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Research Article

Bulbs of Period Two in the Family of Chebyshev-Halley Iterative Methods on Quadratic Polynomials

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The parameter space associated to the parametric family of Chebyshev-Halley on quadratic polynomials shows a dynamical richness worthy of study. This analysis has been initiated by the authors in previous works. Every value of the parameter belonging to the same connected component of the parameter space gives rise to similar dynamical behavior. In this paper, we focus on the search of regions in the parameter space that gives rise to the appearance of attractive orbits of period two.

1. Introduction

The application of iterative methods for solving nonlinear equations $f(z) = 0$, $f : \mathbb{C} \to \mathbb{C}$ gives rise to rational functions whose dynamical behavior provides us with important information about the stability and reliability of the corresponding iterative scheme. The best known iterative method, under the dynamical point of view, is Newton's scheme (see, e.g., [1]).

This study has been extended by different authors to other point-to-point iterative methods for solving nonlinear equations (see, e.g., [2, 3] and more recently, [4–8]). In particular, the authors study the parametric family of Chebyshev-Halley, whose dynamical analysis has been started in [9].

The fixed point operator corresponding to the family of Chebyshev-Halley type methods is

$$G(z,\alpha) = z - \left(1 + \frac{1}{2}\frac{L_f(z)}{1-\alpha L_f(z)}\right)\frac{f(z)}{f'(z)}, \quad (1)$$

where

$$L_f(z) = \frac{f(z)\, f''(z)}{\left(f'(z)\right)^2} \quad (2)$$

and α is a complex parameter. In [9], the authors have begun the study of the dynamics of this operator when it is applied on quadratic polynomial $p(z) = z^2 + c$. For this polynomial, the operator (1) is the rational function:

$$G_p(z,\alpha) = \frac{z^4(-3+2\alpha) + 6cz^2 + c^2(1-2\alpha)}{4z\left(z^2(-2+\alpha) + \alpha c\right)}, \quad (3)$$

depending on parameters α and c.

The parameter c can be removed by applying the conjugacy map

$$h(z) = \frac{z - i\sqrt{c}}{z + i\sqrt{c}}, \quad (4)$$

with the properties $h(\infty) = 1$, $h(i\sqrt{c}) = 0$, and $h(-i\sqrt{c}) = \infty$.

Then, the operator (3) becomes a one-parametric rational function:

$$O_p(z,\alpha) = z^3 \frac{z - 2(\alpha - 1)}{1 - 2(\alpha - 1) z}. \quad (5)$$

As it is known (see [10]), for a rational function $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, on the Riemann sphere $\widehat{\mathbb{C}}$, the *orbit of a point* $z_0 \in \widehat{\mathbb{C}}$ is defined as

$$\left\{z_0, R(z_0), R^2(z_0), \dots, R^n(z_0), \dots\right\}. \quad (6)$$

Depending on the asymptotic behavior of their orbits, a point $z_0 \in \widehat{\mathbb{C}}$ is called a *fixed point* if it satisfies $R(z_0) = z_0$. A *periodic point* z_0 of period $p > 1$ is a point such that $R^p(z_0) = z_0$ and $R^k(z_0) \neq z_0$, $k < p$. A *preperiodic point* is a point z_0 that is not periodic, but there exists a $k > 0$ such that $R^k(z_0)$ is periodic. A *critical point* z_0 is a point where the derivative of rational function vanishes, $R'(z_0) = 0$.

On the other hand, a fixed point z_0 is called *attractor* if $|R'(z_0)| < 1$, *superattractor* if $|R'(z_0)| = 0$, *repulsor* if $|R'(z_0)| > 1$, and *parabolic* if $|R'(z_0)| = 1$. The stability of a periodic orbit is defined by the magnitude (lower than 1 or not) of $|R'(z_1)\cdots R'(z_p)|$, where $\{z_1, \ldots, z_p\}$ are the points of the orbit of period p.

The basin of attraction of an attractor $\bar{z}$ is defined as the set of pre images of any order:

$$\mathscr{A}(\bar{z}) = \left\{z_0 \in \widehat{\mathbb{C}} : R^n(z_0) \longrightarrow \bar{z},\ n \longrightarrow \infty\right\}. \tag{7}$$

The set of points $z \in \widehat{\mathbb{C}}$ such that their families $\{R^n(z)\}_{n\in N}$ are normal in some neighborhood $U(z)$ is the *Fatou set*, $\mathscr{F}(R)$, that is, the Fatou set is composed by the set of points whose orbits tend to an attractor (fixed point, periodic orbit, or infinity). Its complement in $\widehat{\mathbb{C}}$ is the *Julia set*, $\mathscr{J}(R)$; therefore, the Julia set includes all repelling fixed points, periodic orbits, and their pre images. That means that the basin of attraction of any fixed point belongs to the Fatou set. On the contrary, the boundaries of the basins of attraction belong to the Julia set.

The invariant Julia set for Newton's method on quadratic polynomials is the unit circle S^1, and the Fatou set is defined by the two basins of attraction of the superattractor fixed points: 0 and ∞. However, as it can be seen in [11], the Julia set for Chebyshev's method applied to quadratic polynomials is more complicated than for Newton's method. These methods are two elements of the Chebyshev-Halley family. The dynamical study of operator of the Chebyshev-Halley family on quadratic polynomials (3) has been started for the authors in [9, 12].

The rest of the paper is organized as follows. in Section 2, we recall some results about the stability of the strange fixed points of operator $O_p(z,\alpha)$. In Sections 3, 4, and 5, we analyze the black regions of the parameter space involving attractive cycles of period two. We finish the work with some conclusions.

2. Previous Results on Chebyshev-Halley Family

Fixed points of the operator $O_p(z,\alpha)$ are $z = 0$, $z = \infty$, which correspond to the roots of the polynomial and the strange fixed points $z = 1$ and $z = (-3 + 2\alpha \pm \sqrt{5 - 12\alpha + 4\alpha^2})/2$, denoted by $s_1(\alpha)$ and $s_2(\alpha)$, respectively.

Moreover, $z = 0$ and $z = \infty$ are superattractors, and the stability of the other fixed points is established in the following results.

Proposition 1 (see [9, Proposition 1]). *The fixed point $z = 1$ satisfies the following statements.*

(1) *If $|\alpha - (13/6)| < 1/3$, then $z = 1$ is an attractor, and, in particular, it is a superattractor for $\alpha = 2$.*

(2) *If $|\alpha - (13/6)| = 1/3$, then $z = 1$ is a parabolic point.*

(3) *If $|\alpha - (13/6)| > 1/3$, then $z = 1$ is a repulsive fixed point.*

Proposition 2 (see [9, Proposition 2]). *The fixed points $z = s_i(\alpha)$, $i = 1, 2$, satisfy the following statements.*

(i) *If $|\alpha - 3| < 1/2$, then $s_1(\alpha)$ and $s_2(\alpha)$ are two different attractive fixed points. In particular, $s_1(3)$ and $s_2(3)$ are superattractors.*

(ii) *If $|\alpha - 3| = 1/2$, then $s_1(\alpha)$ and $s_2(\alpha)$ are parabolic points. In particular, $s_1(5/2) = s_2(5/2) = 1$.*

(iii) *If $|\alpha - 3| > 1/2$, then $s_1(5/2)$ and $s_2(5/2)$ are repulsive fixed points.*

On the other hand, the critical points of $O_p(z,\alpha)$ are $z = 0$, $z = \infty$ and

$$z = \frac{3 - 4\alpha + 2\alpha^2 \pm \sqrt{-6\alpha + 19\alpha^2 - 16\alpha^3 + 4\alpha^4}}{3(\alpha - 1)}, \tag{8}$$

which are denoted by $c_1(\alpha)$ and $c_2(\alpha)$, respectively.

It is known that there is at least one critical point associated with each invariant Fatou component (see [13]). It is also shown in [9] that the critical points c_i, $i = 1, 2$, are inside the basin of attraction of $z = 1$ when it is attractive $(11/6 < \alpha < 5/2)$ and coincide with $z = 1$ for $\alpha = 2$. Then, they move to the basins of attraction of $s_1(\alpha)$ and $s_2(\alpha)$ when these fixed points become attractive $(5/2 < \alpha < 7/2)$. Critical and fixed points coincide for $\alpha = 3$, and $s_1(3)$ and $s_2(3)$ become superattractors.

The study of the parameter space enables us to analyze the dynamics of the rational function associated to an iterative method: each point of the parameter plane is associated to a complex value of α, that is, to an element of the family. Moreover, every value of α belonging to the same connected component of the parameter space gives rise to subsets of schemes of family with similar dynamical behavior. The parameter space of (3) is shown in Figure 1.

Briefly summarizing the results of [9], we observe a black figure (*the cat set*) with two big disks corresponding to the α values for those fixed points $z = 1$ (the head, $|\alpha - (13/6)| < 1/3$), and $s_1(\alpha)$ and $s_2(\alpha)$ (the body, $|\alpha - 3| < 1/2$) become attractive and have their own basin of attraction, and one critical point is in each basin. The intersection point of the head and the body of the cat is in their common boundary and corresponds to $\alpha = 5/2$. The parameter space inside *the necklace* (the curve similar to a circle that passes through the cat's neck) is topologically equivalent to a disk. The boundary of the cat set is exactly the set of parameters for which the dynamics changes abruptly under small changes of α, that is, the bifurcation loci of the family of Chebyshev-Halley acting on quadratic polynomial.

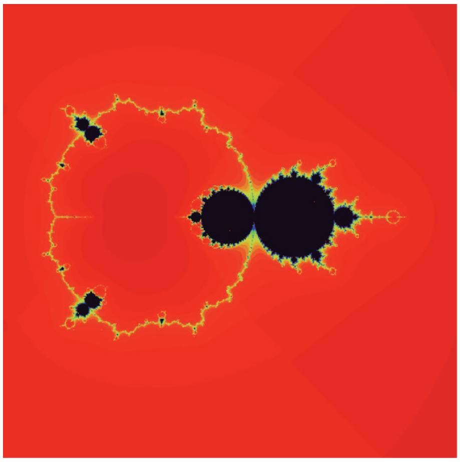


Figure 1: Parameter plane.

Let us stress that the head and the body are surrounded by bulbs, of different sizes, that yield to the appearance of attractive cycles of different periods. In this paper, we focus on the study of all bulbs involving attractive cycles of period 2. As we see in the following sections, these attractive 2-cycles also appear in the small black figures passing through the necklace (*little cats*).

3. Bulbs Involving Attractive Cycles of Period 2

The 2-bulbs consist of values of the parameter which have been associated with an attracting periodic cycle of period two in their respective dynamical planes. Cycles of period 2 satisfy the equation:

$$O_p^2(z,\alpha) = z. \tag{9}$$

The relation $O_p^2(z,\alpha) - z = 0$ can be factorized as

$$z(-1+z)\left(1+3z-2\alpha z+z^2\right) f(z,\alpha)\, g(z,\alpha) = 0, \tag{10}$$

where

$$\begin{aligned} f(z,\alpha) &= 1+(3-2\alpha)z+(3-2\alpha)z^2 \\ &\quad +(3-2\alpha)z^3+z^4, \end{aligned} \tag{11}$$

$$\begin{aligned} g(z,\alpha) &= 1+(3-4\alpha)z+\left(2-6\alpha+4\alpha^2\right)z^2 \\ &\quad +\left(3-6\alpha+4\alpha^2\right)z^3 \\ &\quad +\left(9-22\alpha+20\alpha^2-8\alpha^3\right)z^4 \\ &\quad +\left(3-6\alpha+4\alpha^2\right)z^5+\left(2-6\alpha+4\alpha^2\right)z^6 \\ &\quad +(3-4\alpha)z^7+z^8. \end{aligned} \tag{12}$$

As we have seen in [9], the product $z(-1+z)(1+3z-2\alpha z+z^2)$ yields to the fixed points. So, 2 periodic points come from the roots of $f(z,\alpha)=0$ or $g(z,\alpha)=0$. In the following, we study the bulbs where 2-cycles become attractive, and, by imitating the notation of the Mandelbrot set (see [14]), we call them *2-bulbs*.

In addition, the authors showed in [12] that the strange fixed points $z = 1$ and $s_1(\alpha), s_2(\alpha)$ move from attractors to repulsor in some bifurcation points, and one attractive 2-cycle appears. If we study the dynamical plane for a value of α inside these 2-bulbs, we observe that the Fatou set has a periodic component with two connected components containing the attracting 2-cycle. The two connected components touch at a common point that it is the fixed point from which the attractive cycle comes. Examples of these dynamical planes can be seen in Figures 2, 3, 4, and 5, where the fixed points are identified with little white stars. In these figures we also observe three different Fatou components: the orange one is the attraction basin of $z = 0$, the blue one is the attraction basin of $z = \infty$, and the black one corresponds to the attractive 2-cycle. Let us notice that the strange fixed points are repulsive and they are located on the Julia set.

Let us observe in Figure 4 that the two components, where the 2-cycle is included, touch each other in the strange fixed point $z = 1$.

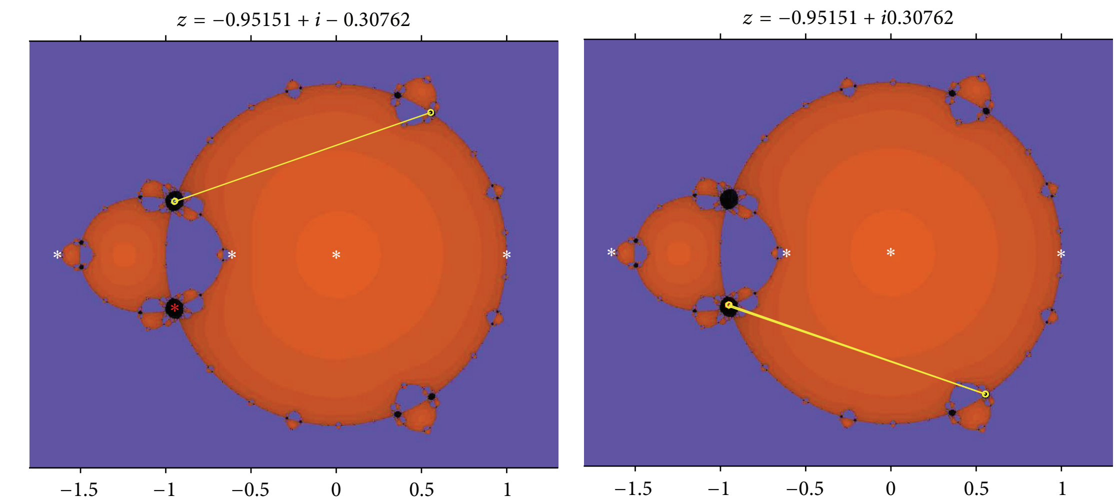
For $\alpha = 3.55$ (Figure 5), two attractive 2-cycles appear, one comes from the strange fixed point $s_1(3.55)$, and the other comes from $s_2(3.55)$.

For $\alpha = 1.687616$ (Figure 3), two attractive 2-cycles appear from the bifurcation of the 2-cycle coming from the fixed point $z = 1$. So, this fixed point is in the boundary of its basin of attraction but not in the boundary of the two immediate basins.

4. Bulbs Coming from $f(z,\alpha)$

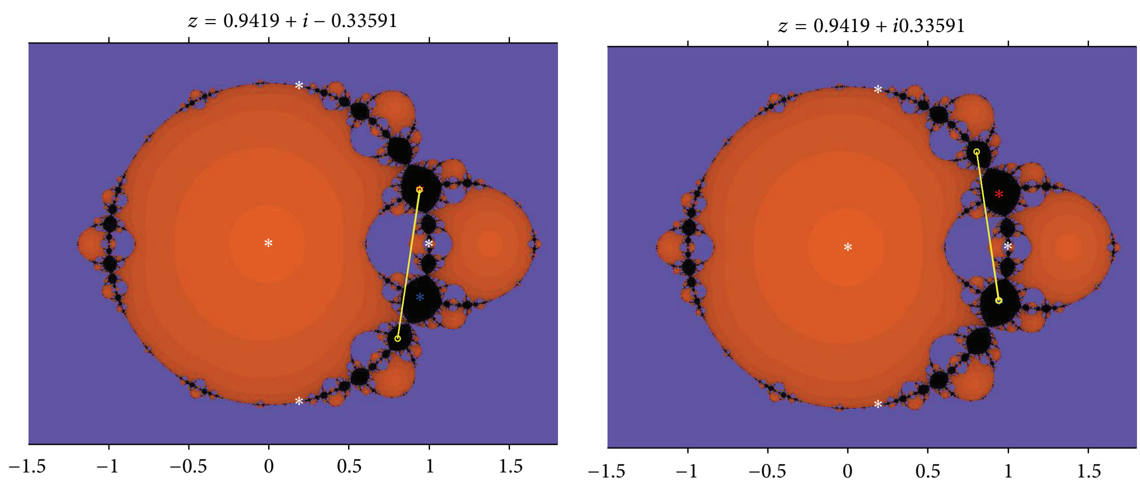
The roots of $f(z,\alpha) = 0$ are

$$\begin{aligned} z_1(\alpha) &= -\frac{3}{4}+\frac{\alpha}{2}+\frac{1}{4}\sqrt{5-4\alpha+4\alpha^2} \\ &\quad -\frac{1}{4}\sqrt{-2-16\alpha+8\alpha^2+(-6+4\alpha)\sqrt{5-4\alpha+4\alpha^2}}, \\ z_2(\alpha) &= -\frac{3}{4}+\frac{\alpha}{2}+\frac{1}{4}\sqrt{5-4\alpha+4\alpha^2} \\ &\quad +\frac{1}{4}\sqrt{-2-16\alpha+8\alpha^2+(-6+4\alpha)\sqrt{5-4\alpha+4\alpha^2}}, \\ z_3(\alpha) &= -\frac{3}{4}+\frac{\alpha}{2}-\frac{1}{4}\sqrt{5-4\alpha+4\alpha^2} \\ &\quad -\frac{1}{4}\sqrt{-2-16\alpha+8\alpha^2-(-6+4\alpha)\sqrt{5-4\alpha+4\alpha^2}}, \\ z_4(\alpha) &= -\frac{3}{4}+\frac{\alpha}{2}-\frac{1}{4}\sqrt{5-4\alpha+4\alpha^2} \\ &\quad +\frac{1}{4}\sqrt{-2-16\alpha+8\alpha^2-(-6+4\alpha)\sqrt{5-4\alpha+4\alpha^2}}, \end{aligned} \tag{13}$$



(a) (b)

FIGURE 2: Two 2-cycles in the dynamical plane for $\alpha = 0.376$.



(a) (b)

FIGURE 3: Two 2-cycles in the dynamical plane for $\alpha = 1.687616$.

where $\{z_1(\alpha), z_2(\alpha)\}$ and $\{z_3(\alpha), z_4(\alpha)\}$ form two cycles of period two:

$$\begin{aligned} O_p\left(z_1(\alpha), \alpha\right) &= z_2(\alpha), \\ O_p\left(z_3(\alpha), \alpha\right) &= z_4(\alpha). \end{aligned} \tag{14}$$

The stability functions of these 2-cycles depends on α:

$$\begin{aligned} S_{12}(\alpha) &= O_p'\left(z_1(\alpha), \alpha\right) \cdot O_p'\left(z_2(\alpha), \alpha\right) \\ &= 2(-1+\alpha)(-3+2\alpha)\left(-9+13\alpha-12\alpha^2+4\alpha^3\right) \\ &\quad \times \sqrt{5-4\alpha+4\alpha^2} + 2\left(63-210\alpha+363\alpha^2\right. \\ &\quad \left. -376\alpha^3+248\alpha^4 -96\alpha^5+16\alpha^6\right), \\ S_{34}(\alpha) &= O_p'\left(z_3(\alpha), \alpha\right) \cdot O_p'\left(z_4(\alpha), \alpha\right) \\ &= -2(-1+\alpha)(-3+2\alpha)\left(-9+13\alpha-12\alpha^2+4\alpha^3\right) \\ &\quad \times \sqrt{5-4\alpha+4\alpha^2} + 2\left(63-210\alpha+363\alpha^2\right. \\ &\quad \left. -376\alpha^3+248\alpha^4 -96\alpha^5+16\alpha^6\right). \end{aligned} \tag{15}$$

We can draw numerically the boundaries where these 2-cycles are parabolic, (see Figure 6). In it, blue points correspond to $|S_{12}(\alpha)| = |O_p'(z_1(\alpha), \alpha) \cdot O_p'(z_2(\alpha), \alpha)| = 1$ and red points represent the stability function $|S_{34}(\alpha)| = |O_p'(z_3(\alpha), \alpha) \cdot O_p'(z_4(\alpha), \alpha)| = 1$.

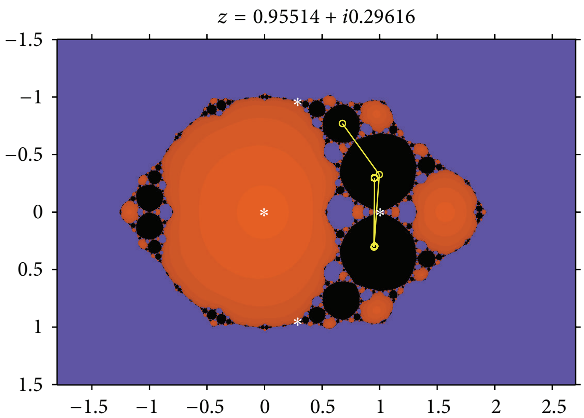


FIGURE 4: Dynamical plane for $\alpha = (11/6) - 0.05$.

The first two drawings of Figure 6 are part of the boundary of the little cats and correspond to values of the parameter where the imaginary part is always different from zero. The third one admits real values of α. As we see in the following result, these real values permit to obtain the "radius" and the center of this bulb (see [12]).

Moreover, we find two disks that embed this bulb.

Proposition 3. *Let $S_{12}(\alpha)$ be the stability function of the 2-cycle $\{z_1(\alpha), z_2(\alpha)\}$. Then, $|S_{12}(\alpha)| < 1$, for $\alpha \in D_1$, such that*

$$C_2 := \{\alpha : |\alpha - m| < 0.064\} \subset D_1 \subset C_1 := \{\alpha : |\alpha - m| < r\}, \tag{16}$$

where $m = (1/2)((11/6) + \alpha^*)$, $r = (1/2)((11/6) - \alpha^*) \approx 0.064617$, and $\alpha^* = (1/6)\sqrt[3]{(134 + 18\sqrt{57})} - (4/(3\sqrt[3]{(134 + 18\sqrt{57})})) + (5/6) \approx 1.7041$.

Proof. The boundary of the bulb satisfies $|S_{12}(\alpha)| = 1$, and it is not a circle, but there exists a corona delimited by two circles C_1 and C_2. These circles are centered in the middle point of α^* and $11/6$, $\alpha_0^* = (1/2)((11/6) + \alpha^*) \approx 1.76871$, and have radii $r = (1/2)((11/6) - \alpha^*) \approx 0.0646$ and $r' = 0.064$, respectively, see Figure 7.

As we see in Figures 7 and 8, the value of the stability function $|S_{12}(\alpha)|$ in these circles is such that $|S_{12}(\alpha)| \geq 1$ if $\alpha \in C_1$ and $|S_{12}(\alpha)| < 1$ if $\alpha \in C_2$. In Figure 8 the red curve corresponds to $|S_{12}(\alpha)|$ when $\alpha \in C_1$ and the blue one corresponds to $\alpha \in C_2$. The value $|S_{12}(\alpha)| = 1$ corresponds to the real values $\alpha = 11/6$ and $\alpha = \alpha^*$. □

The dynamical planes for values of α inside this bulb D_1 are similar to those obtained in Figure 4.

5. Bulbs Coming from $g(z,\alpha)$

A similar study can be made on $g(z, \alpha)$ in order to obtain the other bulbs of period 2 belonging to the cat set. To simplify the study we factorize $g(z, \alpha) = h_1(z, \alpha)h_2(z, \alpha)h_3(z, \alpha)h_4(z, \alpha)$, where $h_i(z, \alpha)$ are polynomials of degree two, $h_i(z, \alpha) = 1 + b_i z + z^2$, $i = 1, 2, 3, 4$. The relationships that the coefficients must satisfy are

$$\begin{gathered} b_1 + b_2 + b_3 + b_4 = 3 - 4\alpha, \\ b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 = 4\alpha^2 - 6\alpha - 2, \\ b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4 = 4\alpha^2 - 6\alpha - 6, \\ b_1b_2b_3b_4 = -8\alpha^3 + 12\alpha^2 - 10\alpha + 7, \end{gathered} \tag{17}$$

whose solution is

$$\begin{aligned} b_1(\alpha) &= \frac{1}{4}\left(3 - 4\alpha - \sqrt{-3 + 8\alpha}\right. \\ &\qquad \left. - \sqrt{2}\sqrt{23 - 16\alpha + 8\alpha^2 + \frac{-3 + 20\alpha - 32\alpha^2}{\sqrt{-3 + 8\alpha}}}\right), \\ b_2(\alpha) &= \frac{1}{4}\left(3 - 4\alpha - \sqrt{-3 + 8\alpha}\right. \\ &\qquad \left. + \sqrt{2}\sqrt{23 - 16\alpha + 8\alpha^2 + \frac{-3 + 20\alpha - 32\alpha^2}{\sqrt{-3 + 8\alpha}}}\right), \\ b_3(\alpha) &= \frac{1}{4}\left(3 - 4\alpha + \sqrt{-3 + 8\alpha}\right. \\ &\qquad \left. - \sqrt{2}\sqrt{23 - 16\alpha + 8\alpha^2 - \frac{-3 + 20\alpha - 32\alpha^2}{\sqrt{-3 + 8\alpha}}}\right), \\ b_4(\alpha) &= \frac{1}{4}\left(3 - 4\alpha + \sqrt{-3 + 8\alpha}\right. \\ &\qquad \left. + \sqrt{2}\sqrt{23 - 16\alpha + 8\alpha^2 - \frac{-3 + 20\alpha - 32\alpha^2}{\sqrt{-3 + 8\alpha}}}\right). \end{aligned} \tag{18}$$

We define the following functions, in terms of the behavior of the cycles

$$\begin{aligned} g_1(z, \alpha) &= h_1(z, \alpha) h_2(z, \alpha) \\ &= 1 + \frac{1}{2}\left(3 - 4\alpha - \sqrt{-3 + 8\alpha}\right) z \\ &\quad + \frac{1}{2}(-1 + 2\alpha)\left(1 + \sqrt{-3 + 8\alpha}\right) z^2 \\ &\quad + \frac{1}{2}\left(3 - 4\alpha - \sqrt{-3 + 8\alpha}\right) z^3 + z^4, \\ g_2(z, \alpha) &= h_3(z, \alpha) h_4(z, \alpha) \\ &= 1 + \frac{1}{2}\left(3 - 4\alpha + \sqrt{-3 + 8\alpha}\right) z \end{aligned}$$

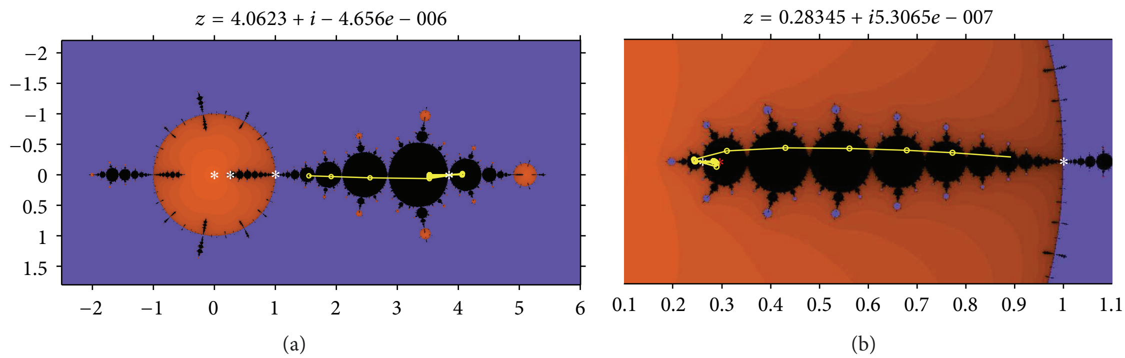


FIGURE 5: Two 2-cycles in the dynamical plane for $\alpha = 3.55$.

$$+\frac{1}{2}(-1+2\alpha)\left(1-\sqrt{-3+8\alpha}\right)z^2 + \frac{1}{2}\left(3-4\alpha+\sqrt{-3+8\alpha}\right)z^3 + z^4. \tag{19}$$

We see in the following sections that the roots of these functions yield to the appearance of attractive 2-cycles.

5.1. Cycles of Period 2 Coming from $g_1(z,\alpha)$. The four solutions of $g_1(z,\alpha) = 0$ are

$$\begin{aligned}
w_1(\alpha) &= \frac{1}{8}\left(-3+4\alpha+\sqrt{-3+8\alpha}-\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}}\right)\\
&\quad -\frac{1}{4}\sqrt{-3-12a+8a^2-\sqrt{-3+8\alpha}+\frac{\sqrt{2}}{2}\left(3-4\alpha-\sqrt{8\alpha-3}\right)\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}}},\\
w_2(\alpha) &= \frac{1}{8}\left(-3+4\alpha+\sqrt{-3+8\alpha}+\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}}\right)\\
&\quad -\frac{1}{4}\sqrt{-3-12a+8a^2-\sqrt{-3+8\alpha}-\frac{\sqrt{2}}{2}\left(3-4\alpha-\sqrt{8\alpha-3}\right)\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}}},\\
w_3(\alpha) &= \frac{1}{8}\left(-3+4\alpha+\sqrt{-3+8\alpha}-\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}}\right)\\
&\quad +\frac{1}{4}\sqrt{-3-12a+8a^2-\sqrt{-3+8\alpha}+\frac{\sqrt{2}}{2}\left(3-4\alpha-\sqrt{8\alpha-3}\right)\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}}},\\
w_4(\alpha) &= \frac{1}{8}\left(-3+4\alpha+\sqrt{-3+8\alpha}+\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}}\right)\\
&\quad +\frac{1}{4}\sqrt{-3-12a+8a^2-\sqrt{-3+8\alpha}-\frac{\sqrt{2}}{2}\left(3-4\alpha-\sqrt{8\alpha-3}\right)\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}}}.
\end{aligned} \tag{20}$$

It is easy to see that $w_1(7/2) = w_2(7/2) = s_1(7/2)$ and $w_3(7/2) = w_4(7/2) = s_2(7/2)$, so there are two 2-cycles coming from this function, $\{w_1(\alpha), w_2(\alpha)\}$ and $\{(w_3(\alpha), w_4(a)\}$. We know (see [9]) that for $\alpha = 7/2$ the strange fixed points, $s_1(7/2) = 2 - \sqrt{3}$ and $s_2(7/2) = 2 + \sqrt{3}$ become parabolic $|O'_p(s_1(7/2),\alpha)| = |O'_p(s_2(7/2),\alpha)| = 1$, and for $\alpha > 7/2$, these strange fixed points are repulsive. As we prove in [12] these two 2-cycles become attractive for real $\alpha \in (7/2, \alpha^{**})$, $\alpha^{**} \approx 3.738271$. In this paper, we are interested in locating the boundaries of the bulb. In the stability study of the cycles $\{w_1(\alpha), w_2(\alpha)\}$ and $\{(w_3(\alpha), w_4(a)\}$, we obtain two 2-bulbs where these 2-cycles are attractive.

The dynamical planes for values of the parameter α inside these 2-bulbs are similar to those obtained in Figures 3 (region D_2) and 5 (region D_3).

Proposition 4. *Let $Sw_{12}(\alpha)$ and $Sw_{34}(\alpha)$ be the stability functions of the 2-cycles $\{w_1(\alpha), w_2(\alpha)\}$ and $\{w_3(\alpha), w_4(\alpha)\}$,* *respectively. Then,* $|Sw_{12}(\alpha)| < 1$ *and* $|Sw_{34}(\alpha)| < 1$*, for* $\alpha \in D_2 \cup D_3$*, such that*

$$\begin{aligned} C_2 &:= \{\alpha : |\alpha - 1.6876| < 0.015\} \\ &\subset D_2 \subset C_1 := \{\alpha : |\alpha - 1.6876| < 0.017\}, \\ C_4 &:= \{\alpha : |\alpha - 3.62| < 0.117\} \\ &\subset D_3 \subset C_3 := \{\alpha : |\alpha - 3.62| < 0.12\}. \end{aligned} \tag{21}$$

Proof. The 2-cycles $\{w_1(\alpha), w_2(\alpha)\}$ and $\{w_3(\alpha), w_4(\alpha)\}$ derive from the strange points $s_1(\alpha)$ and $s_2(\alpha)$ for $\alpha = 7/2$ and $\alpha = \alpha^*$, being attractive for the same values of the parameter. There are two different bulbs where these 2-cycles are attractive, D_2 and D_3. The boundaries of these bulbs satisfy $|Sw_{12}(\alpha)| = |Sw_{34}(\alpha)| = 1$, and they are not circles. Nevertheless, there exist two coronas delimited by the circles (C_1, C_2) and (C_3, C_4), see Figures 9 and 10, surrounding the boundaries of these bulbs.

The stability function on these two circles $|Sw_{12}(\alpha)| = |Sw_{34}(\alpha)|$ is drawn for all of them: $\partial C_1 : \alpha = 1.6876 + 0.017e^{2\pi t i}$ and $\partial C_2 : \alpha = 1.6876 + 0.015e^{2\pi t i}$, $\partial C_3 : \alpha = 3.62 + 0.12e^{2\pi t i}$ and $\partial C_4 : \alpha = 3.62 + 0.117e^{2\pi t i}$, $0 \le t \le 1$, respectively, and they can be seen in Figures 11 and 12, where the color of the stability function is the same as the corresponding circle. Let us notice that the stability function is equal to 1 for $t = \pi$ in the second picture; this point corresponds to $\alpha = 7/2 \in C_4$, and it is precisely the bifurcation point. □

5.2. Cycles of Period 2 Coming from $g_2(z,\alpha)$. The four solutions of $g_2(z, \alpha) = 0$ are

$$\begin{aligned} t_1(\alpha) &= \frac{1}{8}\left(-3 + 4\alpha - \sqrt{-3 + 8\alpha} - \sqrt{2}\sqrt{23 - 16\alpha + 8\alpha^2 - (1 - 4\alpha)\sqrt{-3 + 8\alpha}}\right) \\ &\quad - \frac{1}{4}\sqrt{-3 - 12\alpha + 8\alpha^2 + \sqrt{-3 + 8\alpha} + \frac{\sqrt{2}}{2}\left(3 - 4\alpha + \sqrt{8\alpha - 3}\right)\sqrt{23 - 16\alpha + 8\alpha^2 - (1 - 4\alpha)\sqrt{-3 + 8\alpha}}}, \\ t_2(\alpha) &= \frac{1}{8}\left(-3 + 4\alpha - \sqrt{-3 + 8\alpha} - \sqrt{2}\sqrt{23 - 16\alpha + 8\alpha^2 - (1 - 4\alpha)\sqrt{-3 + 8\alpha}}\right) \\ &\quad + \frac{1}{4}\sqrt{-3 - 12\alpha + 8\alpha^2 + \sqrt{-3 + 8\alpha} + \frac{\sqrt{2}}{2}\left(3 - 4\alpha + \sqrt{8\alpha - 3}\right)\sqrt{23 - 16\alpha + 8\alpha^2 - (1 - 4\alpha)\sqrt{-3 + 8\alpha}}}, \\ t_3(\alpha) &= \frac{1}{8}\left(-3 + 4\alpha - \sqrt{-3 + 8\alpha} + \sqrt{2}\sqrt{23 - 16\alpha + 8\alpha^2 - (1 - 4\alpha)\sqrt{-3 + 8\alpha}}\right) \\ &\quad - \frac{1}{4}\sqrt{-3 - 12\alpha + 8\alpha^2 + \sqrt{-3 + 8\alpha} - \frac{\sqrt{2}}{2}\left(3 - 4\alpha + \sqrt{8\alpha - 3}\right)\sqrt{23 - 16\alpha + 8\alpha^2 - (1 - 4\alpha)\sqrt{-3 + 8\alpha}}}, \\ t_4(\alpha) &= \frac{1}{8}\left(-3 + 4\alpha - \sqrt{-3 + 8\alpha} + \sqrt{2}\sqrt{23 - 16\alpha + 8\alpha^2 - (1 - 4\alpha)\sqrt{-3 + 8\alpha}}\right) \\ &\quad + \frac{1}{4}\sqrt{-3 - 12\alpha + 8\alpha^2 + \sqrt{-3 + 8\alpha} - \frac{\sqrt{2}}{2}\left(3 - 4\alpha + \sqrt{8\alpha - 3}\right)\sqrt{23 - 16\alpha + 8\alpha^2 - (1 - 4\alpha)\sqrt{-3 + 8\alpha}}}. \end{aligned} \tag{22}$$

The study of the stability functions of these roots

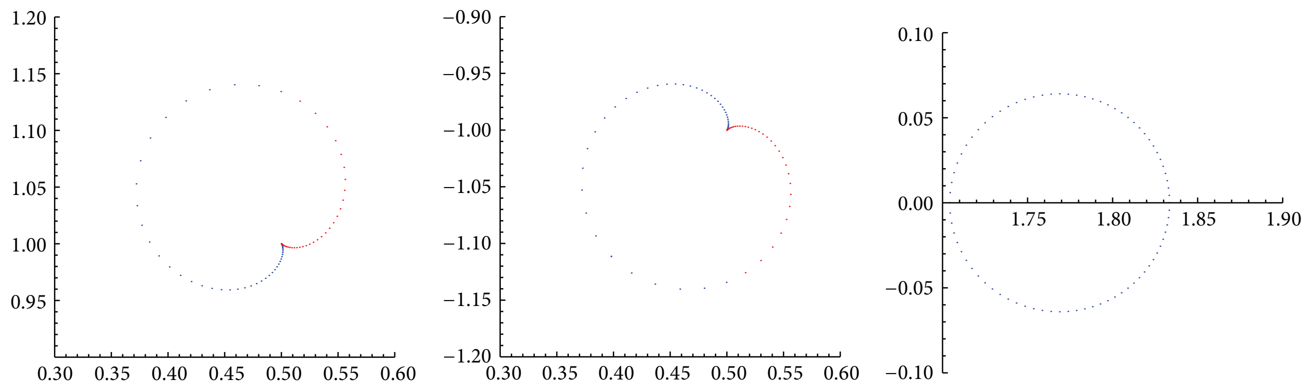
$$\begin{aligned} St_{14}(\alpha) &= O_p'(t_1(\alpha), \alpha)\, O_p'(t_4(\alpha), \alpha), \\ St_{23}(\alpha) &= O_p'(t_2(\alpha), \alpha)\, O_p'(t_3(\alpha), \alpha), \end{aligned} \tag{23}$$

and their boundaries $|St_{14}(\alpha)| = |St_{23}(\alpha)| = 1$ gives three different regions where the 2-cycles $\{t_1(\alpha), t_4(\alpha)\}$ and $\{t_2(\alpha), t_3(\alpha)\}$ are attractive (see Figure 13).

Proposition 5. *Let $St_{14}(\alpha)$ be the stability function of the 2-cycle $\{t_1(\alpha), t_4(\alpha)\}$. Then, $|St_{14}(\alpha)| < 1$, for $\alpha \in D_4$, such that*

$$\begin{aligned} C_2 &:= \{\alpha : |\alpha - 0.376| < 0.001\} \\ &\subset D_4 \subset C_1 := \{\alpha : |\alpha - 0.376| < 0.002\}. \end{aligned} \tag{24}$$

Proof. The 2-cycle $\{t_1(\alpha), t_4(\alpha)\}$ is attractive in three different bulbs. One of them cuts the real line, and its boundary satisfies $|St_{14}(\alpha)| = 1$; as in the previous cases, it is not circle but there exists one corona delimited by the circles (C_1, C_2), (Figure 14).



(a) (b) (c)

FIGURE 6: $|S_{12}(\alpha)| = 1, |S_{34}(\alpha)| = 1$.

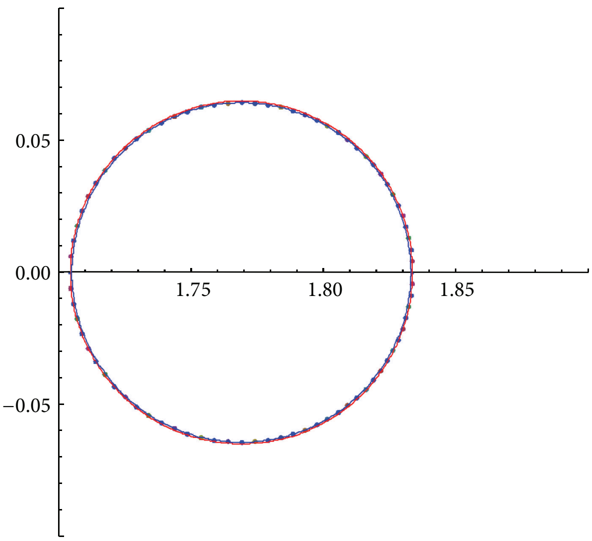


FIGURE 7: Corona delimiting the bulb of period two in the head.

The stability function of these two circles $|St_{14}(\alpha)|$ is drawn for $\partial C_1 : \alpha = 0.376 + 0.02e^{2\pi t i}$ and $\partial C_2 : \alpha = 0.376 + 0.01e^{2\pi t i}$, $0 \leq t \leq 1$, and can be seen in Figure 15 where the colour of stability function is the same as the corresponding circle. The dynamical planes for values of α inside this bulb are similar to those obtained in Figure 2. □

6. Conclusions

The cat set as a parameter space of the Chebyshev-Halley family on quadratic polynomials is dynamically very wealthy, as it happens with Mandelbrot set. The head and the body of the cat set are surrounded by bulbs of different sizes. In this paper, we study those which give rise to attractive cycles of period two. We observe that these attractive 2-cycles exist for many different parameter values, that is, for many different members of the family of iterative methods.

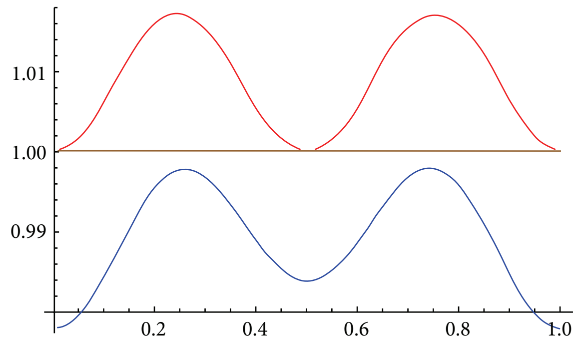


FIGURE 8: Stability function of the 2-cycle $\{z_1(\alpha), z_2(\alpha)\}$ on C_1 and C_2.

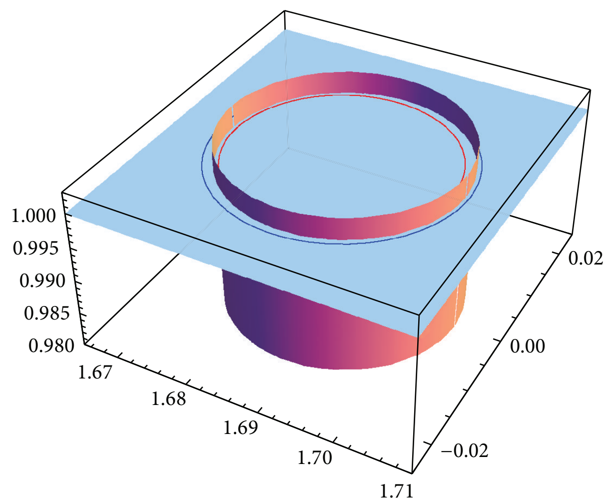


FIGURE 9: Corona delimited by the circles (C_1, C_2).

We can draw the stability functions of all these 2-cycles. Mathematica permits us to draw these stability functions for values between 0 and 1, Figure 16. In it, the big circles correspond to values of α where the fixed points $z = 1$ (left

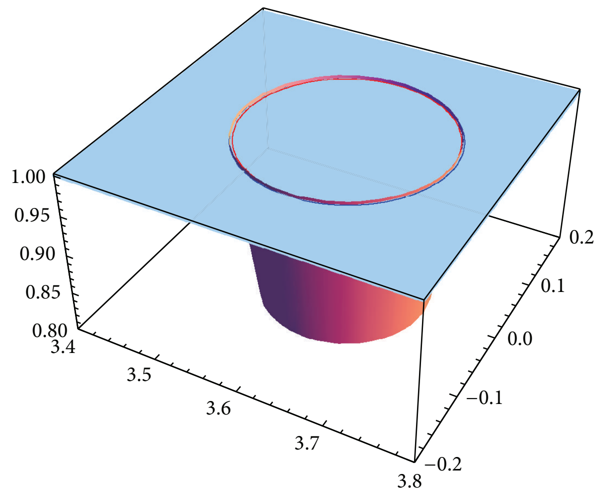


FIGURE 10: Corona delimited by the circles (C_3, C_4).

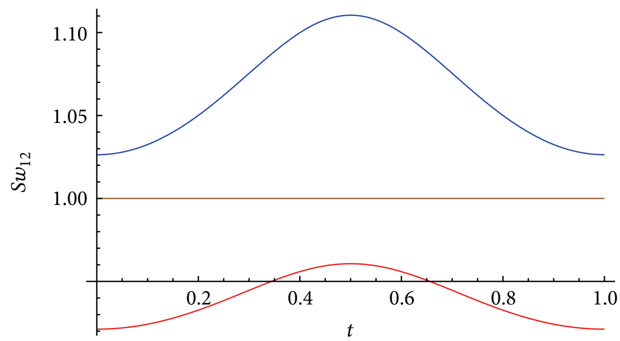


FIGURE 11: Stability function of the 2-cycle $\{w_1(\alpha), w_2(\alpha)\}$ for C_1 and C_2.

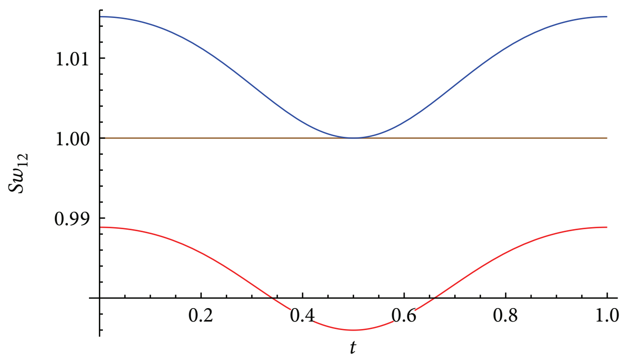


FIGURE 12: Stability function of the 2-cycle $\{w_3(\alpha), w_4(\alpha)\}$ for C_3 and C_4.

one, $|\alpha - (13/6)| < 1/3$), and $z = s_1(\alpha), s_2(\alpha)$ (right one, $|\alpha - 3| < 1/2$), respectively, are attractive. The other figures correspond to the different bulbs where the 2-cycles studied in the previous sections are attractive.

Let us remark that the number of 2-cycles is different depending on the bulb considered. There is only one attractive 2-cycle in the region D_1, whereas there are two 2-cycles in the regions D_2 and D_3. This is because the attractive 2-cycle

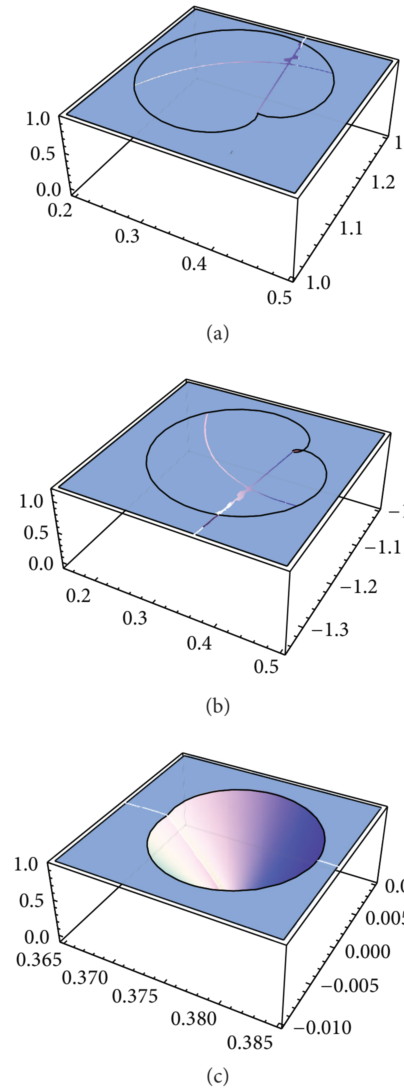


FIGURE 13: $|St_{12}(\alpha)| = 1$ and $|St_{34}(\alpha)| = 1$.

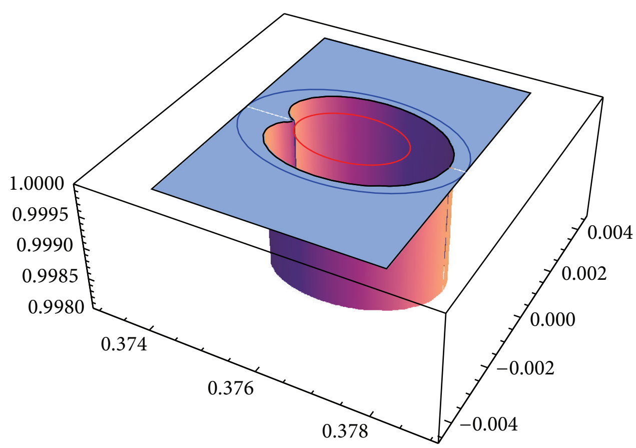


FIGURE 14: $|St_{14}(\alpha)| = 1$ and the corona (C_1, C_2).

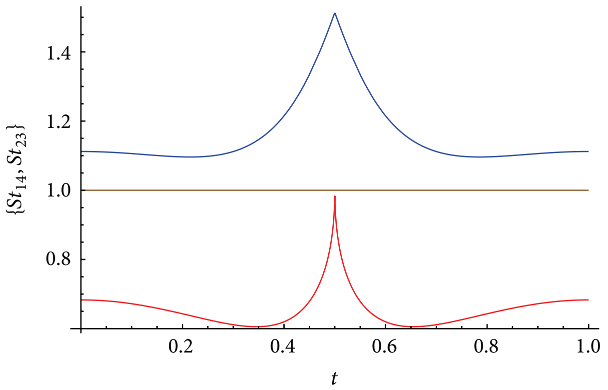


FIGURE 15: Stability function of the 2-cycle $\{t_1(\alpha), t_4(\alpha)\}$ for C_1 and C_2.

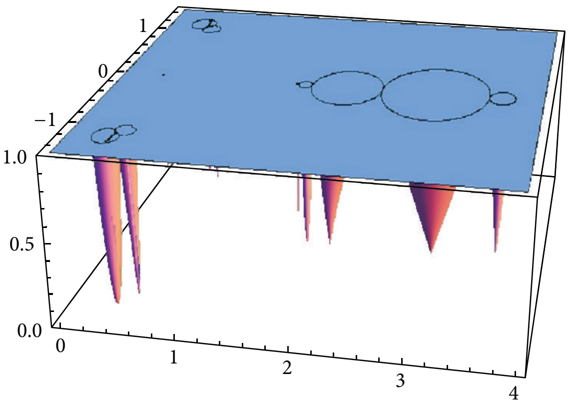


FIGURE 16: Stability functions of cycles of period two.

for $\alpha \in D_1$ comes from the bifurcation of the strange fixed point $z = 1$, and the two attractive 2-cycles for $\alpha \in D_2 \cup D_3$ come from the bifurcation of the strange fixed points $z = s_1(\alpha), s_2(\alpha)$, and the two attractive 2-cycles for $\alpha \in D_2$ come from the bifurcation of the 2-cycle $\{z_1(\alpha), z_2(\alpha)\}$.

Furthermore, by comparing Figures 1 and 16, we can conclude that the little cats on the necklace also correspond to values of α where some attractive 2-cycles appear in the Chebyshev-Halley family.

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