

## Research Article

# Energy Solution to the Chern-Simons-Schrödinger Equations

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We prove that the Chern-Simons-Schrödinger system, under the condition of a Coulomb gauge, has a unique local-in-time solution in the energy space  $H^1(\mathbb{R}^2)$ . The Coulomb gauge provides elliptic features for gauge fields  $A_0, A_j$ . The Koch- and Tzvetkov-type Strichartz estimate is applied with Hardy-Littlewood-Sobolev and Wentz's inequalities.

## 1. Introduction

We study herein the initial value problem of the Chern-Simons-Schrödinger (CSS) equations

$$\begin{aligned} iD_0\phi + D_j D_j \phi &= -\lambda |\phi|^2 \phi, \\ \partial_0 A_1 - \partial_1 A_0 &= -\text{Im}(\bar{\phi} D_2 \phi), \\ \partial_0 A_2 - \partial_2 A_0 &= \text{Im}(\bar{\phi} D_1 \phi), \\ \partial_1 A_2 - \partial_2 A_1 &= -\frac{1}{2} |\phi|^2, \end{aligned} \quad (1)$$

where  $i$  denotes the imaginary unit;  $\partial_0 = \partial/\partial t$ ,  $\partial_1 = \partial/\partial x_1$ , and  $\partial_2 = \partial/\partial x_2$  for  $(t, x_1, x_2) \in \mathbb{R}^{1+2}$ ;  $\phi : \mathbb{R}^{1+2} \rightarrow \mathbb{C}$  is the complex scalar field;  $A_\mu : \mathbb{R}^{1+2} \rightarrow \mathbb{R}$  is the gauge field;  $D_\mu = \partial_\mu + iA_\mu$  is the covariant derivative for  $\mu = 0, 1, 2$ , and  $\lambda > 0$  is a coupling constant representing the strength of interaction potential. The summation convention used involves summing over repeated indices and Latin indices are used to denote 1, 2.

The CSS system of equations was proposed in [1, 2] to deal with the electromagnetic phenomena in planar domains, such as the fractional quantum Hall effect or high-temperature superconductivity. We refer the reader to [3, 4] for more information on the physical nature of these phenomena.

The CSS system exhibits conservation of mass

$$M(t) = \int_{\mathbb{R}^2} |\phi(t, x)|^2 dx = M(0), \quad (2)$$

and the conservation of total energy

$$E(t) = \int_{\mathbb{R}^2} |D_j \phi(t, x)|^2 - \frac{\lambda}{2} |\phi(t, x)|^4 dx = E(0). \quad (3)$$

Note that the terms  $|F|^2 = (1/2)F_{\mu\nu}F^{\mu\nu}$  are missing in (3) when compared to the Maxwell-Schrödinger equations studied in [5].

To figure out the optimal regularity for the CSS system, we observe that the CSS system is invariant under scaling:

$$\begin{aligned} \phi^a(t, x) &= a\phi(a^2 t, ax), & A_j^a(t, x) &= aA_j(a^2 t, ax), \\ A_0^a(t, x) &= a^2 A_0(a^2 t, ax). \end{aligned} \quad (4)$$

Therefore, the scaled critical Sobolev exponent is  $s_c = 0$  for  $\phi$ . In view of (2) we may say that the initial value problem of the CSS system is mass critical.

The CSS system is invariant under the following gauge transformations:

$$\phi \longrightarrow \phi e^{i\chi}, \quad A_\mu \longrightarrow A_\mu - \partial_\mu \chi, \quad (5)$$

where  $\chi : \mathbb{R}^{2+1} \rightarrow \mathbb{R}$  is a smooth function. Therefore, a solution to the CSS system is formed by a class of gauge

equivalent pairs  $(\phi, A_\mu)$ . In this work, we fix the gauge by imposing the *Coulomb gauge* condition of  $\partial_j A_j = 0$ , under which the Cauchy problem of the CSS system may be reformulated as follows:

$$i\partial_t \phi - A_0 \phi + \Delta \phi + 2iA_j \partial_j \phi - A_j^2 \phi = -\lambda |\phi|^2 \phi, \quad (6)$$

$$\partial_1 A_2 - \partial_2 A_1 = -1/2 |\phi|^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0, \quad (7)$$

$$\Delta A_0 = \text{Im}(Q_{12}(\bar{\phi}, \phi)) + \partial_1(A_2 |\phi|^2) - \partial_2(A_1 |\phi|^2), \quad (8)$$

where the initial data  $\phi(0, x) = \phi_0(x)$ . For the formulation of (6)–(8) we refer the reader to Section 3.

The initial value problem of the CSS system was investigated in [6, 7]. It was shown in [6] that the Cauchy problem is locally well posed in  $H^2(\mathbb{R}^2)$ , and that there exists at least one global solution,  $\phi \in L^\infty(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap C_\omega(\mathbb{R}^+; H^1(\mathbb{R}^2))$ , provided that the initial data are made sufficiently small in  $L^2(\mathbb{R}^2)$  by finding regularized equations. They also showed, by deriving a virial identity, that solutions blow up in finite time under certain conditions. Explicit blow-up solutions were constructed in [8] through the use of a pseudo-conformal transformation. The existence of a standing wave solution to the CSS system has also been proved in [9, 10].

The adiabatic approximation of the Chern-Simons-Schrödinger system with a topological boundary condition was studied in [11], which provides a rigorous description of slow vortex dynamics in the near self-dual limit.

Taking the conservation of energy (3) into account, it seems natural to consider the Cauchy problem of the CSS system with initial data  $\phi_0 \in H^1(\mathbb{R}^2)$ . Our purpose here is to supplement the original result of [6] by showing that there is a unique local- in-time solution in the energy space  $H^1(\mathbb{R}^2)$ . We follow a rather direct means of constructing the  $H^1$  solution and prove the uniqueness. We adapt the idea discussed in [12, 13] where a low regularity solution of the modified Schrödinger map (MSM) was studied. In fact, the CSS and MSM systems have several similarities except for the defining equation for  $A_0$ . In the MSM,  $A_0$  can be written roughly as  $R_j R_k (u^2) + |u|^2$ , where  $R_j = \partial_j (-\Delta)^{-1/2}$  denotes the Riesz transform. The local existence of a solution to the MSM was proved in [12] for the initial data in  $H^{s_1}(\mathbb{R}^2)$  with  $s_1 > 1/2$ , and similarly, the uniqueness was proved in [14] for  $H^{s_2}(\mathbb{R}^2)$  with  $s_2 > 3/4$ . To show the existence and uniqueness of the  $H^1$  solution to the CSS system, the estimate of the gauge field,  $A_0$ , is important for situations in which special structures of nonlinear terms in the defining equation for  $A_0$  are used. The following describes our main results.

**Theorem 1.** *Let initial data  $\phi_0$  belong to  $H^1(\mathbb{R}^2)$ . Then, there exists a local-in-time solution,  $\phi$ , to (6)–(8) that satisfies*

$$\begin{aligned} \phi &\in L^\infty([0, T]; H^1(\mathbb{R}^2)) \cap C([0, T]; L^2(\mathbb{R}^2)), \\ J^\delta \phi &\in L^p(0, T; L^q(\mathbb{R}^2)), \end{aligned} \quad (9)$$

where  $0 < \delta < 1/2$ ,  $2 < \delta q$ ,  $1/p + 1/q = 1/2$  and  $J = (1 - \Delta)^{1/2}$ .

**Theorem 2.** *Let  $\phi$  and  $\psi$  be solutions to (6)–(8) on  $(0, T) \times \mathbb{R}^2$  in the distribution sense with the same initial data to that outlined vide supra. Moreover, one assumes that*

$$\begin{aligned} \phi, \psi &\in C([0, T]; L^2(\mathbb{R}^2)), \\ \|\phi\|_{L_T^\infty H^1} &\leq M, \quad \|\psi\|_{L_T^\infty H^1} \leq M, \end{aligned} \quad (10)$$

for some constant  $M > 0$ . One then has  $\|(\phi - \psi)(t, \cdot)\|_{L^2(\mathbb{R}^2)} = 0$  for  $0 \leq t \leq T$ .

We present some preliminaries in Section 2. Theorems 1 and 2 are proved in Sections 3 and 4, respectively. We conclude the current section by providing a few notations. We denote space time derivatives by  $\partial = (\partial_0, \partial_1, \partial_2)$  and  $\nabla$  is used for spacial derivatives. We use the standard Sobolev spaces  $W^{s,p}$ , with the norm  $\|f\|_{W^{s,p}} = \|J^s f\|_{L^p}$  and  $\dot{W}^{s,p}$  with the norm  $\|f\|_{\dot{W}^{s,p}} = \|\nabla^s f\|_{L^p}$ , where  $J = (1 - \Delta)^{1/2}$  and  $|\nabla| = (-\Delta)^{1/2}$ . The space  $H^s$  denotes  $W^{s,2}$ . We define the space time norm as  $\|f\|_{L_T^p L^q} = (\int_0^T \|f(t, \cdot)\|_{L^q(\mathbb{R}^2)}^p dt)^{1/p}$ . We use  $c, C$  to denote various constants. Because we are interested in local solutions, we may assume that  $T \leq 1$ . Thus, we replace the smooth function of  $T, C(T)$  with  $C$ . We also use the convention of writing  $A \leq B$  as shorthand for  $A \leq CB$ .

## 2. Preliminaries

We collect here a few lemmas used for the proof of Theorems 1 and 2. The following lemma is reminiscent of Wenté's inequality (see [15, 16]).

**Lemma 3.** *Let  $f$  and  $g$  be two functions in  $H^1(\mathbb{R}^2)$  and let  $u$  be the solution of*

$$\Delta u = \partial_1 f \partial_2 g - \partial_2 f \partial_1 g \quad \text{in } \mathbb{R}^2, \quad (11)$$

where  $u$  is small at infinity. Then,  $u \in C(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)$  and

$$\|u\|_{L^\infty(\mathbb{R}^2)} + \|\nabla u\|_{L^2(\mathbb{R}^2)} \leq C \|\nabla f\|_{L^2(\mathbb{R}^2)} \|\nabla g\|_{L^2(\mathbb{R}^2)}. \quad (12)$$

The following energy estimate in [17, 18] is used for estimating a solution to the magnetic Schrödinger equation.

**Lemma 4.** *Let  $u$  be a solution of*

$$i\partial_t u + \Delta u + 2i \text{div}(au) = F, \quad (13)$$

where  $a = (a_1(t, x), a_2(t, x))$  and  $a_j$  are real-valued functions. Then, for  $s \geq 0$  there exists an absolute constant  $C_s > 0$  such that

$$\begin{aligned} \|u(t, \cdot)\|_{\dot{H}^s} &\leq \|u(0, \cdot)\|_{\dot{H}^s} \\ &+ C_s \int_0^t (\|\nabla a\|_{\dot{H}^s} \|u\|_{L^\infty} + \|\nabla a\|_{L^\infty} \|u\|_{\dot{H}^s} + \|F\|_{\dot{H}^s}) ds, \end{aligned} \quad (14)$$

wherein one means the homogeneous Sobolev space  $\dot{H}^s$  when  $s > 0$  and simply  $L^2$  when  $s = 0$ .

The following type of Strichartz estimate was used in [19, 20] for the study of the Benjamin-Ono equation. We refer to [12] for the counterpart to the Schrödinger equation.

**Lemma 5.** *Let  $T \leq 1$  and  $v$  be a solution to the equation*

$$i\partial_t v + \Delta v = F_1 + F_2, \quad (t, x) \in (0, T) \times \mathbb{R}^2. \quad (15)$$

Then, for  $\delta \in \mathbb{R}$  and  $\varepsilon > 0$ , one has

$$\|J^\delta v\|_{L_T^p L^q} \lesssim \|v\|_{L_T^\infty H^{\delta+1/2+\varepsilon}} + \|F_1\|_{L_T^2 H^{\delta-1/2}} + \|F_2\|_{L_T^1 H^\delta}, \quad (16)$$

where  $1/p + 1/q = 1/2$  and  $2 \leq q < \infty$ .

We use the following Gagliardo-Nirenberg inequality with the specific constant [21], especially for the proof of Theorem 2.

**Lemma 6.** *For  $2 \leq q < \infty$ , one has*

$$\|u\|_{L^q(\mathbb{R}^2)} \leq (4\pi)^{(2-q)/2q} \left(\frac{q}{2}\right)^{1/2} \|u\|_{L^2(\mathbb{R}^2)}^{2/q} \|\nabla u\|_{L^2(\mathbb{R}^2)}^{1-2/q}. \quad (17)$$

### 3. The Proof of Theorem 1

Theorem 1 is proved in this section. Because the local well-posedness for smooth data is already known in [6], we simply present an *a priori* estimate for the solution to (6)–(8). Let us first explain (8). To derive it, note the following identities:

$$\begin{aligned} \overline{D_\alpha \phi} D_\beta \phi - \overline{D_\beta \phi} D_\alpha \phi &= Q_{\alpha\beta}(\overline{\phi}, \phi) \\ &\quad - i(A_\alpha \partial_\beta (|\phi|^2) - A_\beta \partial_\alpha (|\phi|^2)), \\ D_\alpha D_\beta \phi - D_\beta D_\alpha \phi &= iF_{\alpha\beta} \phi, \end{aligned} \quad (18)$$

where  $Q_{\alpha\beta}(\overline{\phi}, \phi) = \partial_\alpha \overline{\phi} \partial_\beta \phi - \partial_\beta \overline{\phi} \partial_\alpha \phi$  and  $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ . Note that the second-order terms  $\partial_{\alpha\beta} \phi$  are cancelled out. Combined with the above algebra, the equation for  $A_0$  comes from the second and third equations in (1):

$$\begin{aligned} \Delta A_0 &= \partial_1 \operatorname{Im}(\overline{\phi} D_2 \phi) - \partial_2 \operatorname{Im}(\overline{\phi} D_1 \phi) \\ &= \operatorname{Im}(Q_{12}(\overline{\phi}, \phi)) + \partial_1(A_2 |\phi|^2) - \partial_2(A_1 |\phi|^2). \end{aligned} \quad (19)$$

We then have the formulation (6)–(8) in which  $\phi$  is the only dynamical variable and  $A_1, A_2$ , and  $A_0$  are determined through (7) and (8).

The constraint equation  $\partial_1 A_2 - \partial_2 A_1 = -1/2|\phi|^2$  and the Coulomb gauge condition  $\partial_1 A_1 + \partial_2 A_2 = 0$  provide an elliptic feature of  $A = (A_1, A_2)$ ; that is, the components  $A_j$  can be determined from  $\phi$  by solving the elliptic equations

$$\Delta A_1 = \partial_2 \left(\frac{1}{2}|\phi|^2\right), \quad \Delta A_2 = -\partial_1 \left(\frac{1}{2}|\phi|^2\right). \quad (20)$$

Taking into account that the Coulomb gauge condition in Maxwell dynamics deduces a wave equation, the previous

observation was used in [6]. Using (20), we have the following representation of  $A = (A_1, A_2)$ :

$$A_1 = -\frac{1}{4\pi} \left(\frac{x_2}{|x|^2} * |\phi|^2\right), \quad A_2 = \frac{1}{4\pi} \left(\frac{x_1}{|x|^2} * |\phi|^2\right). \quad (21)$$

**3.1. Estimates for  $A$  and  $A_0$ .** We are now ready to estimate several quantities of  $A, A_0$ . Making use of (20) and the representation (21), we obtain the following estimates for  $A$ .

**Proposition 7.** *Let  $s \geq 0$  and  $0 < 2/q < \delta < 1$ . One also assumes that  $2 \leq p < \infty$  if  $s > 0$  or  $2 < p < \infty$  if  $s = 0$ . Then, one has*

$$\begin{aligned} \|\nabla A\|_{\dot{H}^s} &\lesssim \|\phi\|_{L^\infty} \|\phi\|_{\dot{H}^s}, \\ \|\nabla A\|_{L^\infty} &\lesssim \|\phi\|_{L^\infty} \|J^\delta \phi\|_{L^q}, \\ \|A\|_{L^\infty} &\lesssim \|\phi\|_{L^2} \|J^\delta \phi\|_{L^q}, \\ \|\nabla^s A\|_{L^p} &\lesssim \|\phi\|_{L^p} \|\phi\|_{\dot{H}^s}. \end{aligned} \quad (22)$$

*Proof.* The above can be checked by applying Calderon-Zygmund and Hardy-Littlewood-Sobolev inequalities. We refer to [2, Section 2] for the details.  $\square$

To estimate  $A_0$ , the special algebraic structure  $Q_{12}$  and divergence form of the nonlinear terms in (19) are used.

**Proposition 8.** *Let  $A_0$  be the solution of (19). Then, one has*

$$\|A_0\|_{L^\infty} + \|\nabla A_0\|_{L^2} \lesssim (1 + \|\phi\|_{L^2}^2) \|\nabla \phi\|_{L^2}^2. \quad (23)$$

*Proof.* Decompose  $A_0 = A'_0 + A''_0$  as follows:

$$\Delta A'_0 = \operatorname{Im}(Q_{12}(\overline{\phi}, \phi)), \quad (24)$$

$$\Delta A''_0 = \partial_1(A_2 |\phi|^2) - \partial_2(A_1 |\phi|^2). \quad (25)$$

We first estimate the quantity  $\|A_0\|_{L^\infty(\mathbb{R}^2)}$ . Applying Lemma 3 to (24), we deduce that

$$\|A'_0\|_{L^\infty} \lesssim \|\nabla \phi\|_{L^2}^2. \quad (26)$$

To estimate  $\|A''_0\|_{L^\infty(\mathbb{R}^2)}$  we use the Gagliardo-Nirenberg inequality with small  $\varepsilon > 0$ :

$$\begin{aligned} \|u\|_{L^\infty(\mathbb{R}^2)} &\leq C_\varepsilon \|\Delta u\|_{L^{1+\varepsilon}(\mathbb{R}^2)}^\alpha \|u\|_{L^4(\mathbb{R}^2)}^{1-\alpha}, \\ &\text{with } \alpha = (1 + \varepsilon) / (1 + 5\varepsilon). \end{aligned} \quad (27)$$

Applying Hardy-Littlewood-Sobolev's inequality to (25) we deduce

$$\begin{aligned} \|A''_0\|_{L^4} &\leq \|A|\phi|^2\|_{L^{4/3}} \lesssim \|A\|_{L^4} \|\phi\|_{L^4}^2 \\ &\lesssim \|\phi\|_{L^2} \|\phi\|_{L^4}^3 \lesssim \|\phi\|_{L^2}^{5/2} \|\nabla \phi\|_{L^2}^{3/2}, \end{aligned} \quad (28)$$

where Proposition 7 and Lemma 6 are used. We can also derive the following from (25):

$$\|\Delta A_0''\|_{L^{1+\epsilon}} \leq \underbrace{\|\nabla A \cdot |\phi|^2\|_{L^{1+\epsilon}}}_{(i)} + \underbrace{\|A \nabla (|\phi|^2)\|_{L^{1+\epsilon}}}_{(ii)}. \quad (29)$$

The first term can be estimated as follows:

$$(i) \leq \|\nabla A\|_{L^2} \|\phi^2\|_{L^{(2+2\epsilon)/(1-\epsilon)}} \leq \|\phi\|_{L^2}^{2/(1+\epsilon)} \|\nabla \phi\|_{L^2}^{(2+4\epsilon)/(1+\epsilon)}, \quad (30)$$

where  $\|\phi\|_{L^{(4+4\epsilon)/(1-\epsilon)}} \leq \|\phi\|_{L^2}^{(1-\epsilon)/(2+2\epsilon)} \|\nabla \phi\|_{L^2}^{(1+3\epsilon)/(2+2\epsilon)}$  is used. The second term can be estimated as follows:

$$\begin{aligned} (ii) &\leq \|A\|_{L^4} \|\phi\|_{L^{(4+4\epsilon)/(1-3\epsilon)}} \|\nabla \phi\|_{L^2} \\ &\leq \|\phi\|_{L^2} \|\phi\|_{L^4} \|\phi\|_{L^2}^{(1-3\epsilon)/(2+2\epsilon)} \|\nabla \phi\|_{L^2}^{(3+7\epsilon)/(2+2\epsilon)} \\ &\leq \|\phi\|_{L^2}^{2/(1+\epsilon)} \|\nabla \phi\|_{L^2}^{(2+4\epsilon)/(1+\epsilon)}, \end{aligned} \quad (31)$$

where  $\|\phi\|_{L^{(4+4\epsilon)/(1-3\epsilon)}} \leq \|\phi\|_{L^2}^{(1-3\epsilon)/(2+2\epsilon)} \|\nabla \phi\|_{L^2}^{(1+5\epsilon)/(2+2\epsilon)}$  is used. Therefore, we obtain with  $\epsilon = 1/11$ , that is,  $\alpha = 3/4$ ,

$$\|A_0''\|_{L^\infty} \leq \|\phi\|_{L^2}^2 \|\nabla \phi\|_{L^2}^2. \quad (32)$$

Therefore, we conclude that

$$\|A_0\|_{L^\infty} \leq \|\nabla \phi\|_{L^2}^2 (1 + \|\phi\|_{L^2}^2). \quad (33)$$

On the other hand, Lemma 3 shows that

$$\|\nabla A_0'\|_{L^2} \leq \|\nabla \phi\|_{L^2}^2. \quad (34)$$

We also have from (25) that

$$\begin{aligned} \|\nabla A_0''\|_{L^2} &\leq \|A|\phi|^2\|_{L^2} \leq \|A\|_{L^4} \|\phi\|_{L^8}^2 \\ &\leq \|\phi\|_{L^2} \|\phi\|_{L^4} \|\phi\|_{L^8}^2 \leq \|\phi\|_{L^2}^2 \|\nabla \phi\|_{L^2}^2. \end{aligned} \quad (35)$$

Therefore, we have

$$\|\nabla A_0\|_{L^2} \leq \|\nabla \phi\|_{L^2}^2 (1 + \|\phi\|_{L^2}^2). \quad (36)$$

□

**3.2. The Energy Solution to (CSS).** We now prove Theorem 1. Let us define

$$X(T) = \|\phi\|_{L_T^\infty H^1} + \|J^\delta \phi\|_{L_T^p L^q}, \quad (37)$$

where  $0 < \delta < 1/2$ ,  $2 < \delta q$ , and  $1/p + 1/q = 1/2$ . We derive the following estimate:

$$X \leq \|\phi_0\|_{H^1} + T^{1/6} \|\phi_0\|_{L^2} (1 + \|\phi_0\|_{L^2}^2) (X^2 + X^4), \quad (38)$$

from which Theorem 1 is proved by standard argument; see [2, Section 3].

To control  $\|\phi\|_{L_T^\infty H^1}$ , we apply Lemma 4 to the solution of (6)–(8).

**Proposition 9.** Let  $\phi$  be a solution to (6)–(8). Then, one has

$$\begin{aligned} \|\phi\|_{L_T^\infty L^2} &= \|\phi_0\|_{L^2}, \\ \|\phi\|_{L_T^\infty \dot{H}^1} &\leq \|\phi_0\|_{\dot{H}^1} \\ &\quad + (1 + \|\phi_0\|_{L^2}^2) T^{(p-3)/p} \left( \|\phi\|_{L_T^\infty \dot{H}^1}^3 + \|J^\delta \phi\|_{L_T^p L^q}^3 \right), \end{aligned} \quad (39)$$

where  $2 < \delta q$  and  $3 < p < \infty$ .

*Proof.* From the conservation of mass, we derive the first estimate. We apply Lemma 4 to (6) with  $F = A_0 \phi + A_j^2 \phi - \lambda |\phi|^2 \phi$  and  $s = 1$ . Combined with Proposition 7, we have

$$\begin{aligned} \|\nabla A\|_{\dot{H}^1} \|\phi\|_{L^\infty} &\leq \|\phi\|_{W^{\delta,q}}^2 \|\phi\|_{\dot{H}^1}, \\ \|\nabla A\|_{L^\infty} \|\phi\|_{\dot{H}^1} &\leq \|\phi\|_{W^{\delta,q}}^2 \|\phi\|_{\dot{H}^1}, \\ \|A^2 \phi\|_{\dot{H}^1} &\leq \|\phi\|_{L^2}^2 (\|\phi\|_{W^{\delta,q}}^3 + \|\phi\|_{\dot{H}^1}^3), \end{aligned} \quad (40)$$

$$\| |\phi|^2 \phi \|_{\dot{H}^1} \leq \|\phi\|_{W^{\delta,q}}^2 \|\phi\|_{\dot{H}^1},$$

where  $2 < \delta q$ . We are then left to estimate  $\|A_0 \phi\|_{\dot{H}^1}$ . By Proposition 8, we obtain

$$\begin{aligned} \|A_0 \phi\|_{\dot{H}^1} &\leq \|A_0\|_{L^\infty} \|\phi\|_{\dot{H}^1} + \|A_0\|_{\dot{H}^1} \|\phi\|_{L^\infty} \\ &\leq (1 + \|\phi\|_{L^2}^2) (\|\nabla \phi\|_{L^2}^3 + \|J^\delta \phi\|_{L^q}^3). \end{aligned} \quad (41)$$

Combining (40) and (41), we obtain

$$\begin{aligned} \|\phi\|_{L_T^\infty \dot{H}^1} &\leq \|\phi_0\|_{\dot{H}^1} + \int_0^T (1 + \|\phi_0\|_{L^2}^2) (\|\phi\|_{\dot{H}^1}^3 + \|J^\delta \phi\|_{L^q}^3) \\ &\leq \|\phi_0\|_{\dot{H}^1} + (1 + \|\phi_0\|_{L^2}^2) T^{(p-3)/p} \\ &\quad \times \left( \|\phi\|_{L_T^\infty \dot{H}^1}^3 + \|J^\delta \phi\|_{L_T^p L^q}^3 \right), \end{aligned} \quad (42)$$

where  $3 < p < \infty$  and  $T < 1$ . □

To estimate  $\|J^\delta \phi\|_{L_T^p L^q}$ , we apply Lemma 5 to the solution of (6)–(8).

**Proposition 10.** Let  $\phi$  be a solution to (6)–(8). Then, one has

$$\|J^\delta \phi\|_{L_T^p L^q} \leq \|\phi\|_{L_T^\infty H^1} + T^{1/6} \|\phi_0\|_{L^2} (1 + \|\phi_0\|_{L^2}^2) (X^2 + X^4), \quad (43)$$

where  $2 < \delta q$ ,  $3 < p < \infty$  and  $1/p + 1/q = 1/2$ .

*Proof.* Applying Lemma 5 with  $F_1 = A_0 \phi - 2iA_j \partial_j \phi$  and  $F_2 = A^2 \phi - \lambda |\phi|^2 \phi$ , we obtain

$$\begin{aligned} \|J^\delta \phi\|_{L_T^p L^q} &\leq \|\phi\|_{L_T^\infty H^1} + \|A_0 \phi\|_{L_T^2 H^{\delta-1/2}} \\ &\quad + \|A \cdot \nabla \phi\|_{L_T^2 H^{\delta-1/2}} + \|A^2 \phi\|_{L_T^1 H^\delta} + \| |\phi|^2 \phi \|_{L_T^1 H^\delta}, \end{aligned} \quad (44)$$

where  $\delta = 1/2 - \varepsilon$ ,  $3 < p < \infty$  and  $2 < \delta q$ . Considering Proposition 8, we obtain

$$\begin{aligned} & \|A_0\phi\|_{L^2_T H^{\delta-1/2}} \\ & \leq \|A_0\|_{L^\infty_T L^\infty} \|\phi\|_{L^2_T L^2} \leq T^{1/2} \|\phi_0\|_{L^2} (1 + \|\phi_0\|_{L^2}^2) \|\nabla\phi\|_{L^2}^2. \end{aligned} \quad (45)$$

The other terms can be treated, as mentioned in Section 1, by similar arguments to those in [2, Section 3]. Applying Proposition 7, we have

$$\begin{aligned} \|A \cdot \nabla\phi\|_{L^2_T H^{\delta-1/2}} & \leq \|A\|_{L^2_T L^\infty} \|\nabla\phi\|_{L^\infty_T L^2} \\ & \leq \|\phi_0\|_{L^2} T^{(p-2)/2p} \|J^\delta\phi\|_{L^p_T L^q} \|\phi\|_{L^\infty_T H^1}, \end{aligned} \quad (46)$$

$$\begin{aligned} \|A^2\phi\|_{L^1_T H^\delta} & \leq \|A^2\|_{L^1_T L^4} \|J^\delta\phi\|_{L^\infty_T L^4} \\ & \quad + \|A^2\|_{L^2_T W^{\delta,2+\varepsilon}} \|\phi\|_{L^2_T L^{(4+2\varepsilon)/\varepsilon}} \end{aligned} \quad (47)$$

$$\begin{aligned} & \leq T \|\phi_0\|_{L^2}^{3/2} \|\phi\|_{L^\infty_T H^1}^{5/2} \\ & \quad + T^{1/4} \|\phi_0\|_{L^2} \|\phi\|_{L^\infty_T H^1}^2 \|J^\delta\phi\|_{L^p_T L^q}^2, \end{aligned}$$

$$\begin{aligned} \|\phi^3\|_{L^1_T H^\delta} & \leq \|J^\delta\phi\|_{L^\infty_T L^2} \|\phi\|_{L^2_T L^\infty}^2 \\ & \leq T^{(p-2)/p} \|\phi\|_{L^\infty_T H^1} \|\phi\|_{L^p_T L^q}^2. \end{aligned} \quad (48)$$

Plugging estimates (45)–(48) into (44) with  $p > 3$ , we obtain

$$\|J^\delta\phi\|_{L^p_T L^q} \leq \|\phi\|_{L^\infty_T H^1} + T^{1/6} \|\phi_0\|_{L^2} (1 + \|\phi_0\|_{L^2}^2) (X^2 + X^4). \quad (49)$$

□

We finally obtain the estimate (38) by combining Propositions 9 and 10, which proves Theorem 1.

#### 4. The Proof of Theorem 2

In this section, we prove the uniqueness of the solution to (6). The basic rationale is borrowed from [12, 22].

Let  $(\phi, A_0, A)$  and  $(\psi, B_0, B)$  be solutions of (6)–(8) with the same initial data. If we set  $\omega = \phi - \psi$ , then the equation for  $\omega$  is

$$\begin{aligned} i\partial_t\omega + \Delta\omega & = A_0\omega + (A_0 - B_0)\psi - 2iA \cdot \nabla\omega - 2i(A - B) \cdot \nabla\psi \\ & \quad + A^2\omega + (A^2 - B^2)\psi - \lambda|\phi|^2\omega - \lambda(|\phi|^2 - |\psi|^2)\psi. \end{aligned} \quad (50)$$

We will derive

$$\partial_t\|\omega\|_{L^2}^2 \leq q^{1/2} M^2 \|\omega\|_{L^2}^{2-4/q} + qM^{2+4/q} (1 + M^2) \|\omega\|_{L^2}^{2-4/q}, \quad (51)$$

where  $M$  is a constant in Theorem 2 and  $q > 2$ . Then we have

$$\partial_t\|\omega\|_{L^2}^{4/q} \leq \frac{1}{q} (q^{1/2} M^2 + qM^{2+4/q} (1 + M^2)). \quad (52)$$

Considering  $\|\omega(0, \cdot)\|_{L^2} = 0$  and  $2 < q$ , we obtain

$$\|\omega\|_{L^2} \leq (T(M^2 + M^{4+4/q}))^{q/4}. \quad (53)$$

Letting  $q \rightarrow \infty$ , for the time interval satisfying  $T(M^2 + M^{4+4/q}) \leq 1/2$ , we conclude that  $\|\omega(t, \cdot)\|_{L^2} = 0$  for  $0 \leq t \leq T$ , which thus proves Theorem 2.

In the remainder of this section, we derive inequality (51). Multiplying  $\bar{\omega}$  to both sides of (50) and integrating the imaginary part of  $\mathbb{R}^2$ , we have

$$\begin{aligned} \partial_t\|\omega\|_{L^2}^2 & = \int \underbrace{2(A_0 - B_0) \operatorname{Im}(\psi\bar{\omega})}_{(I)} - \underbrace{2A_j\partial_j|\omega|^2}_{(II)} \\ & \quad - \underbrace{4(A_j - B_j) \operatorname{Re}(\partial_j\psi\bar{\omega})}_{(III)} dx \\ & \quad + \int \underbrace{2(A^2 - B^2) \operatorname{Im}(\psi\bar{\omega})}_{(IV)} \\ & \quad - \underbrace{2\lambda(|\phi|^2 - |\psi|^2) \operatorname{Im}(\psi\bar{\omega})}_{(V)} dx. \end{aligned} \quad (54)$$

The integrals (II)–(V), that is, those not containing  $A_0$ , can be controlled by applying similar arguments to those described in [2, Section 4]. Integral (II) can be estimated, considering  $\partial_j A_j = 0$ , by

$$\int -A_j\partial_j|\omega|^2 dx = \int \partial_j A_j |\omega|^2 dx = 0, \quad (55)$$

$$(III), (IV), (V) \leq qM^{2+4/q} (1 + M^2) \|\omega\|_{L^2}^{2-4/q}$$

for which we omit the proof.

We simply present how to control integral (I), for which we have

$$\left| \int (A_0 - B_0) \operatorname{Im}(\psi\bar{\omega}) dx \right| \leq \|A_0 - B_0\|_{L^a} \|\psi\|_{L^b} \|\omega\|_{L^c}, \quad (56)$$

where  $1/a + 1/b + 1/c = 1$ ,  $2 \leq a, b, c$ . Applying Lemma 6, we obtain

$$\begin{aligned} \|\psi\|_{L^b} & \leq b^{1/2} \|\psi\|_{L^2}^{2/b} \|\nabla\psi\|_{L^2}^{1-2/b} \leq b^{1/2} M^{1-2/b}, \\ \|\omega\|_{L^c} & \leq c^{1/2} \|\omega\|_{L^2}^{2/c} \|\nabla\omega\|_{L^2}^{1-2/c} \leq c^{1/2} \|\omega\|_{L^2}^{2/c} M^{1-2/c}. \end{aligned} \quad (57)$$

To control  $\|A_0 - B_0\|_{L^a}$ , we consider the equation for  $A_0 - B_0$

$$\begin{aligned} \Delta(A_0 - B_0) & = \partial_1 \operatorname{Im}(\bar{\phi}\partial_2\phi) - \partial_2 \operatorname{Im}(\bar{\phi}\partial_1\phi) - \partial_1 \operatorname{Im}(\bar{\psi}\partial_2\psi) \\ & \quad + \partial_2 \operatorname{Im}(\bar{\psi}\partial_1\psi) + \partial_1 (A_2|\phi|^2) - \partial_2 (A_1|\phi|^2) \\ & \quad - \partial_1 (B_2|\psi|^2) + \partial_2 (B_1|\psi|^2). \end{aligned} \quad (58)$$

Decomposing  $A_0$  and  $B_0$  as (24) and (25), we have

$$\begin{aligned} \Delta(A'_0 - B'_0) &= \partial_1 \operatorname{Im}(\bar{\phi} \partial_2 \omega) - \partial_2 \operatorname{Im}(\bar{\psi} \partial_1 \omega) \\ &\quad + \partial_1 \operatorname{Im}(\bar{\omega} \partial_2 \psi) - \partial_2 \operatorname{Im}(\bar{\omega} \partial_1 \phi), \end{aligned} \quad (59)$$

$$\begin{aligned} \Delta(A''_0 - B''_0) &= \partial_1 (A_2 (|\phi|^2 - |\psi|^2)) - \partial_2 (A_1 (|\phi|^2 - |\psi|^2)) \\ &\quad + \partial_1 ((A_2 - B_2) |\psi|^2) - \partial_2 ((A_1 - B_1) |\phi|^2). \end{aligned} \quad (60)$$

Taking into account

$$\partial_1 \operatorname{Im}(\bar{\phi} \partial_2 \omega) = \partial_1 (\partial_2 \operatorname{Im}(\bar{\phi} \omega) - \operatorname{Im}(\omega \partial_2 \bar{\phi})), \quad (61)$$

$$\partial_2 \operatorname{Im}(\bar{\psi} \partial_1 \omega) = \partial_2 (\partial_1 \operatorname{Im}(\bar{\psi} \omega) - \operatorname{Im}(\omega \partial_1 \bar{\psi})),$$

we can rewrite the equation for  $A'_0 - B'_0$  as follows:

$$\begin{aligned} \Delta(A'_0 - B'_0) &= \partial_1 (\operatorname{Im}(\bar{\omega} \partial_2 \psi) - \operatorname{Im}(\omega \partial_2 \bar{\phi})) \\ &\quad + \partial_2 (\operatorname{Im}(\omega \partial_1 \bar{\psi}) - \operatorname{Im}(\bar{\omega} \partial_2 \phi)), \end{aligned} \quad (62)$$

where  $\partial_1 \partial_2 \operatorname{Im}(\bar{\phi} \omega) - \partial_2 \partial_1 \operatorname{Im}(\bar{\psi} \omega) = \partial_1 \partial_2 \operatorname{Im}(\bar{\omega} \omega) = 0$  should be noted. Using the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned} \|A'_0 - B'_0\|_{L^a} &\leq \| |x|^{-1} * (\omega \nabla \psi) \|_{L^a} \\ &\leq \|\omega \nabla \psi\|_{L^r} \leq \|\omega\|_{L^s} \|\nabla \psi\|_{L^2}, \end{aligned} \quad (63)$$

where  $1/a = 1/r - 1/2$  and  $1/r = 1/s + 1/2$ , from which we deduce  $a = s$ . Then, we have

$$\|\omega\|_{L^a} \|\nabla \psi\|_{L^2} \leq a^{1/2} \|\omega\|_{L^2}^{2/a} \|\nabla \omega\|_{L^2}^{1-2/a} M \leq a^{1/2} M^{2-2/a} \|\omega\|_{L^2}^{2/a}. \quad (64)$$

The term  $A''_0 - B''_0$  can be bounded as follows:

$$\begin{aligned} \|A''_0 - B''_0\|_{L^a} &\leq \underbrace{\| |x|^{-1} * (|A| (|\phi|^2 - |\psi|^2)) \|_{L^a}}_{(1)} \\ &\quad + \underbrace{\| |x|^{-1} * (|A - B| (|\phi|^2 + |\psi|^2)) \|_{L^a}}_{(2)}. \end{aligned} \quad (65)$$

Since  $\|\phi\|^2 - \|\psi\|^2 \leq (|\phi| + |\psi|)|\omega|$ , we have

$$\begin{aligned} (1) &\leq \| |A| (|\phi| + |\psi|) \|_{L^2} \|\omega\|_{L^a} \\ &\leq \|A\|_{L^6} (\|\phi\|_{L^3} + \|\psi\|_{L^3}) \|\omega\|_{L^a} \end{aligned} \quad (66)$$

$$\begin{aligned} &\leq \|\phi\|_{L^3}^2 (\|\phi\|_{L^3} + \|\psi\|_{L^3}) \|\omega\|_{L^a} \\ &\leq a^{1/2} M^{2-2/a} \|\omega\|_{L^2}^{2/a}. \end{aligned} \quad (67)$$

Since  $|A_j - B_j| \leq |x|^{-1} * ((|\phi| + |\psi|)|\omega|)$ , we may check

$$\begin{aligned} (2) &\leq \|A_j - B_j\|_{L^a} (\|\phi^2\|_{L^2} + \|\psi^2\|_{L^2}) \\ &\leq (\|\phi\|_{L^2} + \|\psi\|_{L^2}) \|\omega\|_{L^a} (\|\phi\|_{L^4}^2 + \|\psi\|_{L^4}^2) \\ &\leq a^{1/2} \|\omega\|_{L^2}^{2/a} \|\nabla \omega\|_{L^2}^{1-2/a} (\|\nabla \phi\|_{L^2} + \|\nabla \psi\|_{L^2}) \\ &\leq a^{1/2} M^{2-2/a} \|\omega\|_{L^2}^{2/a}. \end{aligned} \quad (68)$$

Then, we have

$$\|A_0 - B_0\|_{L^a} \leq a^{1/2} M^{2-2/a} \|\omega\|_{L^2}^{2/a}. \quad (69)$$

Combining estimates (57) and (69), and denoting  $b = q/2$ , we obtain

$$\|A_0 - B_0\|_{L^a} \|\psi\|_{L^b} \|\omega\|_{L^c} \leq (aqc)^{1/2} M^2 \|\omega\|_{L^2}^{2-4/q}, \quad (70)$$

where  $1/a + 2/q + 1/c = 1$ . We then obtain (51) by combining (55) and (70).

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