2^{(n-l-1)}(\eta)\right| + \int_0^{2\pi} \left|y_2^{(n-l)}(s)\right| \mathrm{d}s = \int_0^{2\pi} \left| \frac{1}{1+\lambda} G(s, y_1, x_1) - \frac{\lambda}{1+\lambda} G(s, -y_1, -x_1) \right| \mathrm{d}s$$
$$\le \frac{1}{1+\lambda} \int_0^{2\pi} |G(s, y_1, x_1)| \, \mathrm{d}s + \frac{\lambda}{1+\lambda} \int_0^{2\pi} |G(s, -y_1, -x_1)| \, \mathrm{d}s$$
$$\le \frac{1}{1+\lambda} \int_0^{2\pi} |G(s, y_1, x_1) - G(s,0,0)| \, \mathrm{d}s + \frac{\lambda}{1+\lambda} \int_0^{2\pi} |G(s, -y_1, -x_1) - G(s,0,0)| \, \mathrm{d}s + \frac{1}{1+\lambda} \int_0^{2\pi} |G(s,0,0)| \, \mathrm{d}s + \frac{\lambda}{1+\lambda} \int_0^{2\pi} |G(s,0,0)| \, \mathrm{d}s$$
$$\le \frac{1}{1+\lambda} \int_0^{2\pi} \Big(\beta_1 |y_1(s-\mu_0(s))| + \beta_2 |y_1'(s-\mu_1(s))| + \cdots + \beta_{l+1} |y_1^{(l)}(s-\mu_l(s))| + \beta_{l+2} |x_1(s-\nu_0(s))| + \beta_{l+3} |x_1'(s-\nu_1(s))| + \cdots + \beta_{l+k+2} |x_1^{(k)}(s-\nu_k(s))| \Big) \mathrm{d}s$$
$$+ \frac{\lambda}{1+\lambda} \int_0^{2\pi} \Big(\beta_1 |y_1(s-\mu_0(s))| + \beta_2 |y_1'(s-\mu_1(s))| + \cdots + \beta_{l+1} |y_1^{(l)}(s-\mu_l(s))| + \beta_{l+2} |x_1(s-\nu_0(s))| + \beta_{l+3} |x_1'(s-\nu_1(s))| + \cdots + \beta_{l+k+2} |x_1^{(k)}(s-\nu_k(s))| \Big) \mathrm{d}s + \int_0^{2\pi} |G(s,0,0)| \, \mathrm{d}s$$
$$\le \int_0^{2\pi} \Big(\beta_1 |y_1(s-\mu_0(s))| + \beta_2 |y_1'(s-\mu_1(s))| + \cdots + \beta_{l+1} |y_1^{(l)}(s-\mu_l(s))| + \beta_{l+2} |x_1(s-\nu_0(s))| + \beta_{l+3} |x_1'(s-\nu_1(s))| + \cdots + \beta_{l+k+2} |x_1^{(k)}(s-\nu_k(s))| \Big) \mathrm{d}s + \int_0^{2\pi} |G(s,0,0)| \, \mathrm{d}s$$
$$\le \int_0^{2\pi} \Big(\beta_1 |y_1|_\infty + \beta_2 |y_1'|_\infty + \cdots + \beta_l |y_1^{(l-1)}|_\infty + \beta_{l+1} |y_1^{(l)}|_\infty + \beta_{l+2} |x_1|_\infty + \beta_{l+3} |x_1'|_\infty + \cdots + \beta_{l+k+1} |x_1^{(k-1)}|_\infty + \beta_{l+k+2} |x_1^{(k)}|_\infty \Big) \mathrm{d}s + \int_0^{2\pi} |G(s,0,0)| \, \mathrm{d}s$$

$$\begin{aligned}
&\le 2\pi\beta_1|y_1|_\infty + 2\pi\beta_2|y_1'|_\infty + \cdots + 2\pi\beta_l|y_1^{(l-1)}|_\infty \\
&\quad + 2\pi\beta_{l+1}|y_1^{(l)}|_\infty + 2\pi\beta_{l+2}|x_1|_\infty + 2\pi\beta_{l+3}|x_1'|_\infty + \cdots \\
&\quad + 2\pi\beta_{l+k+1}|x_1^{(k-1)}|_\infty + 2\pi\beta_{l+k+2}|x_1^{(k)}|_\infty \\
&\quad + 2\pi \max_{s\in[0,2\pi]}|G(s,0,0)| \\
&\le 2\pi\beta_1\frac{(2\pi)^l}{2^l}|y_1^{(l)}|_\infty + 2\pi\beta_2\frac{(2\pi)^{l-1}}{2^{l-1}}|y_1^{(l)}|_\infty + \cdots \\
&\quad + 2\pi\beta_l\frac{2\pi}{2}|y_1^{(l)}|_\infty + 2\pi\beta_{l+1}|y_1^{(l)}|_\infty + 2\pi\beta_{l+2}\frac{(2\pi)^k}{2^k}|x_1^{(k)}|_\infty \\
&\quad + 2\pi\beta_{l+3}\frac{(2\pi)^{k-1}}{2^{k-1}}|x_1^{(k)}|_\infty + \cdots + 2\pi\beta_{l+k+1}\frac{2\pi}{2}|x_1^{(k)}|_\infty \\
&\quad + 2\pi\beta_{l+k+2}|x_1^{(k)}|_\infty + 2\pi \max_{s\in[0,2\pi]}|G(s,0,0)| \\
&= 2\sum_{i=1}^{l+1}\pi^{l-i+2}\beta_i|y_1^{(l)}|_\infty + 2\sum_{i=1}^{k+1}\pi^{k-i+2}\beta_{l+i+1}|x_1^{(k)}|_\infty \\
&\quad + 2\pi \max_{s\in[0,2\pi]}|G(s,0,0)|.
\end{aligned} \tag{58}$$

Namely,

$$\begin{aligned}
|y_2^{(n-l-1)}|_\infty &\le \sum_{i=1}^{l+1}\pi^{l-i+2}\beta_i|y_1^{(l)}|_\infty + \sum_{i=1}^{k+1}\pi^{k-i+2}\beta_{l+i+1}|x_1^{(k)}|_\infty \\
&\quad + \pi \max_{s\in[0,2\pi]}|G(s,0,0)|.
\end{aligned} \tag{59}$$

From (1) of Lemmas 4 and 6, one can obtain

$$\begin{aligned}
|x_1^{(k)}(t)| &= |A^{-1}Ax_1^{(k)}(t)| \le \frac{1}{\omega_1}|Ax_1^{(k)}(t)| \\
&= \frac{1}{\omega_1}|(Ax_1)^{(k)}(t)| \le \frac{1}{\omega_1}\varphi_{p'}(|x_2|_\infty), \\
|y_1^{(l)}(t)| &= |B^{-1}By_1^{(l)}(t)| \le \frac{1}{\omega_1}|Bx_1^{(l)}(t)| \\
&= \frac{1}{\omega_1}|(By_1)^{(l)}(t)| \le \frac{1}{\omega_1}\varphi_{q'}(|y_2|_\infty).
\end{aligned} \tag{60}$$

That is,

$$|x_1^{(k)}|_\infty \le \frac{1}{\omega_1}\varphi_{p'}(|x_2|_\infty), \qquad |y_1^{(l)}|_\infty \le \frac{1}{\omega_1}\varphi_{q'}(|y_2|_\infty). \tag{61}$$

From (54) and (55), we can get

$$|x_2|_\infty \le \pi^{m-k-1}|x_2^{(m-k-1)}|_\infty, \qquad |y_2|_\infty \le \pi^{n-l-1}|y_2^{(n-l-1)}|_\infty. \tag{62}$$

With (57)–(62), we have

$$\begin{aligned}
|x_2|_\infty &\le \sum_{i=1}^{k+1}\frac{\pi^{m-i+1}}{\omega_1}\alpha_i|x_2|_\infty^{p'-1} \\
&\quad + \sum_{i=1}^{l+1}\frac{\pi^{m-k+l-i+1}}{\omega_1}\alpha_{k+i+1}|y_2|_\infty^{q'-1} + T_1, \\
|y_2|_\infty &\le \sum_{i=1}^{k+1}\frac{\pi^{n-l+k-i+1}}{\omega_1}\beta_{l+i+1}|x_2|_\infty^{p'-1} \\
&\quad + \sum_{i=1}^{l+1}\frac{\pi^{n-i+1}}{\omega_1}\beta_i|y_2|_\infty^{q'-1} + T_2,
\end{aligned} \tag{63}$$

where

$$\begin{aligned}
T_1 &= \pi^{m-k}\max_{s\in[0,2\pi]}|F(s,0,0)|, \\
T_2 &= \pi^{n-l}\max_{s\in[0,2\pi]}|G(s,0,0)|.
\end{aligned} \tag{64}$$

Let $X_0 = \max\{|x_2|_\infty, |y_2|_\infty\}$. Then we have from (63) that

$$X_0 \le W_1X_0^{p'-1} + W_2X_0^{q'-1} + \max\{T_1, T_2\} \tag{65}$$

in which

$$\begin{aligned}
W_1 &= \max\left\{\sum_{i=1}^{k+1}\frac{\pi^{m-i+1}}{\omega_1}\alpha_i, \sum_{i=1}^{k+1}\frac{\pi^{n-l+k-i+1}}{\omega_1}\beta_{l+i+1}\right\}, \\
W_2 &= \max\left\{\sum_{i=1}^{l+1}\frac{\pi^{m-k+l-i+1}}{\omega_1}\alpha_{k+i+1}, \sum_{i=1}^{l+1}\frac{\pi^{n-i+1}}{\omega_1}\beta_i\right\}.
\end{aligned} \tag{66}$$

Since $1 < p', q' \le 2$, (65) implies from (1) that there exists a positive constant M_0 such that $X_0 \le M_0$; that is,

$$|x_2|_\infty \le M_0, \qquad |y_2|_\infty \le M_0, \tag{67}$$

which implies from (61) that there exist positive constants M_1 and M_2 such that

$$|x_1^{(k)}|_\infty \le M_1, \qquad |y_1^{(l)}|_\infty \le M_2, \tag{68}$$

which implies from (57) and (59) that

$$\begin{aligned}
|x_2^{(m-k-1)}|_\infty &\le \sum_{i=1}^{k+1}\pi^{k-i+2}\alpha_iM_1 + \sum_{i=1}^{l+1}\pi^{l-i+2}\alpha_{k+i+1}M_2 \\
&\quad + \pi\max_{s\in[0,2\pi]}|F(s,0,0)| \equiv M_3, \\
|y_2^{(n-l-1)}|_\infty &\le \sum_{i=1}^{l+1}\pi^{l-i+2}\beta_iM_2 + \sum_{i=1}^{k+1}\pi^{k-i+2}\beta_{l+i+1}M_1 \\
&\quad + \pi\max_{s\in[0,2\pi]}|G(s,0,0)| \equiv M_4.
\end{aligned} \tag{69}$$

Substituting (68), (69) into (52), (53), (54), and (55), we get

$$\begin{aligned} \left|x_1^{(i)}\right|_\infty &\le \frac{T^{k-i}}{2^{k-i}}\left|x_1^{(k)}\right|_\infty \\ &\le \frac{T^{k-i}}{2^{k-i}}M_1 \equiv N_i \quad \text{for } i=0,1,\ldots,k-1, \\ \left|y_1^{(i)}\right|_\infty &\le \frac{T^{l-i}}{2^{l-i}}\left|y_1^{(l)}\right|_\infty \\ &\le \frac{T^{l-i}}{2^{l-i}}M_2 \equiv L_i \quad \text{for } i=0,1,\ldots,l-1, \\ \left|x_2^{(j)}\right|_\infty &\le \frac{T^{m-k-j-1}}{2^{m-k-j-1}}\left|x_2^{(m-k-1)}\right|_\infty \\ &\le \frac{T^{m-k-j-1}}{2^{m-k-j-1}}M_3 \equiv H_j \quad \text{for } j=0,1,\ldots,m-k-2, \\ \left|y_2^{(j)}\right|_\infty &\le \frac{T^{n-l-j-1}}{2^{n-l-j-1}}\left|y_2^{(n-l-1)}\right|_\infty \\ &\le \frac{T^{n-l-j-1}}{2^{n-l-j-1}}M_4 \equiv K_j \quad \text{for } j=0,1,\ldots,n-l-2. \end{aligned} \tag{70}$$

Let

$$\begin{aligned} M &= \sum_{i=0}^{k-1}N_i + \sum_{i=0}^{m-k-2}H_i + \sum_{i=0}^{l-1}L_i + \sum_{i=0}^{n-l-2}K_i \\ &\quad + \sum_{i=1}^{4}M_i + 1 \left(\text{Clearly, } M \text{ is independent of } \lambda\right). \end{aligned} \tag{71}$$

Take

$$\Omega = \left\{x \in \mathbb{X} : \|x\|_{\mathbb{X}} < M\right\}. \tag{72}$$

It is clear that Ω satisfies all the requirements in Lemma 1 and condition (H) is satisfied. In view of all the discussions above, we conclude from Lemma 1 that system (10) has at least one π-antiperiodic solution. This completes the proof. □

Similar to Theorem 7, we can easily obtain the following results.

Theorem 8. *Assume that (H_1)-(H_2) hold. Suppose further that $c = 1, |\tau| = (b/a)\pi$, where a and b are coprime positive integers with b odd, then system (10) has at least one π-antiperiodic solution, if one of the following conditions holds.*

(1) *$d = -1$, $|\sigma| = (\nu/\mu)\pi$, where μ and ν are coprime positive integers with ν odd and μ even, and $W_5 + W_2 < 1$, where*

$$\begin{aligned} W_5 &= \max\left\{\sum_{i=1}^{k+1}\frac{\pi^{m-i+1}}{\omega_2}\alpha_i, \sum_{i=1}^{k+1}\frac{\pi^{n-l+k-i+1}}{\omega_2}\beta_{l+i+1}\right\}, \\ W_2 &= \max\left\{\sum_{i=1}^{l+1}\frac{\pi^{m-k+l-i+1}}{\omega_1}\alpha_{k+i+1}, \sum_{i=1}^{l+1}\frac{\pi^{n-i+1}}{\omega_1}\beta_i\right\}. \end{aligned} \tag{73}$$

(2) *$d = 1$, $|\sigma| = (\nu/\mu)\pi$, where μ and ν are coprime positive integers with ν odd, and $W_5 + W_3 < 1$, where*

$$W_3 = \max\left\{\sum_{i=1}^{l+1}\frac{\pi^{m-k+l-i+1}}{\omega_2}\alpha_{k+i+1}, \sum_{i=1}^{l+1}\frac{\pi^{n-i+1}}{\omega_2}\beta_i\right\}. \tag{74}$$

(3) *$d = 1$, $|\sigma| = \pi$, and $W_5 + W_4 < 1$, where*

$$W_4 = \max\left\{\sum_{i=1}^{l+1}\frac{\pi^{m-k+l-i+1}}{\omega_3}\alpha_{k+i+1}, \sum_{i=1}^{l+1}\frac{\pi^{n-i+1}}{\omega_3}\beta_i\right\}, \tag{75}$$

in which ω_1, ω_2, and ω_3 are constants defined by Lemma 3 or Lemma 5.

Theorem 9. *Assume that (H_1)-(H_2) hold. Suppose further that $c = 1$, $|\tau| = \pi$, then system (10) has at least one π-antiperiodic solution, if one of the following conditions holds.*

(1) *$d = -1$, $|\sigma| = (\nu/\mu)\pi$, where μ and ν are coprime positive integers with ν odd and μ even, and $W_6 + W_2 < 1$, where*

$$\begin{aligned} W_6 &= \max\left\{\sum_{i=1}^{k+1}\frac{\pi^{m-i+1}}{\omega_3}\alpha_i, \sum_{i=1}^{k+1}\frac{\pi^{n-l+k-i+1}}{\omega_3}\beta_{l+i+1}\right\}, \\ W_2 &= \max\left\{\sum_{i=1}^{l+1}\frac{\pi^{m-k+l-i+1}}{\omega_1}\alpha_{k+i+1}, \sum_{i=1}^{l+1}\frac{\pi^{n-i+1}}{\omega_1}\beta_i\right\}. \end{aligned} \tag{76}$$

(2) *$d = 1$, $|\sigma| = (\nu/\mu)\pi$, where μ and ν are coprime positive integers with ν odd, and $W_6 + W_3 < 1$, where*

$$W_3 = \max\left\{\sum_{i=1}^{l+1}\frac{\pi^{m-k+l-i+1}}{\omega_2}\alpha_{k+i+1}, \sum_{i=1}^{l+1}\frac{\pi^{n-i+1}}{\omega_2}\beta_i\right\}. \tag{77}$$

(3) *$d = 1$, $|\sigma| = \pi$, and $W_6 + W_4 < 1$, where*

$$W_4 = \max\left\{\sum_{i=1}^{l+1}\frac{\pi^{m-k+l-i+1}}{\omega_3}\alpha_{k+i+1}, \sum_{i=1}^{l+1}\frac{\pi^{n-i+1}}{\omega_3}\beta_i\right\}, \tag{78}$$

in which ω_1, ω_2, and ω_3 are constants defined by Lemma 3 or Lemma 5.

4. An Example

Example 1. Let $p, q \ge 2$. Then the following third-order (p,q)-Laplacian neutral differential system

$$\begin{aligned} &\left(\varphi_p\left(\left(x(t) + x\left(t-\frac{\pi}{2}\right)\right)'\right)\right)'' \\ &\quad = \sin t + \frac{\sqrt{2}}{\pi^4}x(t-1) + \frac{\sqrt{2}}{\pi^6}y(t-1) + \frac{\sqrt{2}}{\pi^6}y'(t-1), \\ &\left(\varphi_q\left(\left(y(t) + y\left(t-\frac{\pi}{2}\right)\right)''\right)\right)' \\ &\quad = \cos t + \frac{\sqrt{2}}{\pi^6}y(t-1) + \frac{\sqrt{2}}{\pi^4}x(t-1) \end{aligned} \tag{79}$$

has at least one π-antiperiodic solution.

Proof. By calculation, $\omega_1 = \sqrt{2}$, $\alpha_1 = \sqrt{2}/\pi^4$, $\alpha_3 = \alpha_4 = \sqrt{2}/\pi^6$, $\alpha_2 = \alpha_5 = 0$, $\beta_1 = \sqrt{2}/\pi^6$, $\beta_4 = \sqrt{2}/\pi^4$, and $\beta_2 = \beta_3 = \beta_5 = 0$. Therefore,

$$W_1 = \max\left\{\frac{1}{\pi}, \frac{1}{\pi^2}\right\} = \frac{1}{\pi} < \frac{1}{2},$$
$$W_2 = \max\left\{\frac{1}{\pi^2} + \frac{1}{\pi^3}, \frac{1}{\pi^3}\right\} = \frac{1}{\pi^2} + \frac{1}{\pi^3} < \frac{1}{2}. \quad (80)$$

Hence,

$$W_1 + W_2 < \frac{1}{2} + \frac{1}{2} = 1, \quad (81)$$

which implies that case (1) in Theorem 7 holds. It is easy to verify that (H_1)-(H_2) hold and the result follows from Theorem 7. This completes the proof. □

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