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Research Article

New Application of the (G'/G)-Expansion Method for Thin Film Equations

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The (G'/G)-expansion method is used for the first time to find traveling wave solutions for thin film equations, where it is found that the related balance numbers are not the usual positive integers. The closed-form solution obtained via this method is in good agreement with the previously obtained solutions of other researchers. It is also noted that, for appropriate parameters, new solitary waves solutions are found.

1. Introduction

In this paper we are interested in the so-called standard thin film equation of the form

$$u_t = -(u^n u_{xxx})_x, \tag{1}$$

which in general has important applications in geology, biophysics, physics, and engineering (see [1–3]). Also known as the lubrication equation [4], it models the spreading motion of the free surface of a thin film on a solid substrate [5]. In particular, the function $u(x,t)$ is the thickness of the fluid film at position x and time t. Here the parameter n denotes the kind of flow. In the case $n = 1$, the equation models the thickness of a thin film in a Hele-Shaw cell [6]. When $n = 2$, it is the Navier slip thin film equation which arises in the study of wetting films with a free contact line between film and substrate [7]. Furthermore, when $n = 3$, it corresponds to the surface-tension-driven spreading of a thin Newtonian fluid [8].

King [9] introduced a generalization of the above equation given by fourth-order nonlinear degenerate parabolic equations of the following form:

$$u_t = -\left(u^n u_{xxx} + \alpha u^{(n-1)} u_x u_{xx} + \beta u^{(n-2)} (u_x)^3\right)_x, \tag{2}$$

where n, α, and β are constants, while the second is a doubly nonlinear equation:

$$u_t = -\left(u^n |u_{xxx}|^{k-1} u_{xxx}\right)_x, \tag{3}$$

where $k > 0$ is a constant related to the flow. Note that when $\alpha = 0$, $\beta = 0$, and $k = 1$ each of the generalized thin film equations turned into the standard thin film equation (1). One of the most effective direct methods to build traveling wave solution of nonlinear PDEs is the (G'/G)-expansion method, which was first proposed by Wang et al. [10]. It is assumed that the traveling wave solutions can be expressed by a polynomial in (G'/G), where $G = G(\zeta)$ satisfies the following second-order linear ordinary differential equation $G''(\zeta) + \lambda G'(\zeta) + \mu G(\zeta) = 0$, where $\zeta = x - ct$, λ, μ, and c are constants. Until now, (G'/G)-expansion method has been successfully applied to obtain exact solution for a variety of nonlinear PDEs [11–21].

Our main objective in this paper is to apply the (G'/G) method to provide closed-form travelling wave solutions of the generalized thin film equations and also standard thin film equation. To the best of our knowledge, this is the first time this method has been applied to such equations. In solving these equations, we found an instance where the related balance numbers are not the usual positive integers (see Zhang [22]). It is also noted that for appropriate parameters new solitary waves solutions are found. We compare our solutions with the solutions previously obtained by Bertozzi and Pugh [23] and King [9], where they proved the existence solution to thin film equation via separation of variables. The closed-form solution obtained via this method is in good agreement with the solutions reported in [9, 23].

Our paper is organized as follows: in Section 2, we present the summary of the (G'/G)-expansion method, in Section 3, we describe the applications of the (G'/G)-expansion method for two generalization thin film equations, standard thin film equation and a special case, and in Section 4, some conclusions are given.

2. Summary of the (G'/G)-Expansion Method

In this section, we describe the (G'/G)-expansion method for finding traveling wave solutions of nonlinear partial differential equations (PDEs). Suppose that a nonlinear partial differential equation, say in two independent variables x and t, is given by the following:

$$P\left(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \ldots\right) = 0, \tag{4}$$

where $u = u(x,t)$ is an unknown function and P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which highest-order derivatives and nonlinear terms are involved. The procedure of the (G'/G)-expansion method can be presented in the following six steps.

Step 1. To find the traveling wave solutions of (4), we introduce the wave variable

$$u(x,t) = u(\zeta), \quad \zeta = x - ct, \tag{5}$$

where the constant c is generally termed the wave velocity. Substituting (5) into (4), we obtain the following ordinary differential equations (ODE) in ζ (which illustrates a principal advantage of a traveling wave solution, i.e., a PDE is reduced to an ODE) as follows.

$$P\left(u, cu', u', cu'', c^2u'', u'', \ldots\right) = 0. \tag{6}$$

Step 2. If necessary we integrate (6) as many times as possible and set the constants of integration to be zero for simplicity. The solution process for (6) is based on the auxiliary conditions that the dependent variable and its first, second, and higher spatial derivatives tend to zero as $\zeta \to \infty$, that is,

$$u(\zeta \longrightarrow \pm\infty) = 0, \qquad \frac{du(\zeta \longrightarrow \pm\infty)}{d\zeta} = 0,$$
$$\frac{d^2u(\zeta \longrightarrow \pm\infty)}{d\zeta^2} = 0, \ldots. \tag{7}$$

From these conditions, we can take the constants of integration to be zero.

Step 3. We suppose that the solution of nonlinear partial differential equation can be expressed by a polynomial in (G'/G) as follows:

$$u(\zeta) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{8}$$

where $G = G(\zeta)$ satisfies the second-order linear ordinary differential equation as follows:

$$G''(\zeta) + \lambda G'(\zeta) + \mu G(\zeta) = 0. \tag{9}$$

Here the prime denotes the derivative respective to ζ, and a_i, λ, and μ are real constants with $a_m \neq 0$. Using the general solutions of (9), we have the following:

$$\frac{G'}{G} = \begin{cases} -\dfrac{\lambda}{2} + \dfrac{\sqrt{\lambda^2 - 4\mu}}{2} \times \left[\dfrac{c_1 \sinh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\zeta\right) + c_2 \cosh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\zeta\right)}{c_1 \cosh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\zeta\right) + c_2 \sinh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\zeta\right)}\right], & \lambda^2 - 4\mu > 0, \\[2ex] -\dfrac{\lambda}{2} + \dfrac{\sqrt{4\mu - \lambda^2}}{2} \times \left[\dfrac{-c_1 \sin\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\zeta\right) + c_2 \cos\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\zeta\right)}{c_1 \cos\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\zeta\right) + c_2 \sin\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\zeta\right)}\right], & \lambda^2 - 4\mu < 0, \\[2ex] \left(\dfrac{c_2}{c_1 + c_2\zeta}\right) - \dfrac{\lambda}{2}, & \lambda^2 - 4\mu = 0. \end{cases} \tag{10}$$

Step 4. The positive integer m can be accomplished by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (6) as follows: if we define the degree of $u(\zeta)$ as $D[u(\zeta)] = m$, then the degree of other expressions is defined by

$$D\left[u^r\left(\frac{d^q u}{d\zeta^q}\right)^s\right] = mr + s(q+m), \tag{11}$$

where s is an integer. Therefore, we can get the value of m in (8).

Step 5. Substitute (8) into (6) and use (9) and collect all terms with the same order of (G'/G) together, then set each coefficient of this polynomial to zero which yields a set of algebraic equations for a_i, c, λ, and μ.

Step 6. Substitute a_i, c, λ, and μ obtained in Step 5 and the general solution of (9) into (8). Next, depending on the sign of the discriminant $\lambda^2 - 4\mu$, we get the solutions of (6). So, we can obtain exact solutions of the given (4).

The advantages of the approach taken in this paper are as follows.

(i) It will be more important to seek solutions of higher-order nonlinear equations which can be reduced to ODEs of the order greater than 3.

(ii) In the (G'/G)-expansion method, there is no need to apply the initial and boundary conditions at the outset. The method yields a general solution with free parameters which can be identified by the above conditions.

(iii) The general solution obtained by (G'/G)-expansion method without approximation.

(iv) Finally, the solution procedure can be easily implemented in Mathematica or Maple.

3. Application of the (G'/G)-Expansion Method

3.1. Exact Traveling Wave Solution of Standard Thin Film Equation (1). Now we consider (1) which arises in the flow of a surface-tension dominated thin liquid film. Substituting (5) into (1) and integrating the result, and for simplicity equating the integration constant equal to zero, we get the following

$$u^n u''' - cu = 0. \tag{12}$$

Suppose that the solution of (12) can be expressed by a polynomial in (G'/G) as follows:

$$u(\zeta) = E\left(\frac{G'}{G}\right)^m, \tag{13}$$

where E is a real constant to be determined later and G satisfies (9). Balancing between $u^n u'''$ and u, we get $m = -3/n$. Now it is easy to deduce that

$$\begin{aligned} u' &= \frac{3E(G'/G)^{(-3+n)/n}\left((G'/G)^2 + \lambda(G'/G) + \mu\right)}{n}, \\ u'' &= \left(\left(-3E\left(\frac{G'}{G}\right)^2 + \lambda\left(\frac{G'}{G}\right) + \mu\right)\right. \\ &\quad \times\left(\left(\frac{G'}{G}\right)^{-3/n}(n-3) - 3\left(\frac{G'}{G}\right)^{(-3+n)/n}\right. \\ &\quad \left.\left.\times\lambda - \left(\frac{G'}{G}\right)^{(-3+2n)/n}(3\mu + n\mu)\right)\right) \times \left(n^2\right)^{-1}. \end{aligned} \tag{14}$$

With the aid of symbolic computation, substituting (13) along with (9) into (12), and setting the coefficients of all powers of (G'/G) to zero, we obtain the following system of nonlinear algebraic equations for E, c, λ, n, and μ:

$$\begin{aligned} \left(\frac{G'}{G}\right)^{-3/n} &: -27E^{1+n}\lambda\mu + 3E^{1+n}n^2\lambda^2\mu + 27E^{1+n}n\lambda^2\mu \\ &\quad + 81E^{1+n}\mu^2 + 54E^{1+n}\lambda^2\mu + 81E^{1+n}\lambda^2 \\ &\quad + 81E^{1+n}\mu + 3E^{1+n}n^2\lambda^2 + 27E^{1+n}n\mu^2 \\ &\quad + 6E^{1+n}\mu n^2 + 6E^{1+n}\mu^2 n^2 = 0, \\ \left(\frac{G'}{G}\right)^{(-3+n)/n} &: 9E^{1+n}n^2\lambda\mu^2 + 54E^{1+n}\lambda n\mu^2 - 54E^{1+n}\lambda n \\ &\quad + 9E^{1+n}\lambda n^2 + 81E^{1+n}\lambda + 81E^{1+n}\lambda\mu^2 = 0, \\ \left(\frac{G'}{G}\right)^{-3(1+n)/n} &: 6E^{1+n}n^2\lambda\mu^2 + 27E^{1+n}\lambda^3 + 162E^{1+n}\lambda\mu = 0, \\ \left(\frac{G'}{G}\right)^{3(n-1)/n} &: 6E^{1+n}n^2 - 27E^{1+n}n + 27E^{1+n} \\ &\quad - cE^{1+n}n^3 = 0, \\ \left(\frac{G'}{G}\right)^{(-3+2n)/n} &: 27E^{1+n}\mu^3 + 6E^{1+n}\mu^3 n^2 + 27E^{1+n}\mu^3 n^2 \\ &\quad + 27E^{1+n}\mu^3 n = 0. \end{aligned} \tag{15}$$

The solutions of this system are as follows:

$$\lambda = 0, \qquad \mu = 0, \qquad c = \left(\frac{3}{n} - 2\right)\left(\frac{3}{n} - 1\right)\frac{3}{n}E^n, \tag{16}$$

where E and n are arbitrary constants.

Consequently, we obtain the exact traveling wave solution of (1),

$$u(x,t) = u(\zeta) = E\left(\frac{c_2}{c_1 + c_2\zeta}\right)^{-3/n}, \tag{17}$$

where $\zeta = x - ct = x - ((3/n) - 2)((3/n) - 1)(3/n)E^n t$.

If we set $c_1 = 0$ and $c_2 = 1$ in (17), we obtain the solitary wave solution

$$u(x,t) = u(\zeta) = E(\zeta)^{3/n}, \tag{18}$$

where ζ is as above. This is exactly the same solution obtained by Bertozzi and Pugh [23]:

$$u(x,t) = \begin{cases} A(x-ct)^{3/n}, & x > ct, \\ 0, & \text{otherwise}, \end{cases} \tag{19}$$

when $c = ((3/n) - 2)((3/n) - 1)(3/n)A^n$.

We remark that if $n \geq 3$, the solution of the system using the (G'/G) cannot be solved due to the no-slip boundary condition on the liquid solid surface, in one space dimension, a similar result reached by Bertozzi and Pugh [24].

3.2. Exact Traveling Wave Solution of the Generalized Thin Film Equation (2). We study nonnegative solutions of the generalized degenerate fourth-order parabolic equation of thin film equation (2). The solution of (2) as found by King [9] is as follows:

$$u=\left[-\frac{n^3c}{3[(3-n)(3-2n)+3(3-n)\alpha+9\beta]}\left[-(x-ct)^3\right]\right]^{1/n}, \tag{20}$$

requiring $n>0$ and $c>0$ for $8\beta<(1-\alpha)^2$ with $3(\alpha-\nu+3)/4<n<3(\alpha+\nu+3)/4$.

We seek the traveling wave solution of (2) in the form (5). Now upon substituting of (5) into (2), one gets

$$-cu'+\left[u^n u'''+\alpha u^{(n-1)}u'u''+\beta u^{(n-2)}\left(u'\right)^3\right]'=0, \tag{21}$$

and by integrating (21) and, for simplicity, equating the integration constant which is equal to zero, we get

$$-cu+u^n u'''+\alpha u^{(n-1)}u'u''+\beta u^{(n-2)}\left(u'\right)^3=0. \tag{22}$$

Balancing between $u^n u'''$ and u, we get $m=-3/n$. Then, suppose that (21) has the following formal solution:

$$u(\zeta)=E\left(\frac{G'}{G}\right)^{-3/n}, \tag{23}$$

where E is an unknown constant to be determined later.

Substituting (23), along with (9), into (22), and setting the coefficients of $(G'/G)^i$, $(i=0,1,\ldots,5)$ to zero, we obtain a system of nonlinear algebraic equations as follows:

$$\begin{aligned}
\left(\frac{G'}{G}\right)^{3(n-1)/n}&: 27\beta E^{1+n}+27E^{1+n}+6E^{1+n}n^2\\
&\quad-27E^{1+n}n-cEn^3-9\alpha E^{1+n}\\
&\quad+27\alpha E^{1+n}=0,\\
\left(\frac{G'}{G}\right)^{(n-3)/n}&: 3E^{1+n}n^2\lambda^2-27E^{1+n}\lambda^2 n+6E^{1+n}\mu n^2\\
&\quad+6E^{1+n}\mu^2n^2+81E^{1+n}\lambda^2\mu-27E^{1+n}n\mu\\
&\quad+27E^{1+n}n\mu^2+81E^{1+n}\mu+81E^{1+n}\mu^2\\
&\quad+81E^{1+n}\lambda^2+81E^{1+n}\lambda^2\mu+3E^{1+n}\lambda^2 n\mu\\
&\quad+9\alpha E^{1+n}\mu^2 n-9\alpha E^{1+n}n\mu+81\alpha E^{1+n}\lambda^2\mu\\
&\quad-9\alpha E^{1+n}\lambda^2 n\mu+81\beta E^{1+n}\mu+81\beta E^{1+n}\lambda^2\\
&\quad+81\beta E^{1+n}\mu^2+81\alpha E^{1+n}\mu^2+81\alpha E^{1+n}\mu\\
&\quad+81\alpha E^{1+n}\lambda^2=0,\\
\left(\frac{G'}{G}\right)^{-3/n}&: 126E^{1+n}\lambda\mu+27E^{1+n}\lambda^3+162\beta E^{1+n}\lambda\mu\\
&\quad+6E^{1+n}n^2\lambda\mu+162\alpha E^{1+n}\lambda\mu+27\beta E^{1+n}\lambda^2\\
&\quad+27\alpha E^{1+n}\lambda^3=0,\\
\left(\frac{G'}{G}\right)^{(2n-3)/n}&: 81E^{1+n}\lambda\mu^2-54E^{1+n}\lambda\mu+9E^{1+n}\lambda n^2\\
&\quad+81E^{1+n}\lambda\mu^2+9E^{1+n}n^2\lambda\nu^2+54E^{1+n}n\lambda\mu^2\\
&\quad+81\alpha E^{1+n}\lambda\mu^2-18\alpha E^{1+n}n\lambda\mu^2+81E^{1+n}\beta\lambda\\
&\quad+81\alpha\lambda E^{1+n}=0,\\
\left(\frac{G'}{G}\right)^{-3(1+n)/n}&: 27E^{1+n}n\mu^3+27E^{1+n}\mu^3+9\alpha E^{1+n}\mu^3 n\\
&\quad+6E^{1+n}n^2\mu^3\lambda^3+27\beta E^{1+n}\mu^3\\
&\quad+27\alpha E^{1+n}\mu^3=0,
\end{aligned} \tag{24}$$

with the solutions

$$\begin{gathered}\lambda=0,\qquad \mu=0,\\ c=\frac{3E^n\left[(3-n)(3-2n)+3(3-n)\alpha+9\beta\right]}{n^3},\end{gathered} \tag{25}$$

where β, n, and α are arbitrary constants. Hence, we obtain the exact traveling wave solution of (2) as follows:

$$u(\zeta)=E\left[\frac{c_2}{c_1+c_2\zeta}\right]^{-3/n}. \tag{26}$$

For the comparison between our solution (26) with that of King's as given in (20), first we assume $c_1=0$ and $c_2=1$ and then we get the same as that of King's (20) if we take c as in (25) in (20).

3.3. Exact Traveling Wave Solution of the Second Generalization of the Thin Film Equation (3). This part is primarily concerned with the Cauchy problem for the doubly degenerate equation (3). King [9] gave the solution of (3) in the form:

$$u=-c|F|^{k-1}F\left[-(x-ct)^{3k/(n+k-1)}\right], \tag{27}$$

where $F=(k+n-1)^3/(3k(2k+1-n)(k+2-2n))$ and here $k=m$ as in King [9].

The traveling wave variable (5) permits us to convert (3) into an ordinary differential equation as follows:

$$-cu + u^n\left|u'''\right|^{k-1}u''' = 0. \tag{28}$$

Considering the homogeneous balance between $u^n|u'''|^{k-1}u'''$ and u in (28), we obtain $m = -3k/(n+k-1)$. Therefore, we can write the following:

$$u(\zeta) = E\left(\frac{G'}{G}\right)^{-3k/(n+k-1)}, \tag{29}$$

for the traveling wave solutions of (28). By substituting (29) together with (9) into (28), clearing the denominator, and setting the coefficients of $(G'/G)^i$, $(i = 0, 1, \ldots, 7)$ to zero, we have the following algebraic system for E, λ, μ, n, and k:

$$\begin{aligned}
\left(\frac{G'}{G}\right)^{-3n/(n+k-1)} &: 6E^{1+n}k + 15E^{1+n}k^2 + 6E^{1+n}k^3 \\
&\quad - 15E^{1+n}k^2n + 6E^{1+n}kn^2 \\
&\quad - 12E^{1+n}kn - cE = 0, \\
\left(\frac{G'}{G}\right)^{-k(5n+5k-2)/(n+k-1)} &: -18E^{1+n}k\lambda n\mu^2 + 9E^{1+n}k\lambda\mu^2n^2 \\
&\quad + 72E^{1+n}k^2n\mu^2\lambda^2 + 9E^{1+n}k\lambda\mu^2 \\
&\quad - 72E^{1+n}\lambda k^2\mu^2 \\
&\quad + 144E^{1+n}k^3\lambda\mu^2 = 0, \\
\left(\frac{G'}{G}\right)^{-k(4n+4k-1)/(n+k-1)} &: -6E^{1+n}\lambda^2n\mu + 33E^{1+n}\mu\lambda^2k^2 \\
&\quad + 6E^{1+n}k\mu^2 + 114E^{1+n}k^3\mu^2 \\
&\quad + 3E^{1+n}\mu\lambda^2n^2 + 6E^{1+n}k\mu^2n^2 \\
&\quad + 111E^{1+n}\mu\lambda^2k^3 + 39E^{1+n}n\mu^2k^2 \\
&\quad - 39E^{1+n}k^2\mu^2 - 33E^{1+n}\mu\lambda^2k^2 \\
&\quad + 3E^{1+n}k\mu\lambda^2 - 12E^{1+n}kn\mu^2 = 0, \\
\left(\frac{G'}{G}\right)^{-3k(n+k)/(n+k-1)} &: 6E^{1+n}k\lambda\mu n^2 - 12E^{1+n}k\lambda\mu n \\
&\quad + 12E^{1+n}\lambda\mu nk^2 + 27E^{1+n}k^3\lambda^3 \\
&\quad + 6E^{1+n}k\lambda\mu + 168E^{1+n}\lambda\mu k^3 \\
&\quad - 12E^{1+n}\lambda\mu k^2 = 0, \\
\left(\frac{G'}{G}\right)^{-k(n+k+2)/(n+k-1)} &: 36E^{1+n}\lambda k^3 + 9E^{1+n}k\lambda \\
&\quad + 36E^{1+n}\lambda k^2 - 18E^{1+n}kn\lambda \\
&\quad + E^{1+n}k\lambda n^2 - 36E^{1+n}\lambda nk^2 = 0, \\
\left(\frac{G'}{G}\right)^{-k(2n+2k+1)/(n+k-1)} &: 21E^{1+n}\lambda^2k^2 + 57E^{1+n}\lambda^2k^3 \\
&\quad + 3E^{1+n}k\mu + 3E^{1+n}k\lambda^2 \\
&\quad + 15E^{1+n}\mu k^2 + 60E^{1+n}\mu k^3 \\
&\quad - 6E^{1+n}kn\lambda^2 + 6E^{1+n}k\mu n^2 \\
&\quad - 15E^{1+n}n\mu k^2 - 21E^{1+n}nk^2 \\
&\quad + 3E^{1+n}kn^2\lambda^2 - 12E^{1+n}kn\mu = 0, \\
\left(\frac{G'}{G}\right)^{-3k(2n+2k-1)/(n+k-1)} &: 60E^{1+n}k^3\mu^3 - 234E^{1+n}k^3\mu^6 \\
&\quad + 6E^{1+n}k\mu^3n^2 - 12E^{1+n}kn\mu^3 \\
&\quad + 39E^{1+n}n\mu^3k^2 = 0.
\end{aligned} \tag{30}$$

Solving this algebraic system by the use of Maple, we get the solutions for (3) as follows:

$$\lambda = 0, \qquad \mu = 0, \qquad c = 3E^nk(2k+1-n)(k+2-2n), \tag{31}$$

where k, n, and E are arbitrary constants. From (31) and (29), we obtain the exact traveling wave solution as follows:

$$u(x,t) = u(\zeta) = E\left[\frac{c_2}{c_1 + c_2\zeta}\right]^{-3k/(n+k-1)}. \tag{32}$$

Choosing $c_1 = 0$ and $c_2 = 1$ in (32), we get

$$u(x,t) = u(\zeta) = E(\zeta)^{-3k/(n+k-1)}. \tag{33}$$

Now this is exactly the same as (27) if we substitute c as given in (31) into (27).

4. Conclusion

In this paper, we provide another instance of the applications of the (G'/G)-expansion method to the still very limited case whereby the balance numbers are not positive integers; see Zhang [22]. We have obtained some new exact traveling wave solutions of the thin film equation and its two generalizations. The solitary wave solutions are derived from these functions when the parameters are taken as special values. The Zhang technique [22] used in this paper is more effective and more general than that originally proposed by Wang et al. [10]. In all the general solutions (17), (26), and (32), we have the additional arbitrary constants c_1 and c_2. We note that the special case $c_1 = 0$ and $c_2 = 1$ reproduced the results of Bertozzi and Pugh [23] and King [9] with an appropriate choice of c. The new type of exact traveling wave solutions obtained in this paper for thin film equation and its two generalizations could be of beneficial use in future studies.

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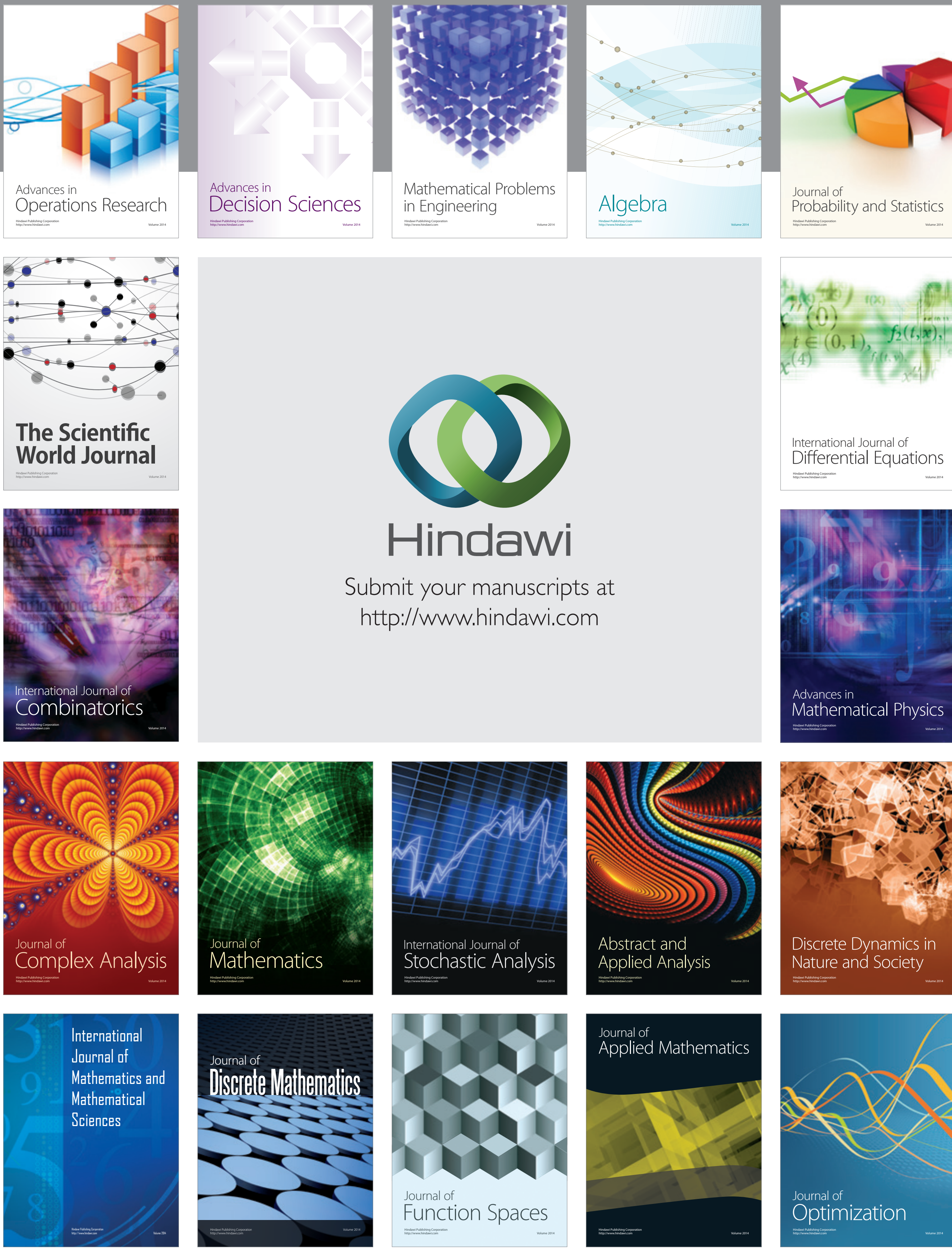
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