

Research Article

Some Identities on the High-Order q -Euler Numbers and Polynomials with Weight 0

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We construct the N th order nonlinear ordinary differential equation related to the generating function of q -Euler numbers with weight 0. From this, we derive some identities on q -Euler numbers and polynomials of higher order with weight 0.

1. Introduction

As a well-known definition, the Euler polynomial $E_n(x)$ is given by

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \quad (1)$$

In the special case, $x = 0$, $E_n(0) = E_n$ is the n th Euler number.

From (1), we note that

$$E_0 = 1, \quad (E + 1)^n + E_n = 0, \quad \text{if } n > 0, \quad (2)$$

with the usual convention of replacing E^n by E_n (see [1–16]).

In the viewpoint of the q -extension of (1) and (2), let us consider the following q -Euler number and polynomial:

$$\frac{2}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{t^n}{n!}, \quad (3)$$

$$\tilde{E}_{0,q} = \frac{2}{1+q}, \quad q(\tilde{E}_q + 1)^n + \tilde{E}_{n,q} = 0, \quad \text{if } n > 0, \quad (4)$$

with the usual convention of replacing \tilde{E}_q^n by $\tilde{E}_{n,q}$.

Equation (3) is called the generating function of q -Euler polynomial with weight 0. In the case $x = 0$, $\tilde{E}_{n,q}(0) = \tilde{E}_{n,q}$ is the n th q -Euler number with weight 0 (see [5, 11]).

Throughout this paper, let q be a complex number with $|q| < 1$. As $q \rightarrow 1$, we obtain (1) and (2) from (3) and (4).

The generating function of Eulerian polynomial $H_n(x | u)$ is defined by

$$\frac{1-u}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(x | u) \frac{t^n}{n!}, \quad (5)$$

where $u \in \mathbb{C}$ with $u \neq 1$. In the special case, $x = 0$, $H_n(0 | u) = H_n(u)$ is called the n th Eulerian number (see [1–3]). Sometimes that is called the n th Frobenius-Euler number (see [9–11, 15]).

From (1) and (5), we note that $H_n(x | -1) = E_n(x)$. From (5), we have

$$H_0(u) = 1, \quad H_n(1 | u) - uH_n(u) = (1-u)\delta_{0,n}, \quad (6)$$

where $\delta_{n,k}$ is Kronecker symbol (see [9–11]).

For $N \in \mathbb{N}$, the q -Euler polynomial of order N is defined by the generating function as follows:

$$\begin{aligned} G_q^N(t, x) &= \underbrace{\left(\frac{2}{qe^t + 1} \right) \times \cdots \times \left(\frac{2}{qe^t + 1} \right)}_{N\text{-times}} e^{xt} \\ &= \sum_{n=0}^{\infty} \tilde{E}_{n,q}^{(N)}(x) \frac{t^n}{n!}. \end{aligned} \quad (7)$$

In the special case, $x = 0$, $\tilde{E}_{n,q}^{(N)}(0) = \tilde{E}_{n,q}^{(N)}$ is called the n th q -Euler number of order N with weight 0 (see [5, 11]).

In [9], Kim derived some identities between the sums of products of Frobenius-Euler polynomials and Frobenius-Euler polynomials of higher order. The main idea is to construct nonlinear ordinary differential equations with respect to t which are closely related to the generating function of Frobenius-Euler polynomial. In [3], Choi considered nonlinear ordinary differential equations with respect to u not t .

In this paper, we construct nonlinear ordinary differential equations with respect to q . The purpose of this paper is to give some new identities on the high order q -Euler numbers and polynomials with weight 0 by using the differential equations of q .

2. Construction of Nonlinear Differential Equations

We define

$$G = G(q) = \frac{1}{qe^t + 1}, \tag{8}$$

$$G^N(t, x) = \underbrace{G \times \dots \times G}_{N\text{-times}} e^{xt} \quad \text{for } N \in \mathbb{N}.$$

From (7) and (8), we note that

$$G_q^N(t, x) = 2^N G^N(t, x) = 2^N G^N e^{xt}. \tag{9}$$

By differentiating (8) with respect to q , we get

$$G^{(1)} = \frac{dG}{dq} = -\frac{qe^t + 1 - 1}{q(qe^t + 1)^2} = -\frac{G}{q} + \frac{G^2}{q}, \tag{10}$$

$$qG^{(1)} + G = G^2.$$

By differentiating (10) with respect to q , we get

$$q^2 G^{(2)} + 4qG^{(1)} + 2G = 2!G^3, \tag{11}$$

where $G^{(N)} = d^N G/dq^N$.

By the derivative of (11) with respect to q , we have

$$q^3 G^{(3)} + 9q^2 G^{(2)} + 18qG^{(1)} + 3!G = 3!G^4. \tag{12}$$

Continuing this process, we get

$$(N - 1)!G^N = \sum_{k=0}^{N-1} a_k(N) q^k G^{(k)}. \tag{13}$$

Let us consider the derivative of (13) with respect to q to find the coefficient $a_k(N)$ in (13).

By (10), we get

$$\begin{aligned} q \frac{d}{dq} \left((N - 1)! G^N \right) &= N! G^{N-1} q G^{(1)} \\ &= N! G^{N-1} (-G + G^2) \\ &= N! G^{N+1} - N(N - 1)! G^N. \end{aligned} \tag{14}$$

From (13) and (14), we get

$$\begin{aligned} N! G^{N+1} &= N(N - 1)! G^N \\ &+ \sum_{k=0}^{N-1} k a_k(N) q^k G^{(k)} \\ &+ \sum_{k=1}^N a_{k-1}(N) q^k G^{(k)}, \end{aligned} \tag{15}$$

where $N! G^{N+1} = \sum_{k=0}^N a_k(N + 1) q^k G^{(k)}$.

By comparing coefficients on both sides of (15), we obtain the following recurrence relations:

$$a_0(N + 1) = N a_0(N), \quad a_N(N + 1) = a_{N-1}(N), \tag{16}$$

$$a_k(N + 1) = N a_k(N) + k a_k(N) + a_{k-1}(N), \tag{17}$$

for $1 \leq k \leq N - 1$ and $a_k(N) = 0$.

From the first part of (16), we have

$$\begin{aligned} a_0(N + 1) &= N a_0(N) \\ &= N(N - 1) a_0(N - 1) \\ &= \dots = N! a_0(2). \end{aligned} \tag{18}$$

By (10) and (13), we have

$$\begin{aligned} qG^{(1)} + G &= G^2 = \sum_{k=0}^1 a_k(2) q^k G^{(k)} \\ &= a_0(2) G + a_1(2) q G^{(1)}. \end{aligned} \tag{19}$$

From (18) and (19), we get

$$a_0(2) = 1, \quad a_1(2) = 1, \quad a_0(N) = (N - 1)!. \tag{20}$$

From the second part of (16), we have

$$a_N(N + 1) = a_{N-1}(N) = \dots = a_1(2) = 1. \tag{21}$$

To find $a_k(N)$ in (13) from (17), we set

$$g(t, s) = \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_k(N) \frac{t^N}{N!} s^k, \tag{22}$$

where $|t| < 1$ (see [9]).

From (17) and (22), we have

$$\begin{aligned} \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_{k+1}(N + 1) \frac{t^N}{N!} s^k \\ &= \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} N a_{k-1}(N) \frac{t^N}{N!} s^k \\ &+ \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} (k + 1) a_{k+1}(N) \frac{t^N}{N!} s^k + g(t, s). \end{aligned} \tag{23}$$

From the left hand side of (23), we have

$$\begin{aligned} & \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_{k+1} (N+1) \frac{t^N}{N!} s^k \\ &= \frac{1}{s} \sum_{N \geq 2} \sum_{1 \leq k \leq N-1} a_k (N) \frac{t^{N-1}}{(N-1)!} s^k \\ &= \frac{1}{s} \sum_{N \geq 2} \left(\sum_{0 \leq k \leq N-1} a_k (N) \frac{t^{N-1}}{(N-1)!} s^k - a_0 (N) \frac{t^{N-1}}{(N-1)!} \right) \\ &= \frac{1}{s} \left(\sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_k (N) \frac{t^{N-1}}{(N-1)!} s^k - a_0 (1) - \sum_{N \geq 2} t^{N-1} \right) \\ &= \frac{1}{s} \left(g_t + \frac{1}{t-1} \right), \end{aligned} \tag{24}$$

where $g_t = \partial g / \partial t$. From the first term of the right hand side of (23), we have

$$\begin{aligned} & \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} N a_{k+1} (N) \frac{t^N}{N!} s^k \\ &= \frac{t}{s} \sum_{N \geq 1} \sum_{1 \leq k \leq N-1} a_k (N) \frac{t^{N-1}}{(N-1)!} s^k \\ &= \frac{t}{s} \left(\sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_k (N) \frac{t^{N-1}}{(N-1)!} s^k \right. \\ & \quad \left. - \sum_{N \geq 1} \frac{a_0 (N)}{(N-1)!} t^{N-1} \right) \\ &= \frac{t}{s} \left(g_t + \frac{1}{t-1} \right). \end{aligned} \tag{25}$$

From the second term of the right hand side of (23), we have

$$\begin{aligned} & \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} (k+1) a_{k+1} (N) \frac{t^N}{N!} s^k \\ &= \sum_{N \geq 1} \sum_{1 \leq k \leq N} k a_k (N) \frac{t^N}{N!} s^{k-1} \\ &= \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} k a_k (N) \frac{t^N}{N!} s^{k-1} = g_s, \end{aligned} \tag{26}$$

where $g_s = \partial g / \partial s$.

From (22)–(26), we obtain the following initial value problem quasilinear first-order partial differential equation:

$$\begin{aligned} (t-1) g_t + s g_s &= -s g - 1, \quad |t| < 1, \\ g(0, s) &= 0, \quad s \in \mathbb{R}. \end{aligned} \tag{27}$$

We consider Cauchy problem for the following first-order quasilinear partial differential equation:

$$\begin{aligned} P(x, y, z) z_x + Q(x, y, z) z_y &= R(x, y, z), \\ z(x_0(t), y_0(t)) &= z_0(t), \quad t \in I, \end{aligned} \tag{28}$$

where I is some interval.

We know that (28) has a unique solution under some conditions as follows.

Theorem A (see [17, page 65]). *Suppose that P, Q , and R are of class C^1 in a domain Ω of \mathbb{R}^3 containing the point (x_0, y_0, z_0) and suppose that*

$$P(x_0, y_0, z_0) \frac{dy_0(t_0)}{dt} - Q(x_0, y_0, z_0) \frac{dx_0(t_0)}{dt} \neq 0. \tag{29}$$

Then in a neighborhood U of (x_0, y_0) there exists a unique solution of (28) at every point of initial curve contained in U .

Since (27) satisfies (29) and regularity conditions, there exists a unique solution of (27).

It is customary to write (27) in the form

$$\frac{dt}{t-1} = \frac{ds}{s} = \frac{dg}{-sg-1}, \tag{30}$$

$$t = 0, \quad s = p, \quad g = 0. \tag{31}$$

Since $dt/(t-1) = ds/s$ is separable, we get

$$u_1(t, s, g) = \frac{1-t}{s}. \tag{32}$$

u_1 is a solution of partial differential equation of (27).

From (30), we get the linear equation

$$\frac{dg}{ds} = -g - \frac{1}{s}. \tag{33}$$

By the integrating factor method, we have

$$u_2(t, s, g) = e^s g + E_i(s). \tag{34}$$

The exponential integral $E_i(s)$ is defined by

$$\begin{aligned} E_i(s) &= \int_{-\infty}^s \frac{e^r}{r} dr \\ &= \gamma + \ln |s| + \sum_{n=1}^{\infty} \frac{s^n}{n \cdot n!}, \quad (s \in \mathbb{R}, s \neq 0), \end{aligned} \tag{35}$$

where γ is Euler constant.

u_2 is another solution of partial differential equation of (27), and u_1 and u_2 are linearly independent.

From the parameterized initial conditions (31), (33), and (34), we get

$$u_2 = E_i \left(\frac{1}{u_1} \right), \quad e^s g + E_i(x) = E_i \left(\frac{s}{1-t} \right). \tag{36}$$

Thus, from (35) and (36), we obtain the following unique solution of (27):

$$g(t, s) = e^{-s} \left(-\ln |1 - t| + \sum_{n=1}^{\infty} \frac{s^n}{n \cdot n!} \left(\left(\frac{1}{1-t} \right)^n - 1 \right) \right). \tag{37}$$

Moreover, if we choose another initial condition

$$g(t, 0) = \sum_{N \geq 1} a_0(N) \frac{t^N}{N!} = \sum_{N \geq 1} \frac{t^N}{N} \tag{38}$$

from (20) and (22), then (37) satisfies it.

We note that

$$\begin{aligned} \left(\frac{1}{1-t} \right)^n - 1 &= \underbrace{\left(\sum_{l_1 \geq 0} t^{l_1} \right) \times \dots \times \left(\sum_{l_n \geq 0} t^{l_n} \right)}_{n\text{-times}} - 1 \\ &= \sum_{N \geq 1} \left(\sum_{l_1 + \dots + l_n = N} t^N \right) \\ &= \sum_{N \geq 1} \binom{n + N - 1}{N} t^N. \end{aligned} \tag{39}$$

By (37) and (39), we get

$$\begin{aligned} g(t, s) &= \left(\sum_{k \geq 0} \frac{(-1)^k}{k!} s^k \right) \left(\sum_{N \geq 1} \frac{t^N}{N} \right) \\ &+ \left(\sum_{k \geq 0} \frac{(-1)^k}{k!} s^k \right) \\ &\times \left(\sum_{n \geq 1} \frac{s^n}{n \cdot n!} \sum_{N \geq 1} \binom{n + N - 1}{N} t^N \right) \\ &= \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} \frac{(-1)^k}{N \cdot k!} t^N s^k + \sum_{N \geq 1} \sum_{k \geq N} \frac{(-1)^k}{N \cdot k!} t^N s^k \\ &+ \sum_{N \geq 1} \sum_{1 \leq k \leq N-1} \left(\sum_{l=1}^k \frac{(-1)^{k-l}}{(k-l)! l \cdot l!} \binom{l + N - 1}{N} \right) t^N s^k \\ &+ \sum_{N \geq 1} \sum_{k \geq N} \left(\sum_{l=1}^k \frac{(-1)^{k-l}}{(k-l)! l \cdot l!} \binom{l + N - 1}{N} \right) t^N s^k. \end{aligned} \tag{40}$$

It is known that

$$\begin{aligned} (-1)^l \binom{l + N - 1}{N} &= \frac{l}{N} \binom{-N}{l}, \\ \sum_{l=0}^k \binom{k}{l} \binom{-N}{l} &= \binom{k + N}{k}. \end{aligned} \tag{41}$$

In the case of $k \geq N$ in (40), from (41), we get

$$\begin{aligned} \frac{(-1)^k}{N \cdot k!} + \sum_{l=1}^k \frac{(-1)^{k-l}}{(k-l)! l \cdot l!} \binom{l + N - 1}{N} \\ = (-1)^k \left(\frac{1}{N \cdot k!} + \frac{1}{N \cdot k!} \sum_{l=1}^k \binom{k}{l} \binom{-N}{l} \right) \\ = (-1)^k \frac{1}{N \cdot k!} \left(1 + \binom{k - N}{k} - 1 \right) = 0. \end{aligned} \tag{42}$$

By (40) and (41), we get

$$\begin{aligned} g(t, s) &= \sum_{N \geq 1} \sum_{1 \leq k \leq N-1} (-1)^k \frac{(N-1)!}{k!} \binom{k - N}{k} \frac{t^N}{N!} s^k \\ &+ \sum_{N \geq 1} (N-1)! \frac{t^N}{N!} \\ &= \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} (N - k - 1)! \binom{N - 1}{k} \frac{t^N}{N!} s^k, \end{aligned} \tag{43}$$

where $\binom{k - N}{k} = (-1)^k \binom{N - 1}{k}$. Thus, by (22) and (43), we get

$$a_k(N) = (N - k - 1)! \binom{N - 1}{k}. \tag{44}$$

Therefore, by (13) and (44), we obtain the following theorem.

Theorem 1. For $q \in \mathbb{C}$ with $|q| < 1$ and $N \in \mathbb{N}$, one can consider the following nonlinear $(N - 1)$ th order ordinary differential equation with respect to q :

$$\begin{aligned} G^N(q) &= \frac{1}{(N-1)!} \sum_{k=0}^{N-1} (N - k - 1)! \binom{N - 1}{k} q^k G^{(k)} \\ &= \sum_{k=0}^{N-1} \frac{1}{k!} \binom{N - 1}{k} q^k G^{(k)}, \end{aligned} \tag{45}$$

where $G^{(k)} = d^k G^{(q)} / dq^k$ and $G^N(q) = \underbrace{G(q) \times \dots \times G(q)}_{N\text{-times}}$.

Then $G(q) = 1/(qe^t + 1)$ is a solution of (45).

Let us define $G^{(k)}(t, x) = G^{(k)}(q)e^{xt}$. Then we obtain the following corollary.

Corollary 2. For $N \in \mathbb{N}$, one considers

$$\begin{aligned} G^N(t, x) &= \frac{1}{(N-1)!} \sum_{k=0}^{N-1} (N - k - 1)! \binom{N - 1}{k} q^k G^{(k)}(t, x) \\ &= \sum_{k=0}^{N-1} \frac{1}{k!} \binom{N - 1}{k} q^k G^{(k)}(t, x). \end{aligned} \tag{46}$$

Then $G(t, x) = e^{xt}/(qe^t + 1)$ is a solution of (46).

3. Identities on the High-Order q -Euler Numbers and Polynomials with Weight 0

From (3), (7), and (8), we get

$$G^N(q) = \frac{1}{2^N} \underbrace{\left(\frac{2}{qe^t + 1} \right) \times \cdots \times \left(\frac{2}{qe^t + 1} \right)}_{N\text{-times}} \tag{47}$$

$$= \frac{1}{2^N} \sum_{n=0}^{\infty} \tilde{E}_{n,q}^{(N)} \frac{t^n}{n!},$$

$$G(q) = \frac{1}{2} \frac{2}{qe^t + 1} = \frac{1}{2} \sum_{n=0}^{\infty} \tilde{E}_{n,q} \frac{t^n}{n!}.$$

From (47), we note that

$$G^{(k)} = \frac{d^k G(q)}{dq^k} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{d^k \tilde{E}_{n,q}}{dq^k} \frac{t^n}{n!}. \tag{48}$$

Therefore, by (47), (48), and (45), we obtain the following theorem.

Theorem 3. For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, one has

$$\tilde{E}_{n,q}^{(N)} = 2^{N-1} \sum_{k=0}^{N-1} \frac{1}{k!} \binom{N-1}{k} q^k \frac{d^k \tilde{E}_{n,q}}{dq^k}. \tag{49}$$

From (48), we get

$$G^{(k)}(t, x) = G^{(k)}(q) e^{xt}$$

$$= \left(\frac{1}{2} \sum_{n=0}^{\infty} \frac{d^k \tilde{E}_{n,q}}{dq^k} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{x^n t^n}{n!} \right) \tag{50}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \frac{1}{2} \binom{n}{l} x^{n-l} \frac{d^k \tilde{E}_{l,q}}{dq^k} \right) \frac{t^n}{n!}.$$

Therefore, by (7), (47), and (50), we obtain the following corollary.

Corollary 4. For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, one has

$$\tilde{E}_{n,q}^{(N)}(x) = 2^{N-1} \sum_{k=0}^{N-1} \frac{1}{k!} \binom{N-1}{k} q^k \sum_{l=0}^n \binom{n}{l} x^{n-l} \frac{d^k \tilde{E}_{l,q}}{dq^k}. \tag{51}$$

From (3) and (7), we get

$$\sum_{n=0}^{\infty} \tilde{E}_{n,q}^{(N)} \frac{t^n}{n!} = \underbrace{\left(\frac{2}{qe^t + 1} \right) \times \cdots \times \left(\frac{2}{qe^t + 1} \right)}_{N\text{-times}}$$

$$= \left(\sum_{l_1=0}^{\infty} \tilde{E}_{l_1,q} \frac{t^{l_1}}{l_1!} \right) \times \cdots \times \left(\sum_{l_N=0}^{\infty} \tilde{E}_{l_N,q} \frac{t^{l_N}}{l_N!} \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l_1+\cdots+l_N=n} \frac{n! \tilde{E}_{l_1,q} \cdots \tilde{E}_{l_N,q}}{l_1! \cdots l_N!} \right) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l_1+\cdots+l_N=n} \binom{n}{l_1, \dots, l_N} \tilde{E}_{l_1,q} \cdots \tilde{E}_{l_N,q} \right) \frac{t^n}{n!}. \tag{52}$$

Therefore, by (49) and (52), we obtain the following corollary.

Corollary 5. For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, one has

$$\sum_{l_1+\cdots+l_N=n} \binom{n}{l_1, \dots, l_N} \tilde{E}_{l_1,q} \cdots \tilde{E}_{l_N,q} \tag{53}$$

$$= 2^{N-1} \sum_{k=0}^{N-1} \frac{1}{k!} \binom{N-1}{k} q^k \frac{d^k \tilde{E}_{n,q}}{dq^k}.$$

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