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Research Article

Some Bounded Linear Integral Operators and Linear Fredholm Integral Equations in the Spaces $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ and $H_{\alpha,\delta}((a,b),X)$

İsmet Özdemir,[1] Ali M. Akhmedov,[2] and Ö. Faruk Temizer[1]

[1] *İnönü Üniversitesi, Eğitim Fakültesi, A-Blok, Posta Kodu: 44280, Malatya, Turkey*
[2] *Baku State University, Department of Mech. & Math., Z. Khalilov Street, 23, P.O. Box 370145, Baku, Azerbaijan*

Correspondence should be addressed to İsmet Özdemir; ismet.ozdemir@inonu.edu.tr

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The spaces $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),\mathbb{R})$ and $H_{\alpha,\delta}((a,b),\mathbb{R})$ were defined in ((Hüseynov (1981)), pages 271–277). Some singular integral operators on Banach spaces were examined, (Dostanic (2012)), (Dunford (1988), pages 2419–2426 and (Plamenevskiy (1965)). The solutions of some singular Fredholm integral equations were given in (Babolian (2011), Okayama (2010), and Thomas (1981)) by numerical methods. In this paper, we define the sets $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ and $H_{\alpha,\delta}((a,b),X)$ by taking an arbitrary Banach space X instead of $\mathbb{R}$, and we show that these sets which are different from the spaces given in (Dunford (1988)) and (Plamenevskiy (1965)) are Banach spaces with the norms $\|\cdot\|_{\alpha,\delta,\gamma}$ and $\|\cdot\|_{\alpha,\delta}$. Besides, the bounded linear integral operators on the spaces $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ and $H_{\alpha,\delta}((a,b),X)$, some of which are singular, are derived, and the solutions of the linear Fredholm integral equations of the form $f(s)=\phi(s)+\lambda\int_a^b A(s,t)f(t)dt$, $f(s)=\phi(s)+\lambda\int_a^b A(t,s)f(t)dt$ and $f(s,t)=\phi(s,t)+\lambda\int_a^b A(s,t)f(t,s)dt$ are investigated in these spaces by analytical methods.

1. Preliminaries, Background, and Notation

The approximate solution of the singular integral equation

$$u(s)-\frac{1}{\pi}p.v.\int_{-1}^{1}\frac{k(s,t)\,u(t)}{t-s}dt=f(t) \tag{1}$$

was obtained in [1], where $k(s,t)$ is a real-valued kernel, $f(s)$ is a given function, and $u(s)$ is the unknown function.

High-order numerical methods for the singular Fredholm integral equations of the form:

$$\lambda u(t)-p.v.\int_a^b|t-s|^{p-1}k(t,s)\,u(s)\,ds=g(t),\qquad a\le t\le b \tag{2}$$

were developed, where $\lambda\neq 0$, $0<p<1$, k and g are given functions, and u is the unknown function. Equations of this form often arise in practical applications such as Dirichlet problems, mathematical problems of radiative equilibrium, and radiative heat transfer problems, [2].

The polar kernel of integral equations was introduced in [3, 4]. This singular kernel is in the following form:

$$k(x,y)=\frac{g(x,y)}{(x-y)^{\alpha}}+h(x,y),\qquad 0<\alpha\le 1, \tag{3}$$

where the first term of this kernel is weakly singular and g and h are bounded on the square $s=[-1,1]\times[-1,1]$ and $g(x,y)\neq 0$. With $g=1,h=0$, we have the special case of the above kernel:

$$k(x,y)=\frac{1}{(x-y)^{\alpha}},\qquad 0<\alpha\le 1, \tag{4}$$

see [5]. One of the weakly singular integral and integrodifferential equations with this kernel was given in [6–8]. The solution of the singular integral equation of the form:

$$\mu(x)\phi(x)+\lambda(x)\,p.v.\int_{-1}^{1}\frac{\phi(y)}{(y-x)^{\alpha}}dy=f(x),\qquad |x|<1,\ 0<\alpha\leq 1 \tag{5}$$

was examined in [5] by numerical methods, where $\mu(x)\neq 0, \lambda(x)\neq 0$ and $\mu(x)$, $\lambda(x)$, $f(x) \in L^2[-1,1]$ are given functions and $\phi(x)$ is the unknown function to be determined.

The integral operator defined by

$$A: L_2(0,T)\longrightarrow L_2(0,T),\qquad (Af)(x)=\int_0^T k(x-y)f(y)\,dy \tag{6}$$

was studied in [9].

An integral equation of the form:

$$f(s)=\phi(s)+\lambda\int_a^b A(s,t)f(t)\,dt \tag{7}$$

is called Fredholm integral equation of the second type. Here, $[a,b]$ is a given interval, f is a function on $[a,b]$ which is unknown, and λ is a parameter. The kernel A of the equation is a given function on the square $[a,b]\times[a,b]$ and ϕ is a given function on $[a,b]$.

Now, we may give some required definitions and theorems.

Definition 1 (see [10, page 41]). Let X be a Banach space and (S,Σ,μ) be a finite measure space. Then $f: S\rightarrow X$ is called measurable simple function if there exist $x_1,x_2,\ldots,x_n\in X$ and $E_1,E_2,\ldots,E_n\in\Sigma$ such that $f=\sum_{i=1}^{n}x_i\chi_{E_i}$, where

$$\chi_{E_i}(w)=\begin{cases}1, & w\in E_i\\ 0, & w\notin E_i.\end{cases} \tag{8}$$

Definition 2 (see [11, page 88]). The Bochner integral of a simple function f given by Definition 1 with respect to μ on $E\in\Sigma$ is defined by

$$\int_E f\,d\mu=\sum_{i=1}^{n}x_i\mu(E_i\cap E). \tag{9}$$

Definition 3 (see [12, page 201]). Let X be a Banach space with the norm $\|\cdot\|_X$ and (S,Σ,μ) be a finite measure space. Then $g: S\rightarrow X$ is called strongly measurable if there exists a sequence (f_n) of X-valued simple functions defined on S such that

$$\lim_n\|f_n(t)-g(t)\|_X=0;\quad t\in S-N,\quad \mu(N)=0. \tag{10}$$

Theorem 4 (see [11, page 88]). *All continuous functions $f: S\rightarrow X$ are strongly measurable.*

Theorem 5 (see [13, page 336]). *Let X be a Banach space with the norm $\|\cdot\|_X$ and (S,Σ,μ) be a finite measure space. If $g: S\rightarrow X$ is strongly measurable, then the scalar function*

$$h: S\longrightarrow\mathbb{R},\qquad h(t)=\|g(t)\|_X \tag{11}$$

is Σ-measurable.

Definition 6 (see [11, page 88]). Let (S,Σ,μ) be a finite measure space and X be a Banach space, and then the Bochner integral of a strongly measurable function $g: S\rightarrow X$ is the strongly limit of the Bochner integral of an approximating sequence (f_n) of simple functions satisfying (10). That is,

$$\lim_n\int_E f_n\,d\mu=\int_E g\,d\mu. \tag{12}$$

Theorem 7 (see [12, page 202]). *Let (S,Σ,μ) be a finite measure space, X be a Banach space with the norm $\|\cdot\|_X$, and $g: S\rightarrow X$ be a strongly measurable function. If $\int_E g\,d\mu$ exists, then*

$$\left\|\int_E g\,d\mu\right\|_X\leq\int_E\|g\|_X\,d\mu. \tag{13}$$

Theorem 8 (see [12, page 203]). *Let (S,Σ,μ) be a finite measure space and X be a Banach space with the norm $\|\cdot\|_X$. The Bochner integral $\int_E g\,d\mu$ exists if and only if g is strongly measurable and $\int_E\|g\|_X\,d\mu<\infty$.*

Theorem 9 (see [14, page 82]). *Let X be a real or complex Banach space with the norm $\|\cdot\|_X$. Let a,b,α,δ be the real numbers satisfying $-\infty<a$, $b<\infty$, $0<\alpha$, $\delta<1$ and M be a nonnegative constant. The set $C^{0,\alpha,\delta}((a,b)\times(a,b),X)$ consisting of all functions $A:(a,b)\times(a,b)\rightarrow X$ fulfilling the conditions:*

$$\|A(s,t)\|_X\leq M,\qquad \|A(s+\Delta s,t+\Delta t)-A(s,t)\|_X\leq M\left(|\Delta s|^{\alpha}+|\Delta t|^{\delta}\right) \tag{14}$$

for all $s,t,s+\Delta s,t+\Delta t\in(a,b)$ is a linear space with the algebraic operations:

$$(A+B)(s,t)=A(s,t)+B(s,t),\qquad (\lambda A)(s,t)=\lambda A(s,t),\quad \lambda\in K,\ (K=\mathbb{R}\ \text{or}\ K=\mathbb{C}), \tag{15}$$

and $C^{0,\alpha,\delta}((a,b)\times(a,b),X)$ is a Banach space with the norm

$$\|A\|_{0,\alpha,\delta}=\max\left\{\sup_{s,t\in(a,b)}\|A(s,t)\|_X, m_A\right\}, \tag{16}$$

where

$$m_A=\sup_{\substack{s,t,s+\Delta s,t+\Delta t\in(a,b)\\ \Delta s\neq 0\vee\Delta t\neq 0}}\frac{\|A(s+\Delta s,t+\Delta t)-A(s,t)\|_X}{|\Delta s|^{\alpha}+|\Delta t|^{\delta}}. \tag{17}$$

Again, let X be a real or complex Banach space with the norm $\|\cdot\|_X$. Let a,b,α be the real numbers with

$-\infty < a,b < \infty$, $0 < \alpha < 1$ and M also be a nonnegative constant. Let A be an X-valued function defined on (a,b) such that

$$\|A(s)\|_X \le M, \quad \|A(s+\Delta s) - A(s)\|_X \le M(\Delta s)^{\alpha} \tag{18}$$

for all $s, s+\Delta s \in (a,b)$ with $\Delta s \ge 0$. By $C^{0,\alpha}((a,b),X)$, we denote the set of all functions A satisfying (18). $C^{0,\alpha}((a,b),X)$ is a linear space with the algebraic operations:

$$(A+B)(s) = A(s) + B(s), \quad (\lambda A)(s) = \lambda A(s), \quad \lambda \in K, \ (K = \mathbb{R} \text{ or } K = \mathbb{C}), \tag{19}$$

and $C^{0,\alpha}((a,b),X)$ is a Banach space with the norm:

$$\|A\|_{0,\alpha} = \max\left\{\sup_{s \in (a,b)} \|A(s)\|_X, \sup_{\substack{s+\Delta s \in (a,b) \\ \Delta s \ne 0}} \frac{\|A(s+\Delta s) - A(s)\|_X}{(\Delta s)^{\alpha}}\right\}. \tag{20}$$

$C^{0,\alpha,\delta}((a,b)\times(a,b),X)$ and $C^{0,\alpha}((a,b),X)$ are called a Hölder space. The functions spaces which are similar to $C^{0,\alpha,\delta}((a,b)\times(a,b),X)$ and $C^{0,\alpha}((a,b),X)$ were investigated in [15, pages 2419–2426], [16, pages 25–51], and [17, pages 18–33]. The class $G_1(h,e)$ of the functions k satisfying the equalities

$$k(x,y) = O\left(|x-y|^{-m} h(|x-y|)\right) \tag{21}$$

and

$$k(x',y) - k(x'',y) = O\left(\frac{h\left(p_y(x',x'')\right) e\left(|x'-x''|\right)}{p_y^m(x',x'') e\left(p_y(x'-x'')\right)}\right) \tag{22}$$

was introduced in [18], where functions $h, e : (0,\infty) \to (0,\infty)$ are increasing,

$$p_y(x',x'') = \min\left\{|y-x'|, |y-x''|\right\} \tag{23}$$

and m is a natural number.

Theorem 10 (see [14, page 16]). *Let $T : X \to X$ be a bounded linear operator mapping a Banach space X into itself with $\|T\| < 1$ and $I : X \to X$ denote the identity operator. Then $I - T$ has a bounded inverse operator on X which is given by the Neumann series:*

$$(I-T)^{-1} = \sum_{k=0}^{\infty} T^k \tag{24}$$

which satisfies

$$\left\|(I-T)^{-1}\right\| \le \frac{1}{1-\|T\|}. \tag{25}$$

The iterated operators T^n are defined by $T^0 = I$ and $T^n = TT^{n-1}$. The series in the right of (24) is convergent in the norm on $B(X,X)$.

2. The Main Results

2.1. The Spaces $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ and $H_{\alpha,\delta}((a,b),X)$. In this section, we determine essentially the spaces $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ and $H_{\alpha,\delta}((a,b),X)$.

Theorem 11. *Let X be a real or complex Banach space with the norm $\|\cdot\|_X$, a,b,α,δ,γ be real numbers with $-\infty < a$, $b < \infty$, $0 < \alpha$, $\alpha+\gamma < 1$, $0 < \gamma < \delta < 1$, and M be a nonnegative constant. Then the set $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ of all functions*

$$A : \begin{array}{ccc}(a,b)\times(a,b) & \longrightarrow & X, \\ (s,t) & \longrightarrow & A(s,t)\end{array} \tag{26}$$

satisfying the inequalities

$$\|A(s,t)\|_X \le \frac{M}{(t-a)^{\alpha}}, \tag{27}$$

$$\|\Delta A\|_X \le \frac{M\left(|\Delta s|^{\delta} + (\Delta t)^{\gamma}\right)}{(t-a)^{\alpha+\gamma}}, \quad \Delta A = A(s+\Delta s, t+\Delta t) - A(s,t) \tag{28}$$

for all $s, t, s+\Delta s, t+\Delta t \in (a,b)$ with $\Delta t \ge 0$ is a linear space with the usual algebraic operations addition and scalar multiplication defined by

$$(A+B)(s,t) = A(s,t) + B(s,t), \quad (\lambda A)(s,t) = \lambda A(s,t), \quad \lambda \in K, \ (K = \mathbb{R} \text{ or } K = \mathbb{C}), \tag{29}$$

and $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ is a Banach space with the norm

$$\|A\|_{\alpha,\delta,\gamma} = \max\left\{p_A, k_A\right\}, \tag{30}$$

where p_A and k_A are defined by

$$p_A = \sup_{s,t \in (a,b)} (t-a)^{\alpha} \|A(s,t)\|_X, \tag{31}$$

$$k_A = \sup_{\substack{s,t,s+\Delta s,t+\Delta t \in (a,b) \\ \Delta s \ne 0 \vee \Delta t \ne 0}} \frac{(t-a)^{\alpha+\gamma}\|\Delta A\|_X}{|\Delta s|^{\delta} + (\Delta t)^{\gamma}}. \tag{32}$$

Proof. Let X be a Banach space with the norm $\|\cdot\|_X$. It is known that the set $\mathscr{A} = \{A \mid A : (a,b)\times(a,b) \to X\}$ is a linear space with operations in $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$. Also, it is obvious that $A+B, \lambda A \in H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$. On the other hand, since

$$\begin{aligned} &H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X) \\ &= \{A \mid A : (a,b)\times(a,b) \longrightarrow X, A \text{ satisifes (27) and (28)}\} \\ &\subset \mathscr{A}, \end{aligned} \tag{33}$$

$H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ is a linear space.

Furthermore, $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ is a normed space with the norm $\|\cdot\|_{\alpha,\delta,\gamma}$. Indeed, consider the following.

(N1) It is clear that $\|A\|_{\alpha,\delta,\gamma} \ge 0$ for all $A \in H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$.

(N2) If $A \in H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ and $\|A\|_{\alpha,\delta,\gamma}=0$, then $A(s,t)=0$ for all $s,t\in(a,b)$, since

$$\|A\|_{\alpha,\delta,\gamma}=\max\{p_A,k_A\}=0. \tag{34}$$

So $A=0$. On the other hand, if $A=0$, it is found that $\|A\|_{\alpha,\delta,\gamma}=0$ by (30), (31) and (32). Hence, the proposition "$\|A\|_{\alpha,\delta,\gamma}=0$ if and only if $A=0$" is true.

(N3) Let $\lambda\in K$, ($K=\mathbb{R}$ or $K=\mathbb{C}$). Since

$$\begin{aligned} p_{(\lambda A)} &= \sup_{s,t\,\in\,(a,b)} (t-a)^{\alpha}\|(\lambda A)(s,t)\|_X = |\lambda|\,p_A, \\ k_{(\lambda A)} &= \sup_{\substack{s,t,s+\Delta s,t+\Delta t\,\in\,(a,b)\\ \Delta s\neq 0\vee\Delta t\neq 0}} \frac{(t-a)^{\alpha+\gamma}\|\Delta(\lambda A)\|_X}{|\Delta s|^{\delta}+(\Delta t)^{\gamma}} = |\lambda|\,k_A \end{aligned} \tag{35}$$

by (31) and (32),

$$\begin{aligned} \|\lambda A\|_{\alpha,\delta,\gamma} &= \max\{p_{(\lambda A)},k_{(\lambda A)}\} = |\lambda|\max\{p_A,k_A\} \\ &= |\lambda|\,\|A\|_{\alpha,\delta,\gamma}. \end{aligned} \tag{36}$$

(N4) If $A,B\in H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$,

$$\begin{aligned} p_{(A+B)} &= \sup_{s,t\,\in\,(a,b)} (t-a)^{\alpha}\|(A+B)(s,t)\|_X \\ &\leq \sup_{s,t\,\in\,(a,b)} (t-a)^{\alpha}\|A(s,t)\|_X \\ &\quad + \sup_{s,t\,\in\,(a,b)} (t-a)^{\alpha}\|B(s,t)\|_X \\ &= p_A+p_B \end{aligned} \tag{37}$$

by (31) and

$$\begin{aligned} k_{(A+B)} &= \sup_{\substack{s,t,s+\Delta s,t+\Delta t\,\in\,(a,b)\\ \Delta s\neq 0\vee\Delta t\neq 0}} \frac{(t-a)^{\alpha+\gamma}\|\Delta(A+B)\|_X}{|\Delta s|^{\delta}+(\Delta t)^{\gamma}} \\ &\leq \sup_{\substack{s,t,s+\Delta s,t+\Delta t\,\in\,(a,b)\\ \Delta s\neq 0\vee\Delta t\neq 0}} \frac{(t-a)^{\alpha+\gamma}\|\Delta A\|_X}{|\Delta s|^{\delta}+(\Delta t)^{\gamma}} \\ &\quad + \sup_{\substack{s,t,s+\Delta s,t+\Delta t\,\in\,(a,b)\\ \Delta s\neq 0\vee\Delta t\neq 0}} \frac{(t-a)^{\alpha+\gamma}\|\Delta B\|_X}{|\Delta s|^{\delta}+(\Delta t)^{\gamma}} \\ &= k_A+k_B \end{aligned} \tag{38}$$

by (32). Hence,

$$\begin{aligned} \|A+B\|_{\alpha,\delta,\gamma} &= \max\{p_{(A+B)},k_{(A+B)}\} \\ &\leq \max\{p_A+p_B,k_A+k_B\}. \end{aligned} \tag{39}$$

From the inequality

$$\max\{p_A+p_B,k_A+k_B\}\leq\max\{p_A,k_A\}+\max\{p_B,k_B\} \tag{40}$$

which holds for all real numbers $p_A,p_B,k_A,k_B\geq 0$, we obtain $\|A+B\|_{\alpha,\delta,\gamma}\leq\|A\|_{\alpha,\delta,\gamma}+\|B\|_{\alpha,\delta,\gamma}$.

As a result, $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ is a normed space with the norm $\|\cdot\|_{\alpha,\delta,\gamma}$.

The space $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ is a Banach space with respect to $\|\cdot\|_{\alpha,\delta,\gamma}$. To see this we consider an arbitrary Cauchy sequence (A_n) in $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$, and we show that (A_n) converges to a function $A\in H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$. Since (A_n) is Cauchy, for every $\varepsilon>0$, there exists $n_0(\varepsilon)\in\mathbb{N}$ such that

$$\|A_n-A_m\|_{\alpha,\delta,\gamma}<\varepsilon,\quad (m,n>n_0(\varepsilon)). \tag{41}$$

So from (41),

$$\begin{aligned} &\sup_{s,t\,\in\,(a,b)} (t-a)^{\alpha}\|A_n(s,t)-A_m(s,t)\|_X<\varepsilon, \\ &\sup_{\substack{s,t,s+\Delta s,t+\Delta t\,\in\,(a,b)\\ \Delta s\neq 0\vee\Delta t\neq 0}} \frac{(t-a)^{\alpha+\gamma}\|\Delta(A_n-A_m)\|_X}{|\Delta s|^{\delta}+(\Delta t)^{\gamma}}<\varepsilon, \end{aligned} \tag{42}$$

where

$$\begin{aligned} \Delta(A_n-A_m) = {} & (A_n-A_m)(s+\Delta s,t+\Delta t) \\ & -(A_n-A_m)(s,t). \end{aligned} \tag{43}$$

By (42), while $\Delta s\neq 0$ or $\Delta t\neq 0$, we have

$$\|(t-a)^{\alpha}A_n(s,t)-(t-a)^{\alpha}A_m(s,t)\|_X<\varepsilon, \tag{44}$$

$$\frac{(t-a)^{\alpha+\gamma}\|\Delta(A_n-A_m)\|_X}{|\Delta s|^{\delta}+(\Delta t)^{\gamma}}<\varepsilon,\quad (m,n>n_0(\varepsilon)) \tag{45}$$

for all $s,t,s+\Delta s,\ t+\Delta t\ \in\ (a,b)$, $(\Delta t\geq 0)$. We see by (44) that $((t-a)^{\alpha}A_n(s,t))$ is Cauchy in X. Since X is complete, there exists a unique $x\in X$ such that

$$\lim_{n\to\infty}(t-a)^{\alpha}A_n(s,t)=x\quad\text{or}\quad\lim_{n\to\infty}A_n(s,t)=\frac{1}{(t-a)^{\alpha}}x. \tag{46}$$

The limit $x\in X$ depends on the choice of $s,t\in(a,b)$. This defines a function:

$$\begin{aligned} A:\ (a,b)\times(a,b) &\longrightarrow X, \\ (s,t) &\longrightarrow A(s,t), \end{aligned} \tag{47}$$

where

$$\lim_{n\to\infty}A_n(s,t)=A(s,t). \tag{48}$$

Now, we want to show that $\lim_{n\to\infty}\|A_n - A\|_{\alpha,\delta,\gamma} = 0$ and $A \in H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$. By (48), the continuity of the norm gives together with (44) and (45) that

$$\begin{aligned}(t-a)^{\alpha}\|A_n(s,t)-A(s,t)\|_X &= (t-a)^{\alpha}\left\|A_n(s,t)-\lim_{m\to\infty}A_m(s,t)\right\|_X\\ &= \lim_{m\to\infty}(t-a)^{\alpha}\|A_n(s,t)-A_m(s,t)\|_X\\ &< \varepsilon,\end{aligned} \tag{49}$$

$$\begin{aligned}\frac{(t-a)^{\alpha+\gamma}\|\Delta(A_n-A)\|_X}{|\Delta s|^{\delta}+(\Delta t)^{\gamma}} &= \frac{(t-a)^{\alpha+\gamma}\|\Delta A_n-\lim_{m\to\infty}\Delta A_m\|_X}{|\Delta s|^{\delta}+(\Delta t)^{\gamma}}\\ &= \lim_{m\to\infty}\frac{(t-a)^{\alpha+\gamma}\|\Delta(A_n-A_m)\|_X}{|\Delta s|^{\delta}+(\Delta t)^{\gamma}}\\ &< \varepsilon\end{aligned} \tag{50}$$

for all $s,t,s+\Delta s,t+\Delta t \in (a,b)$ with $\Delta t \geq 0$ such that $\Delta s \neq 0$ or $\Delta t \neq 0$. Since $\|A_n - A\|_{\alpha,\delta,\gamma} < \varepsilon$ for all $n > n_0(\varepsilon)$ by (49) and (50), we derive $\lim_{n\to\infty}\|A_n - A\|_{\alpha,\delta,\gamma} = 0$. Furthermore, since (A_n) is bounded, there exists the nonnegative constant C such that $\|A_n\|_{\alpha,\delta,\gamma} \leq C$ which yields

$$\begin{aligned}&\sup_{s,t\,\in\,(a,b)}(t-a)^{\alpha}\|A_n(s,t)\|_X \leq C,\\ &\sup_{\substack{s,t,s+\Delta s,t+\Delta t\,\in\,(a,b)\\ \Delta s\neq 0\vee\Delta t\neq 0}}\frac{(t-a)^{\alpha+\gamma}\|\Delta A_n\|_X}{|\Delta s|^{\delta}+(\Delta t)^{\gamma}} \leq C\end{aligned} \tag{51}$$

for all $n \in \mathbb{N}$. Thus, it is obtained by (51) that

$$\|A_n(s,t)\|_X \leq \frac{C}{(t-a)^{\alpha}}, \tag{52}$$

$$\begin{aligned}&\|A_n(s+\Delta s,t+\Delta t)-A_n(s,t)\|_X\\ &\quad\leq \frac{C\left(|\Delta s|^{\delta}+(\Delta t)^{\gamma}\right)}{(t-a)^{\alpha+\gamma}}\end{aligned} \tag{53}$$

for all $s,t,s+\Delta s,t+\Delta t \in (a,b)$ with $\Delta t \geq 0$ and $n \in \mathbb{N}$. By (52) and (49),

$$\begin{aligned}\|A(s,t)\|_X &= \|A_n(s,t)+A(s,t)-A_n(s,t)\|_X\\ &\leq \|A_n(s,t)\|_X+\|A_n(s,t)-A(s,t)\|_X\\ &\leq \frac{C+\varepsilon}{(t-a)^{\alpha}}.\end{aligned} \tag{54}$$

By (53) and (50),

$$\begin{aligned}\|\Delta A\|_X &= \|\Delta A+\Delta A_n-\Delta A_n\|_X\\ &\leq \|\Delta A_n\|_X+\|\Delta(A_n-A)\|_X\\ &\leq \frac{(C+\varepsilon)\left(|\Delta s|^{\delta}+(\Delta t)^{\gamma}\right)}{(t-a)^{\alpha+\gamma}}\end{aligned} \tag{55}$$

for all $\varepsilon > 0$ and $n > n_0(\varepsilon)$. Therefore, we obtain from (54) and (55) that there exists the nonnegative constant C such that

$$\begin{aligned}\|A(s,t)\|_X &\leq \frac{C}{(t-a)^{\alpha}},\\ \|A(s+\Delta s,t+\Delta t)-A(s,t)\|_X &\leq \frac{C\left(|\Delta s|^{\delta}+(\Delta t)^{\gamma}\right)}{(t-a)^{\alpha+\gamma}}.\end{aligned} \tag{56}$$

Hence, $A \in H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$. This step completes the proof. □

Theorem 12. *The inclusion*

$$C^{0,\delta,\gamma}((a,b)\times(a,b),X) \subset H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X) \tag{57}$$

and the inequality

$$\|A\|_{\alpha,\delta,\gamma} \leq \max\left\{\|A\|_{0,\delta,\gamma}(b-a)^{\alpha}, \|A\|_{0,\delta,\gamma}(b-a)^{\alpha+\gamma}\right\} \tag{58}$$

hold, where $\|A\|_{0,\delta,\gamma}$ is defined by (16).

Proof. If $A \in C^{0,\delta,\gamma}((a,b)\times(a,b),X)$, then by (16) and (17) there exists the constant $\|A\|_{0,\delta,\gamma}$ satisfying the inequalities:

$$\begin{aligned}&\|A(s,t)\|_X \leq \frac{\|A\|_{0,\delta,\gamma}(b-a)^{\alpha}}{(t-a)^{\alpha}},\\ &\|A(s+\Delta s,t+\Delta t)-A(s,t)\|_X\\ &\quad\leq \frac{\|A\|_{0,\delta,\gamma}(b-a)^{\alpha+\gamma}\left(|\Delta s|^{\delta}+(\Delta t)^{\gamma}\right)}{(t-a)^{\alpha+\gamma}}\end{aligned} \tag{59}$$

for all $s,t,s+\Delta s,t+\Delta t \in (a,b)$ with $\Delta t \geq 0$. By taking

$$M_1 = \max\left\{\|A\|_{0,\delta,\gamma}(b-a)^{\alpha}, \|A\|_{0,\delta,\gamma}(b-a)^{\alpha+\gamma}\right\}, \tag{60}$$

it is obtained by (59) that $A \in H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ and $\|A\|_{\alpha,\delta,\gamma} \leq M_1$. That is,

$$C^{0,\delta,\gamma}((a,b)\times(a,b),X) \subset H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X), \tag{61}$$

as desired. □

Example 13. Let us take $\mathbb{R}$ instead of X and define the function A as

$$A : (a,b)\times(a,b) \longrightarrow \mathbb{R}, \quad A(s,t) = s+t. \tag{62}$$

Then, we have

$$\begin{aligned}&\|A(s,t)\|_{\mathbb{R}} = |A(s,t)| \leq 2|b| = M_1,\\ &\|A(s+\Delta s,t+\Delta t)-A(s,t)\|_{\mathbb{R}}\\ &\quad= |A(s+\Delta s,t+\Delta t)-A(s,t)|\\ &\quad\leq |\Delta s|+|\Delta t|\\ &\quad= |\Delta s|^{\delta}|\Delta s|^{1-\delta}+|\Delta t|^{\gamma}|\Delta t|^{1-\gamma}\\ &\quad\leq \max\left\{(b-a)^{1-\delta},(b-a)^{1-\gamma}\right\}\left(|\Delta s|^{\delta}+|\Delta t|^{\gamma}\right)\\ &\quad= M_2\left(|\Delta s|^{\delta}+|\Delta t|^{\gamma}\right)\end{aligned} \tag{63}$$

for all $s, t, s+\Delta s, t+\Delta t \in (a,b)$. Thus, we obtain by (63) that $A \in C^{0,\delta,\gamma}((a,b)\times(a,b), X)$ and $\|A\|_{0,\delta,\gamma} \le \max\{M_1, M_2\} = M$. From Theorem 12, we conclude that $A \in H_{\alpha,\delta,\gamma}((a,b)\times(a,b), X)$ and

$$\|A\|_{\alpha,\delta,\gamma} \le \max\left\{M(b-a)^{\alpha}, M(b-a)^{\alpha+\gamma}\right\}. \tag{64}$$

Theorem 14. *The function $A \in H_{\alpha,\delta,\gamma}((a,b)\times(a,b), X)$ is continuous with respect to the Euclidean metric on $(a,b)\times(a,b) \subset \mathbb{R}^2$.*

Proof. Let $(s_0, t_0) \in (a,b)\times(a,b)$ and d be usual metric on $\mathbb{R}^2$. Then, we wish to show that the function $A : ((a,b)\times(a,b), d) \to (X, \|\cdot\|_X)$ is continuous. It is clear that

$$\begin{aligned}&\lim_{(s,t)\to(s_0,t_0)} d((s,t),(s_0,t_0)) \\ &\quad = \lim_{(s,t)\to(s_0,t_0)} \sqrt{(s-s_0)^2+(t-t_0)^2} = 0.\end{aligned} \tag{65}$$

Besides, since

$$\begin{aligned}0 &\le |s-s_0| \le d((s,t),(s_0,t_0)), \\ 0 &\le |t-t_0| \le d((s,t),(s_0,t_0)),\end{aligned} \tag{66}$$

we have by (65) that $s \to s_0$ and $t \to t_0$. Thus, since the equalities:

$$\begin{aligned}&\|A(s,t)-A(s_0,t_0)\|_X \\ &\quad = \|A(s_0+s-s_0, t_0+t-t_0)-A(s_0,t_0)\|_X,\end{aligned} \tag{67}$$

$$\begin{aligned}&\|A(s,t)-A(s_0,t_0)\|_X \\ &\quad = \|A(s_0,t_0)-A(s,t)\|_X \\ &\quad = \|A(s+s_0-s, t+t_0-t)-A(s,t)\|_X\end{aligned} \tag{68}$$

hold, there exists the nonnegative constant M by (28) that the inequalities:

$$\begin{aligned}0 &\le \|A(s,t)-A(s_0,t_0)\|_X \\ &\le \frac{M\left(|s-s_0|^{\delta}+(t-t_0)^{\gamma}\right)}{(t_0-a)^{\alpha+\gamma}}, \quad (t \ge t_0) \\ 0 &\le \|A(s,t)-A(s_0,t_0)\|_X \le \frac{M\left(|s_0-s|^{\delta}+(t_0-t)^{\gamma}\right)}{(t-a)^{\alpha+\gamma}}, \\ &\qquad (t < t_0)\end{aligned} \tag{69}$$

hold from (67) and (68), respectively. By (69), we have

$$\lim_{(s,t)\to(s_0,t_0)} \|A(s,t)-A(s_0,t_0)\|_X = 0. \tag{70}$$

Hence, the function $A \in H_{\alpha,\delta,\gamma}((a,b)\times(a,b), X)$ is continuous at the arbitrary point $(s_0,t_0) \in (a,b)\times(a,b)$ which means that it is continuous on $(a,b)\times(a,b)$. □

Theorem 15. *Let X be a real or complex Banach space with the norm $\|\cdot\|_X$, a, b, α, δ be real numbers such that $-\infty < a, b < \infty$, $0 < \alpha, \delta, \alpha+\delta < 1$, and M be a nonnegative constant. The set $H_{\alpha,\delta}((a,b), X)$ of all functions satisfying*

$$\begin{aligned}\|A(s)\|_X &\le \frac{M}{(s-a)^{\alpha}}, \\ \|A(s+\Delta s)-A(s)\|_X &\le \frac{M(\Delta s)^{\delta}}{(s-a)^{\alpha+\delta}}\end{aligned} \tag{71}$$

for all $s, t, s+\Delta s \in (a,b)$ with $\Delta s \ge 0$ is a linear space with the usual algebraic operations addition and scalar multiplication

$$\begin{aligned}(A+B)(s) &= A(s)+B(s), \\ (\lambda A)(s) &= \lambda A(s), \quad \lambda \in K, \ (K=\mathbb{R} \text{ or } K=\mathbb{C}),\end{aligned} \tag{72}$$

and $H_{\alpha,\delta}((a,b), X)$ is a Banach space with the norm:

$$\begin{aligned}\|A\|_{\alpha,\delta} = \max &\left\{ \sup_{s\in(a,b)} (s-a)^{\alpha}\|A(s)\|_X, \right. \\ &\left. \sup_{\substack{s,s+\Delta s \in (a,b)\\ \Delta s \ne 0}} \frac{(s-a)^{\alpha+\delta}\|\Delta A\|_X}{(\Delta s)^{\delta}} \right\},\end{aligned} \tag{73}$$

where $\Delta A = A(s+\Delta s)-A(s)$.

Theorem 16. *The function $A \in H_{\alpha,\delta}((a,b), X)$ is continuous with respect to the Euclidean metric on $(a,b) \subset \mathbb{R}$.*

Theorem 17. *The inclusion*

$$C^{0,\delta}((a,b), X) \subset H_{\alpha,\delta}((a,b), X) \tag{74}$$

and the inequality

$$\|A\|_{\alpha,\delta} \le \max\left\{\|A\|_{0,\delta}(b-a)^{\alpha}, \|A\|_{0,\delta}(b-a)^{\alpha+\delta}\right\} \tag{75}$$

hold, where $\|A\|_{0,\delta}$ is the norm in the space $C^{0,\delta}((a,b), X)$.

Since the proofs of Theorems 15–17 are completely similar to that of Theorems 11–14, we leave them to the reader.

Lemma 18. *Let X be a real or complex Banach algebra with the norm $\|\cdot\|_X$ and define B_n by*

$$B_n(s,t) = A_1(s,t)A_2(s,t)\cdots A_n(s,t); \quad s,t \in (a,b), \tag{76}$$

where

$$\begin{aligned}&A_i \in H_{\alpha_i,\delta_i,\gamma}((a,b)\times(a,b), X), \quad (i=1,2,\ldots,n), \\ &\alpha_1+\alpha_2+\cdots+\alpha_n+\gamma < 1 \quad \text{for } n \ge 1.\end{aligned} \tag{77}$$

Then, if

$$\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\}, \quad (n \geq 1),$$
$$D_1 = 1,$$
$$D_n = \max\left\{(b-a)^{\min\{\delta_1,\delta_2,\ldots,\delta_{n-1}\}-\delta}, (b-a)^{\delta_n-\delta}\right\}, \quad (n \geq 2), \tag{78}$$

$$C_n = D_1 D_2 \cdots D_n, \tag{79}$$

then $B_n \in H_{\alpha_1+\alpha_2+\cdots+\alpha_n,\delta,\gamma}((a,b)\times(a,b),X)$ *and*

$$\|B_n\|_{\alpha_1+\alpha_2+\cdots+\alpha_n,\delta,\gamma} \leq 2^{n-1}C_n\|A_1\|_{\alpha_1,\delta_1,\gamma}\|A_2\|_{\alpha_2,\delta_2,\gamma}\cdots\|A_n\|_{\alpha_n,\delta_n,\gamma}. \tag{80}$$

Proof. We use the induction method. If $A_1 \in H_{\alpha_1,\delta_1,\gamma}((a,b)\times(a,b),X)$, then

$$\delta = \min\{\delta_1\} = \delta_1, \qquad B_1(s,t) = A_1(s,t),$$
$$B_1 \in H_{\alpha_1,\delta_1,\gamma}((a,b)\times(a,b),X),$$
$$\|B_1\|_{\alpha_1,\delta_1,\gamma} = C_1\|A_1\|_{\alpha_1,\delta_1,\gamma}. \tag{81}$$

Thus, Lemma is true for $n = 1$.

Assume that if

$$A_i \in H_{\alpha_i,\delta_i,\gamma}((a,b)\times(a,b),X), \quad (i = 1,2,\ldots,k),$$
$$\delta = \min\{\delta_1,\delta_2,\ldots,\delta_k\}, \tag{82}$$

then $B_k \in H_{\alpha_1+\alpha_2+\cdots+\alpha_k,\delta,\gamma}((a,b)\times(a,b),X)$ and

$$\|B_k\|_{\alpha_1+\alpha_2+\cdots+\alpha_k,\delta,\gamma} \leq 2^{k-1}C_k\|A_1\|_{\alpha_1,\delta_1,\gamma}\|A_2\|_{\alpha_2,\delta_2,\gamma}\cdots\|A_k\|_{\alpha_k,\delta_k,\gamma}. \tag{83}$$

Now, let $A_i \in H_{\alpha_i,\delta_i,\gamma}((a,b)\times(a,b),X)$, $(i = 1,2,\ldots,k+1)$ and $\delta = \min\{\delta_1,\delta_2,\ldots,\delta_{k+1}\}$. Then, we must show that $B_{k+1} \in H_{\alpha_1+\alpha_2+\cdots+\alpha_{k+1},\delta,\gamma}((a,b)\times(a,b),X)$ and

$$\|B_{k+1}\|_{\alpha_1+\alpha_2+\cdots+\alpha_{k+1},\delta,\gamma} \leq 2^{k}C_{k+1}\|A_1\|_{\alpha_1,\delta_1,\gamma}\|A_2\|_{\alpha_2,\delta_2,\gamma}\cdots\|A_{k+1}\|_{\alpha_{k+1},\delta_{k+1},\gamma}. \tag{84}$$

Then,

$$B_{k+1}(s,t) = A_1(s,t)A_2(s,t)\cdots A_k(s,t)A_{k+1}(s,t) = B_k(s,t)A_{k+1}(s,t). \tag{85}$$

□

(1) If $b-a \leq 1$, then we have from $0 \leq |\Delta s| < b-a \leq 1$ that

$$|\Delta s|^{\delta_{k+1}}, |\Delta s|^{\min\{\delta_1,\delta_2,\ldots,\delta_k\}} \leq |\Delta s|^{\delta};$$
$$\delta = \min\{\delta_1,\delta_2,\ldots,\delta_{k+1}\}. \tag{86}$$

Furthermore,

$$\begin{aligned}\|B_{k+1}(s,t)\|_X &= \|B_k(s,t)A_{k+1}(s,t)\|_X \leq \|B_k(s,t)\|_X\|A_{k+1}(s,t)\|_X \\ &\leq \frac{\|B_k\|_{\alpha_1+\alpha_2+\cdots+\alpha_k,\delta,\gamma}\|A_{k+1}\|_{\alpha_{k+1},\delta_{k+1},\gamma}}{(t-a)^{\alpha_1+\alpha_2+\cdots+\alpha_k}(t-a)^{\alpha_{k+1}}} \\ &\leq \frac{2^{k-1}C_k\|A_1\|_{\alpha_1,\delta_1,\gamma}\cdots\|A_{k+1}\|_{\alpha_{k+1},\delta_{k+1},\gamma}}{(t-a)^{\alpha_1+\alpha_2+\cdots+\alpha_{k+1}}}\end{aligned} \tag{87}$$

for all $s,t \in (a,b)$. Since $D_k = 1$ for all $k \in \mathbb{N}$, $C_k = 1$. So, we have

$$\|B_{k+1}(s,t)\|_X \leq \frac{2^kC_{k+1}\|A_1\|_{\alpha_1,\delta_1,\gamma}\cdots\|A_{k+1}\|_{\alpha_{k+1},\delta_{k+1},\gamma}}{(t-a)^{\alpha_1+\alpha_2+\cdots+\alpha_{k+1}}} \tag{88}$$

from (87). Additionally,

$$\begin{aligned}\|\Delta B_{k+1}\|_X &= \|B_k(s+\Delta s,t+\Delta t)A_{k+1}(s+\Delta s,t+\Delta t) \\ &\quad - B_k(s,t)A_{k+1}(s,t)\|_X \\ &= \|(\Delta B_k)A_{k+1}(s+\Delta s,t+\Delta t) + B_k(s,t)\Delta A_{k+1}\|_X \\ &\leq \|\Delta B_k\|_X\|A_{k+1}(s+\Delta s,t+\Delta t)\|_X \\ &\quad + \|B_k(s,t)\|_X\|\Delta A_{k+1}\|_X.\end{aligned} \tag{89}$$

By (89) and (83), we get

$$\begin{aligned}\|\Delta B_{k+1}\|_X &\leq \frac{\|B_k\|_{\alpha_1+\alpha_2+\cdots+\alpha_k,\delta,\gamma}\|A_{k+1}\|_{\alpha_{k+1},\delta_{k+1},\gamma}\left(|\Delta s|^{\delta}+(\Delta t)^{\gamma}\right)}{(t-a)^{\alpha_1+\alpha_2+\cdots+\alpha_k+\gamma}(t+\Delta t-a)^{\alpha_{k+1}}} \\ &\quad + \frac{\|B_k\|_{\alpha_1+\alpha_2+\cdots+\alpha_k,\delta,\gamma}\|A_{k+1}\|_{\alpha_{k+1},\delta_{k+1},\gamma}\left(|\Delta s|^{\delta_{k+1}}+(\Delta t)^{\gamma}\right)}{(t-a)^{\alpha_1+\alpha_2+\cdots+\alpha_k}(t-a)^{\alpha_{k+1}+\gamma}} \\ &\leq \frac{2^{k-1}C_k\|A_1\|_{\alpha_1,\delta_1,\gamma}\cdots\|A_{k+1}\|_{\alpha_{k+1},\delta_{k+1},\gamma}\left(|\Delta s|^{\delta}+(\Delta t)^{\gamma}\right)}{(t-a)^{\alpha_1+\alpha_2+\cdots+\alpha_{k+1}+\gamma}} \\ &\quad + \frac{2^{k-1}C_k\|A_1\|_{\alpha_1,\delta_1,\gamma}\cdots\|A_{k+1}\|_{\alpha_{k+1},\delta_{k+1},\gamma}\left(|\Delta s|^{\delta_{k+1}}+(\Delta t)^{\gamma}\right)}{(t-a)^{\alpha_1+\alpha_2+\cdots+\alpha_{k+1}+\gamma}},\end{aligned} \tag{90}$$

where $\delta = \min\{\delta_1,\delta_2,\ldots,\delta_k\}$. Thus, we conclude that

$$\|\Delta B_{k+1}\|_X \leq \frac{2^kC_{k+1}\|A_1\|_{\alpha_1,\delta_1,\gamma}\cdots\|A_{k+1}\|_{\alpha_{k+1},\delta_{k+1},\gamma}\left(|\Delta s|^{\delta}+(\Delta t)^{\gamma}\right)}{(t-a)^{\alpha_1+\alpha_2+\cdots+\alpha_{k+1}+\gamma}} \tag{91}$$

from (86) and (90), where $\delta = \min\{\delta_1,\delta_2,\ldots,\delta_{k+1}\}$.

(2) If $b - a > 1$, then $1 < D_{k+1}$, and since $C_{k+1} = C_k D_{k+1}$ for all $k \in \mathbb{N}$, we have $1 \le C_k < C_{k+1}$. That is, (88) is obtained from (87). Besides,

(i) if $0 \le |\Delta s| < 1$, then (91) is derived from (86) and (90).

(ii) If $1 \le |\Delta s| < b - a$, since $0 < |\Delta s|/(b-a) < 1$, we get

$$|\Delta s|^{\delta_{k+1}} \le (b-a)^{\delta_{k+1}-\delta} |\Delta s|^{\delta},$$
$$|\Delta s|^{\min\{\delta_1,\delta_2,\ldots,\delta_k\}} \le (b-a)^{\min\{\delta_1,\delta_2,\ldots,\delta_k\}-\delta} |\Delta s|^{\delta}, \tag{92}$$

by taking $|\Delta s|/(b-a)$ instead of $|\Delta s|$ in (86). Let

$$D_{k+1} = \max\left\{(b-a)^{\min\{\delta_1,\delta_2,\ldots,\delta_k\}-\delta}, (b-a)^{\delta_{k+1}-\delta}\right\}. \tag{93}$$

Since $1 < D_{k+1}$, (91) is derived from (90). By (1) and (2), we show that (88) and (91) hold. Therefore, $B_{k+1} \in H_{\alpha_1+\alpha_2+\cdots+\alpha_{k+1},\delta,\gamma}((a,b)\times(a,b),X)$ and

$$\|B_{k+1}\|_{\alpha_1+\alpha_2+\cdots+\alpha_{k+1},\delta,\gamma} \le 2^k C_{k+1} \|A_1\|_{\alpha_1,\delta_1,\gamma} \|A_2\|_{\alpha_2,\delta_2,\gamma} \cdots \|A_{k+1}\|_{\alpha_{k+1},\delta_{k+1},\gamma}. \tag{94}$$

So, Lemma 18 holds for all $n \in \mathbb{N}$.

2.2. Some Bounded Linear Integral Operators on the Spaces $H_{\alpha,\delta,\gamma}((a,b)\times(a,b), X)$ and $H_{\alpha,\delta}((a,b)\times(a,b), X)$. Hereafter, by C_n, we mean the constant defined by (79), and, by the integral of the Banach valued functions, we mean the Bochner integral unless stated otherwise.

Theorem 19. *Let X be a real or complex Banach algebra with the norm $\|\cdot\|_X$ and $A \in H_{\alpha_1,\delta_1,\gamma_1}((a,b)\times(a,b),X)$.*

(i) Then, the integral operator

$$T_1 : C^{0,\alpha_2}((a,b),X) \longrightarrow C^{0,\delta_1}((a,b),X) \tag{95}$$

defined by

$$(T_1 f)(s) = \int_a^b A(s,t) f(t)\,dt; \quad f \in C^{0,\alpha_2}((a,b),X), \quad s \in (a,b) \tag{96}$$

is bounded with

$$\|T_1\| \le M(\alpha_1,\gamma_1)\|A\|_{\alpha_1,\delta_1,\gamma_1}, \tag{97}$$

that is, $T_1 \in B(C^{0,\alpha_2}((a,b),X), C^{0,\delta_1}((a,b),X))$.

(ii) The operator

$$T_2 : H_{\alpha_2,\delta_2}((a,b),X) \longrightarrow H_{\alpha_1,\gamma_1}((a,b),X) \tag{98}$$

defined by

$$(T_2 f)(s) = \int_a^b A(t,s) f(t)\,dt; \quad f \in H_{\alpha_2,\delta_2}((a,b),X), \quad s \in (a,b) \tag{99}$$

is bounded with

$$\|T_2\| \le M(\alpha_2)\|A\|_{\alpha_1,\delta_1,\gamma_1}, \tag{100}$$

that is, $T_2 \in B(H_{\alpha_2,\delta_2}((a,b),X), H_{\alpha_1,\gamma_1}((a,b),X))$.

(iii) Suppose that $\alpha_1 + \alpha_2 + \gamma_1$ and $\alpha_1 + \delta_1 < 1$. Then, the operator

$$T_3 : H_{\alpha_2,\delta_2}((a,b),X) \longrightarrow H_{\alpha_1,\delta_1}((a,b),X) \tag{101}$$

defined by

$$(T_3 f)(s) = \int_a^b A(s,t) f(t)\,dt; \quad f \in H_{\alpha_2,\delta_2}((a,b),X), \quad s \in (a,b) \tag{102}$$

is bounded with

$$\|T_3\| \le M(\alpha_1,\alpha_2,\delta_1,\gamma_1)\|A\|_{\alpha_1,\delta_1,\gamma_1}, \tag{103}$$

that is, $T_3 \in B(H_{\alpha_2,\delta_2}((a,b),X), H_{\alpha_1,\delta_1}((a,b),X))$, where the constants $M(c)$, $M(d,e)$, and $M(x,y,z,t)$ are given by

$$M(c) = \frac{(b-a)^{1-c}}{1-c}, \tag{104}$$

$$M(d,e) = \max\{M(d), M(d+e)\}, \tag{105}$$

$$M(x,y,z,t) = \max\left\{(b-a)^x M(x+y), (b-a)^{x+z} M(x+y+t)\right\} \tag{106}$$

for all the real numbers c, d, e, x, y, z, t such that $c \neq 1$.

Proof. (i) We have

$$\|A(s,t) f(t)\|_X \le \frac{\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{0,\alpha_2}}{(t-a)^{\alpha_1}},$$
$$\|[A(s+\Delta s,t) - A(s,t)] f(t)\|_X \le \frac{\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{0,\alpha_2}(\Delta s)^{\delta_1}}{(t-a)^{\alpha_1+\gamma_1}} \tag{107}$$

for all $A \in H_{\alpha_1,\delta_1,\gamma_1}((a,b)\times(a,b),X)$, $f \in C^{0,\alpha_2}((a,b),X)$ and $s, t, s+\Delta s \in (a,b)$ with $\Delta s \ge 0$. By (107), it is obtained that

$$\|(T_1 f)(s)\|_X \le \|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{0,\alpha_2}\int_a^b \frac{dt}{(t-a)^{\alpha_1}} = M(\alpha_1)\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{0,\alpha_2},$$
$$\|\Delta(T_1 f)\|_X \le \|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{0,\alpha_2}(\Delta s)^{\delta_1}\int_a^b \frac{dt}{(t-a)^{\alpha_1+\gamma_1}} = M(\alpha_1+\gamma_1)\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{0,\alpha_2}(\Delta s)^{\delta_1} \tag{108}$$

such that

$$\Delta(T_1 f) = (T_1 f)(s+\Delta s) - (T_1 f)(s). \tag{109}$$

Thus, it is concluded from (108) that $T_1 f \in C^{0,\delta_1}((a,b),X)$ and

$$\|T_1 f\|_{\delta_1} \le M(\alpha_1,\gamma_1)\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{0,\alpha_2}. \quad (110)$$

Besides, the norm of T_1 holds the inequality

$$\|T_1\| = \sup_{\|f\|_{0,\alpha_2}=1}\|T_1 f\|_{\delta_1} \le M(\alpha_1,\gamma_1)\|A\|_{\alpha_1,\delta_1,\gamma_1} \quad (111)$$

from (110). Also, since T_1 is linear, it is an element of the space $B(C^{0,\alpha_2}((a,b),X), C^{0,\delta_1}((a,b),X))$.

On the other hand:

(ii) The inequalities

$$\|A(t,s)f(t)\|_X \le \|A(t,s)\|_X\|f(t)\|_X \le \frac{\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{\alpha_2,\delta_2}}{(s-a)^{\alpha_1}(t-a)^{\alpha_2}}, \quad (112)$$

$$\begin{aligned}&\|[A(t,s+\Delta s)-A(t,s)]f(t)\|_X\\ &\quad\le \|A(t,s+\Delta s)-A(t,s)\|_X\|f(t)\|_X\\ &\quad\le \frac{\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{\alpha_2,\delta_2}(\Delta s)^{\gamma_1}}{(t-a)^{\alpha_2}(s-a)^{\alpha_1+\gamma_1}}\end{aligned} \quad (113)$$

hold for all $A \in H_{\alpha_1,\delta_1,\gamma_1}((a,b)\times(a,b),X)$, $f \in H_{\alpha_2,\delta_2}((a,b),X)$, $s,t,s+\Delta s \in (a,b)$ with $\Delta s \ge 0$.

So, by (112),

$$\begin{aligned}&\|(T_2 f)(s)\|_X\\ &\quad\le \frac{\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{\alpha_2,\delta_2}}{(s-a)^{\alpha_1}}\int_a^b \frac{dt}{(t-a)^{\alpha_2}}\\ &\quad= \frac{M(\alpha_2)\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{\alpha_2,\delta_2}}{(s-a)^{\alpha_1}}\end{aligned} \quad (114)$$

and by (113),

$$\begin{aligned}&\|\Delta(T_2 f)\|_X\\ &\quad\le \frac{\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{\alpha_2,\delta_2}(\Delta s)^{\gamma_1}}{(s-a)^{\alpha_1+\gamma_1}}\int_a^b \frac{dt}{(t-a)^{\alpha_2}}\\ &\quad= \frac{M(\alpha_2)\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{\alpha_2,\delta_2}(\Delta s)^{\gamma_1}}{(s-a)^{\alpha_1+\gamma_1}},\end{aligned} \quad (115)$$

where $\Delta(T_2 f) = (T_2 f)(s+\Delta s) - (T_2 f)(s)$. By usage of (114) and (115), we get that $T_2 f \in H_{\alpha_1,\gamma_1}((a,b),X)$ and

$$\|T_2 f\|_{\alpha_1,\gamma_1} \le M(\alpha_2)\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{\alpha_2,\delta_2}. \quad (116)$$

Thus, T_2 is bounded with

$$\|T_2\| = \sup_{\|f\|_{\alpha_2,\delta_2}=1}\|T_2 f\|_{\alpha_1,\gamma_1} \le M(\alpha_2)\|A\|_{\alpha_1,\delta_1,\gamma_1} \quad (117)$$

from (116). Furthermore T_2 is linear, and, hence, $T_2 \in B(H_{\alpha_2,\delta_2}((a,b),X), H_{\alpha_1,\gamma_1}((a,b),X))$.

(iii) It is clear that

$$\begin{aligned}&\|A(s,t)f(t)\|_X \le \|A(s,t)\|_X\|f(t)\|_X \le \frac{\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{\alpha_2,\delta_2}}{(t-a)^{\alpha_1+\alpha_2}},\\ &\|[A(s+\Delta s,t)-A(s,t)]f(t)\|_X\\ &\quad\le \|A(s+\Delta s,t)-A(s,t)\|_X\|f(t)\|_X\\ &\quad\le \frac{\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{\alpha_2,\delta_2}(\Delta s)^{\delta_1}}{(t-a)^{\alpha_1+\alpha_2+\gamma_1}}\end{aligned} \quad (118)$$

for all $f \in H_{\alpha_2,\delta_2}((a,b),X)$, $s,t,s+\Delta s \in (a,b)$ with $\Delta s \ge 0$.

By (118),

$$\begin{aligned}&\|(T_3 f)(s)\|_X\\ &\quad\le \int_a^b \|A(s,t)f(t)\|_X dt\\ &\quad\le \|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{\alpha_2,\delta_2}\int_a^b \frac{dt}{(t-a)^{\alpha_1+\alpha_2}}\\ &\quad\le \frac{(b-a)^{\alpha_1}M(\alpha_1+\alpha_2)\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{\alpha_2,\delta_2}}{(s-a)^{\alpha_1}},\end{aligned} \quad (119)$$

and by taking

$$\Delta(T_3 f) = (T_3 f)(s+\Delta s) - (T_3 f)(s), \quad (120)$$

$$\begin{aligned}&\|\Delta(T_3 f)\|_X\\ &\quad\le \int_a^b \|[A(s+\Delta s,t)-A(s,t)]f(t)\|_X dt\\ &\quad\le \|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{\alpha_2,\delta_2}(\Delta s)^{\delta_1}\int_a^b \frac{dt}{(t-a)^{\alpha_1+\alpha_2+\gamma_1}}\\ &\quad\le \frac{(b-a)^{\alpha_1+\delta_1}M(\alpha_1+\alpha_2+\gamma_1)\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{\alpha_2,\delta_2}(\Delta s)^{\delta_1}}{(s-a)^{\alpha_1+\delta_1}}.\end{aligned} \quad (121)$$

Thus, we have by (119) and (121) that $T_3 f \in H_{\alpha_1,\delta_1}((a,b),X)$ and

$$\|T_3 f\|_{\alpha_1,\delta_1} \le M(\alpha_1,\alpha_2,\delta_1,\gamma_1)\|A\|_{\alpha_1,\delta_1,\gamma_1}\|f\|_{\alpha_2,\delta_2}. \quad (122)$$

So, by (122), T_3 is a bounded linear operator and also

$$\|T_3\| = \sup_{\|f\|_{\alpha_2,\delta_2}=1}\|T_3 f\|_{\alpha_1,\delta_1} \le M(\alpha_1,\alpha_2,\delta_1,\gamma_1)\|A\|_{\alpha_1,\delta_1,\gamma_1}. \quad (123)$$

Therefore, $T_3 \in B(H_{\alpha_2,\delta_2}((a,b),X), H_{\alpha_1,\delta_1}((a,b),X))$. □

Hereafter, by $M(c)$, $M(d,e)$, and $M(x,y,z,t)$ for all the real numbers c,d,e,x,y,z,t with $c \ne 1$, we denote the constants given by (104), (105), and (106), respectively.

Theorem 20. *Let X be a real or complex Banach algebra with the norm $\|\cdot\|_X$ and $A \in H_{\alpha_1,\delta_1,\gamma_1}((a,b)\times(a,b),X)$ such that $\gamma_2 < \delta_1$. Then, T given by*

$$(TB)(s_1,s_2) = \int_a^b A(s_1,t)\,B(t,s_2)\,dt; \tag{124}$$

$$B \in H_{\alpha_2,\delta_2,\gamma_2}((a,b)\times(a,b),X), \quad s_1,s_2 \in (a,b)$$

is a linear operator from $H_{\alpha_2,\delta_2,\gamma_2}((a,b)\times(a,b),X)$ into $H_{\alpha_2,\delta_1,\gamma_2}((a,b)\times(a,b),X)$, and, also, T is bounded with

$$\|T\| \le N\|A\|_{\alpha_1,\delta_1,\gamma_1}, \tag{125}$$

where

$$N = \max\{M(\alpha_1), \max\{(b-a)^{\gamma_2} M(\alpha_1+\gamma_1), M(\alpha_1)\}\}. \tag{126}$$

Proof. The function C_{s_1,s_2} given by

$$C_{s_1,s_2}(t) = A(s_1,t)\,B(t,s_2), \quad t \in (a,b) \tag{127}$$

is continuous for each $s_1,s_2 \in (a,b)$. Therefore, the function $C_{s_1,s_2} : (a,b) \to X$ is strongly measurable by Theorem 4, and, hence, the function $\|C_{s_1,s_2}\|_X : (a,b) \to \mathbb{R}$ is Lebesgue measurable from Theorem 5. Since

$$\begin{aligned}\left\|C_{s_1,s_2}(t)\right\|_X &= \|A(s_1,t)\,B(t,s_2)\|_X \\ &\le \frac{\|A\|_{\alpha_1,\delta_1,\gamma_1}\|B\|_{\alpha_2,\delta_2,\gamma_2}}{(t-a)^{\alpha_1}(s_2-a)^{\alpha_2}}; \quad s_1,s_2,t \in (a,b),\end{aligned} \tag{128}$$

Lebesgue integral $\int_a^b \|C_{s_1,s_2}(t)\|_X dt$ exists, and, so, Bochner integral in (124) exists by Theorem 8. Now, by taking $N_1 = \|A\|_{\alpha_1,\delta_1,\gamma_1}\|B\|_{\alpha_2,\delta_2,\gamma_2}$, we get

$$\begin{aligned}\left\|(TB)(s_1,s_2)\right\|_X &\le \int_a^b \left\|C_{s_1,s_2}(t)\right\|_X dt \\ &\le \frac{N_1}{(s_2-a)^{\alpha_2}} \int_a^b \frac{dt}{(t-a)^{\alpha_1}} \\ &= \frac{M(\alpha_1)\,N_1}{(s_2-a)^{\alpha_2}}\end{aligned} \tag{129}$$

from (128). Let define

$$\begin{aligned} D_t(s_1,s_2) &= A(s_1,t)\,B(t,s_2), \\ \Delta D_t &= D_t(s_1+\Delta s_1, s_2+\Delta s_2) - D_t(s_1,s_2),\end{aligned} \tag{130}$$

$$\begin{aligned}\Delta_{1,t}A &= A(s_1+\Delta s_1,t) - A(s_1,t), \\ \Delta_{t,2}B &= B(t,s_2+\Delta s_2) - B(t,s_2),\end{aligned} \tag{131}$$

$$\Delta(TB) = (TB)(s_1+\Delta s_1, s_2+\Delta s_2) - (TB)(s_1,s_2), \tag{132}$$

for all $\Delta s_2 \ge 0$ and $t, s_1, s_2, s_1+\Delta s_1, s_2+\Delta s_2 \in (a,b)$. Since

$$\Delta D_t = \left[(\Delta_{1,t}A)\,B(t,s_2+\Delta s_2) + A(s_1,t)\,\Delta_{t,2}B\right], \tag{133}$$

we have

$$\begin{aligned}\left\|\Delta D_t\right\|_X &\le (b-a)^{\gamma_2} N_1 \frac{\left|\Delta s_1\right|^{\delta_1}}{(t-a)^{\alpha_1+\gamma_1}(s_2-a)^{\alpha_2+\gamma_2}} \\ &\quad + N_1 \frac{(\Delta s_2)^{\gamma_2}}{(t-a)^{\alpha_1}(s_2-a)^{\alpha_2+\gamma_2}}.\end{aligned} \tag{134}$$

So, we obtain $\|\Delta(TB)\|_X \le \int_a^b \|\Delta D_t\|_X dt$ which yields

$$\|\Delta(TB)\|_X \le \frac{N_2\|A\|_{\alpha_1,\delta_1,\gamma_1}\|B\|_{\alpha_2,\delta_2,\gamma_2}\left(\left|\Delta s_1\right|^{\delta_1} + (\Delta s_2)^{\gamma_2}\right)}{(s_2-a)^{\alpha_2+\gamma_2}}, \tag{135}$$

where $N_2 = \max\{(b-a)^{\gamma_2} M(\alpha_1+\gamma_1), M(\alpha_1)\}$. Since $\max\{M(\alpha_1), N_2\} = N$, it is found by (129) and (135) that

$$\begin{aligned}\left\|(TB)(s_1,s_2)\right\|_X &\le \frac{N\|A\|_{\alpha_1,\delta_1,\gamma_1}\|B\|_{\alpha_2,\delta_2,\gamma_2}}{(s_2-a)^{\alpha_2}}, \\ \|\Delta(TB)\|_X &\le \frac{N\|A\|_{\alpha_1,\delta_1,\gamma_1}\|B\|_{\alpha_2,\delta_2,\gamma_2}\left(\left|\Delta s_1\right|^{\delta_1} + (\Delta s_2)^{\gamma_2}\right)}{(s_2-a)^{\alpha_2+\gamma_2}}.\end{aligned} \tag{136}$$

Hence, by (136), $TB \in H_{\alpha_2,\delta_1,\gamma_2}((a,b)\times(a,b),X)$ and

$$\|TB\|_{\alpha_2,\delta_1,\gamma_2} \le N\|A\|_{\alpha_1,\delta_1,\gamma_1}\|B\|_{\alpha_2,\delta_2,\gamma_2}. \tag{137}$$

That is, T is bounded by (137) and

$$\|T\| = \sup_{\|B\|_{\alpha_2,\delta_2,\gamma_2}=1} \|TB\|_{\alpha_2,\delta_1,\gamma_2} \le N\|A\|_{\alpha_1,\delta_1,\gamma_1}. \tag{138}$$

Clearly, T is linear. So, $T \in B(H_{\alpha_2,\delta_2,\gamma_2}((a,b),X), H_{\alpha_2,\delta_1,\gamma_2}((a,b)\times(a,b),X))$.

□

2.3. The Solutions of the Linear Fredholm Integral Equations in the Spaces $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ and $H_{\alpha,\delta}((a,b)\times(a,b),X)$. We have the following.

Theorem 21. *Let X be a real or complex Banach algebra and define the function A by*

$$A(s,t) = A_1(s,t)\,A_2(s,t)\cdots A_n(s,t); \quad s,t \in (a,b) \tag{139}$$

such that

$$\begin{aligned} &A_i \in H_{\alpha_i,\delta_i,\gamma}((a,b)\times(a,b), X) \quad \forall i \in \{1,2,\ldots,n\}, \\ &\delta = \min\{\delta_1,\delta_2,\ldots,\delta_n\}, \quad \alpha_1+\alpha_2+\cdots+\alpha_n+\gamma < 1 \\ &\qquad \text{with } n \in \mathbb{N}.\end{aligned} \tag{140}$$

(i) Then, the linear Fredholm integral equation of the form

$$f(s) = \phi(s) + \lambda \int_a^b A(s,t)\,f(t)\,dt \tag{141}$$

has a unique solution f in the space $C^{0,\delta}((a,b),X)$ with

$$\|f\|_{\delta} \leq \frac{1}{1-|\lambda| 2^{n-1} C_n M(\alpha_1+\alpha_2+\cdots+\alpha_n, \gamma)\|A_1\|_{\alpha_1,\delta_1,\gamma}\|A_2\|_{\alpha_2,\delta_2,\gamma}\cdots\|A_n\|_{\alpha_n,\delta_n,\gamma}}\|\phi\|_{\delta}, \quad (142)$$

where $\phi \in C^{0,\delta}((a,b),X)$ and λ is a real or complex parameter satisfying the inequality:

$$|\lambda| < \frac{1}{2^{n-1} C_n M(\alpha_1+\alpha_2+\cdots+\alpha_n, \gamma)\|A_1\|_{\alpha_1,\delta_1,\gamma}\|A_2\|_{\alpha_2,\delta_2,\gamma}\cdots\|A_n\|_{\alpha_n,\delta_n,\gamma}}. \quad (143)$$

(ii) The solution of the equation

$$f(s) = \phi(s) + \lambda \int_a^b A(t,s) f(t)\, dt \quad (144)$$

uniquely exists in the space $H_{\alpha_1+\alpha_2+\cdots+\alpha_n,\gamma}((a,b),X)$. Besides,

$$\|f\|_{\alpha_1+\alpha_2+\cdots+\alpha_n,\gamma} \leq \frac{1}{1-|\lambda| 2^{n-1} C_n L}\|\phi\|_{\alpha_1+\alpha_2+\cdots+\alpha_n,\gamma}, \quad (145)$$

where $\phi \in H_{\alpha_1+\alpha_2+\cdots+\alpha_n,\gamma}((a,b),X)$, λ is a real or complex parameter satisfying the inequality:

$$\begin{aligned}|\lambda| &< \frac{1}{2^{n-1}C_n L}, \\ L &= M(\alpha_1+\alpha_2+\cdots+\alpha_n) \\ &\quad \times \|A_1\|_{\alpha_1,\delta_1,\gamma}\|A_2\|_{\alpha_2,\delta_2,\gamma}\cdots\|A_n\|_{\alpha_n,\delta_n,\gamma}.\end{aligned} \quad (146)$$

(iii) Suppose that

$$2(\alpha_1+\alpha_2+\cdots+\alpha_n)+\gamma \text{ and } \alpha_1+\alpha_2+\cdots+\alpha_n+\delta < 1. \quad (147)$$

Then, (141) has a unique solution f in the space $H_{\alpha_1+\alpha_2+\cdots+\alpha_n,\delta}((a,b),X)$ such that

$$\|f\|_{\alpha_1+\alpha_2+\cdots+\alpha_n,\delta} \leq \frac{1}{1-|\lambda| 2^{n-1} C_n K}\|\phi\|_{\alpha_1+\alpha_2+\cdots+\alpha_n,\delta}, \quad (148)$$

where $\phi \in H_{\alpha_1+\alpha_2+\cdots+\alpha_n,\delta}((a,b),X)$,

$$\begin{aligned}K &= M(\alpha_1+\alpha_2+\cdots+\alpha_n, \alpha_1+\alpha_2+\cdots+\alpha_n, \delta, \gamma) \\ &\quad \times \|A_1\|_{\alpha_1,\delta_1,\gamma}\|A_2\|_{\alpha_2,\delta_2,\gamma}\cdots\|A_n\|_{\alpha_n,\delta_n,\gamma},\end{aligned} \quad (149)$$

and λ is a real or complex parameter satisfying

$$|\lambda| < \frac{1}{2^{n-1}C_n K}. \quad (150)$$

Proof. (ii) Equation (144) may be rewritten as

$$f(s) - \lambda \int_a^b A(t,s) f(t)\, dt = \phi(s) \quad (151)$$

or

$$((I-\lambda T) f)(s) = \phi(s), \quad s \in (a,b) \quad (152)$$

which yields

$$(I-\lambda T) f = \phi, \quad (153)$$

where T is defined by

$$(Tf)(s) = \int_a^b A(t,s) f(t)\, dt, \quad f \in H_{\alpha_1+\alpha_2+\cdots+\alpha_n,\gamma}((a,b),X), \quad (154)$$

and I is an identity operator on $H_{\alpha_1+\alpha_2+\cdots+\alpha_n,\gamma}((a,b),X)$.

By Lemma 18, we have that $A \in H_{\alpha_1+\alpha_2+\cdots+\alpha_n,\delta,\gamma}((a,b)\times(a,b),X)$ and

$$\begin{aligned}&\|A\|_{\alpha_1+\alpha_2+\cdots+\alpha_n,\delta,\gamma} \\ &\quad \leq 2^{n-1}C_n\|A_1\|_{\alpha_1,\delta_1,\gamma}\|A_2\|_{\alpha_2,\delta_2,\gamma}\cdots\|A_n\|_{\alpha_n,\delta_n,\gamma}.\end{aligned} \quad (155)$$

Also, we derive by Theorem 19 that $T \in B(H_{\alpha_1+\alpha_2+\cdots+\alpha_n,\gamma}((a,b),X), H_{\alpha_1+\alpha_2+\cdots+\alpha_n,\gamma}((a,b),X))$ and

$$\|T\| \leq \|A\|_{\alpha_1+\alpha_2+\cdots+\alpha_n,\delta,\gamma} M(\alpha_1+\cdots+\alpha_n) \quad (156)$$

from (117), and so

$$\begin{aligned}\|\lambda T\| \leq\ & |\lambda| 2^{n-1} C_n M(\alpha_1+\cdots+\alpha_n) \\ & \times \|A_1\|_{\alpha_1,\delta_1,\gamma}\|A_2\|_{\alpha_2,\delta_2,\gamma}\cdots\|A_n\|_{\alpha_n,\delta_n,\gamma}.\end{aligned} \quad (157)$$

Since $\|\lambda T\| < 1$ by (146), the inverse operator

$$\begin{aligned}(I-\lambda T)^{-1} &: H_{\alpha_1+\alpha_2+\cdots+\alpha_n,\gamma}((a,b),X) \\ &\longrightarrow H_{\alpha_1+\alpha_2+\cdots+\alpha_n,\gamma}((a,b),X)\end{aligned} \quad (158)$$

exists and the norm of $I-\lambda T$ satisfies the inequality:

$$\left\|(I-\lambda T)^{-1}\right\| \leq \frac{1}{1-|\lambda|\, 2^{n-1} C_n M\left(\alpha_1+\alpha_2+\cdots+\alpha_n\right)\left\|A_1\right\|_{\alpha_1,\delta_1,\gamma}\left\|A_2\right\|_{\alpha_2,\delta_2,\gamma}\cdots\left\|A_n\right\|_{\alpha_n,\delta_n,\gamma}} \tag{159}$$

from Theorem 10. So, (144) has a unique solution f in the space $H_{\alpha_1+\alpha_2+\cdots+\alpha_n,\gamma}((a,b),X)$ such that

$$\begin{aligned} f &= (I-\lambda T)^{-1}\phi, \\ \|f\|_{\alpha_1+\alpha_2+\cdots+\alpha_n,\gamma} &= \left\|(I-\lambda T)^{-1}\phi\right\|_{\alpha_1+\alpha_2+\cdots+\alpha_n,\gamma} \\ &\leq \left\|(I-\lambda T)^{-1}\right\| \|\phi\|_{\alpha_1+\alpha_2+\cdots+\alpha_n,\gamma}, \end{aligned} \tag{160}$$

and this also completes the proof of the second part.

The proof of the first and third parts of Theorem can be completed by the similar way to that of the second part, using Lemma 18, Theorem 19, (111), Lemma 18, Theorem 19, and (123), respectively. □

Theorem 22. *Let X be a real or complex Banach algebra. Then Fredholm integral equation of the form*

$$f(s_1,s_2) = \phi(s_1,s_2) + \lambda \int_a^b A(s_1,t)\, f(t,s_2)\, dt \tag{161}$$

has a unique solution f in $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ with

$$\|f\|_{\alpha,\delta,\gamma} \leq \frac{1}{1-|\lambda|\, N\|A\|_{\alpha,\delta,\gamma}} \|\phi\|_{\alpha,\delta,\gamma}, \tag{162}$$

where $A,\phi \in H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$, N is the constant given in Theorem 20, that is,

$$N = \max\left\{M(\alpha), \max\left\{(b-a)^{\gamma} M(\alpha+\gamma), M(\alpha)\right\}\right\}, \tag{163}$$

and λ is a real or complex parameter fulfilling

$$|\lambda| < \frac{1}{N\|A\|_{\alpha,\delta,\gamma}}. \tag{164}$$

Proof. Equation (161) can be expressed as

$$f(s_1,s_2) - \lambda \int_a^b A(s_1,t)\, f(t,s_2)\, dt = \phi(s_1,s_2) \tag{165}$$

or

$$((I-\lambda T) f)(s_1,s_2) = \phi(s_1,s_2); \quad s_1,s_2 \in (a,b) \tag{166}$$

which implies

$$(I-\lambda T) f = \phi, \tag{167}$$

where T is defined by

$$\begin{aligned} (Tf)(s_1,s_2) &= \int_a^b A(s_1,t)\, f(t,s_2)\, dt, \\ f &\in H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X), \end{aligned} \tag{168}$$

and I is an identity operator on $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$.

By Theorem 20, we have that T is in the space $B(H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X), H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X))$, and

$$\|\lambda T\| \leq |\lambda|\, N\|A\|_{\alpha,\delta,\gamma}. \tag{169}$$

Since $\|\lambda T\| < 1$ by (164), the inverse operator

$$\begin{aligned} (I-\lambda T)^{-1} : H_{\alpha,\delta,\gamma}&((a,b)\times(a,b),X) \\ &\longrightarrow H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X) \end{aligned} \tag{170}$$

exists and the norm of $I-\lambda T$ satisfies the inequality:

$$\begin{aligned} \left\|(I-\lambda T)^{-1}\right\| &\leq \frac{1}{1-\|\lambda T\|} \\ &\leq \frac{1}{1-|\lambda|\, N\|A\|_{\alpha,\delta,\gamma}} \end{aligned} \tag{171}$$

from Theorem 10. So, (161) has a unique solution f in the space $H_{\alpha,\delta,\gamma}((a,b)\times(a,b),X)$ such that

$$\begin{aligned} f &= (I-\lambda T)^{-1}\phi, \\ \|f\|_{\alpha,\delta,\gamma} &= \left\|(I-\lambda T)^{-1}\phi\right\|_{\alpha,\delta,\gamma} \\ &\leq \left\|(I-\lambda T)^{-1}\right\| \|\phi\|_{\alpha,\delta,\gamma} \end{aligned} \tag{172}$$

which completes the proof. □

2.4. The Bounded Singular Integral Operators on the Space $H_{\alpha,\delta}((a,b),X)$.

Lemma 23. *Let $A \in H_{\alpha,\delta}((a,b),X)$, $s \in I_p = (a, a+p(b-a)/(p+2))$, and $0 \leq \Delta s < (s-a)/p$ with $p \in (1,\infty)$. Then the improper integral given by*

$$\Gamma(A)(s) = p.v. \int_a^b \frac{A(t)}{t-s} dt, \quad s \in I_p \tag{173}$$

exists, and there exists the nonnegative constant $M(\alpha,\delta,p)$ depending only on α, δ, and p such that

$$\begin{aligned} \|\Gamma(A)(s)\|_X &\leq \frac{M(\alpha,\delta,p)\,\|A\|_{\alpha,\delta}}{(s-a)^{\alpha}}, \\ \|\Gamma(A)(s+\Delta s) - \Gamma(A)(s)\|_X &\leq \frac{M(\alpha,\delta,p)\,\|A\|_{\alpha,\delta}(\Delta s)^{\delta}}{(s-a)^{\alpha+\delta}} \end{aligned} \tag{174}$$

hold for all $s, s+\Delta s \in (a,b)$, where $\|A\|_{\alpha,\delta}$ is defined by (73).

Hereafter, by I_p, we mean the interval $I_p = (a, a+p(b-a)/(p+2))$ with $1 < p < \infty$.

Proof. We can write

$$\Gamma(A)(s) = \int_a^{s-((s-a)/p)} h_s(t)A(t)\,dt + \int_{s-((s-a)/p)}^{s+((s-a)/p)} h_s(t)A(t)\,dt + \int_{s+((s-a)/p)}^{b} h_s(t)A(t)\,dt \quad (175)$$

for all $s \in I_p$. Here, h_x is defined for each $x \in (a,b)$ by $h_x(t) = 1/(t-x)$, $t \in (a,b)$ and $t \neq x$. Since the functions $A \in H_{\alpha,\delta}((a,b),X)$ and h_x are continuous for each $x \in (a,b)$, the function B_x defined by $B_x(t) = h_x(t)A(t)$ is continuous, and, so, it is strongly measurable by Theorem 4, where $t \in (a,b)$ and $t \neq x$.

Since

$$\|h_s(t)A(t)\|_X \le |h_s(t)|\,\|A(t)\|_X \le \frac{|h_s(t)|\,\|A\|_{\alpha,\delta}}{(t-a)^\alpha} \quad (176)$$

and $s-t > (s-a)/p > 0$ for all $t \in (a, s-(s-a)/p)$, we get $1/(s-t) < p/(s-a)$. So,

$$\int_a^{s-((s-a)/p)} \frac{dt}{(t-a)^\alpha|t-s|} \le \frac{p}{s-a}\int_a^{s-((s-a)/p)}\frac{dt}{(t-a)^\alpha} = \frac{p}{s-a}\int_0^{(p-1)(s-a)/p} u^{-\alpha}du = \frac{p^\alpha(p-1)^{1-\alpha}}{(1-\alpha)(s-a)^\alpha}. \quad (177)$$

By (176),

$$\left\|\int_a^{s-((s-a)/p)} h_s(t)A(t)\,dt\right\|_X \le \int_a^{s-((s-a)/p)}\|h_s(t)A(t)\|_X dt \le \frac{p^\alpha(p-1)^{1-\alpha}\|A\|_{\alpha,\delta}}{(1-\alpha)(s-a)^\alpha}. \quad (178)$$

On the other hand, since $(s-a)/p > 0$, we get $\int_{s-(s-a)/p}^{s+(s-a)/p} dt/(t-s) = 0$. Thus,

$$\int_{s-((s-a)/p)}^{s+((s-a)/p)} h_s(t)A(t)\,dt = \int_{s-((s-a)/p)}^{s+((s-a)/p)} h_s(t)[A(t)-A(s)]\,dt = \int_{s-((s-a)/p)}^{s} h_s(t)[A(t)-A(s)]\,dt + \int_{s}^{s+((s-a)/p)} h_s(t)[A(t)-A(s)]\,dt. \quad (179)$$

Since $t \in (s-(s-a)/p, s)$, we obtain $s-t>0$, $t-a > (p-1)(s-a)/p > 0$ and $1/(t-a) < p/(p-1)(s-a)$. So, we have by (28) that

$$|h_s(t)|\,\|A(t)-A(s)\|_X = |h_s(t)|\,\|A(s)-A(t)\|_X = -h_s(t)\|A(t+s-t)-A(t)\|_X \le \frac{\|A\|_{\alpha,\delta}(s-t)^{\delta-1}}{(t-a)^{\alpha+\delta}} \le \frac{p^{\alpha+\delta}\|A\|_{\alpha,\delta}(s-t)^{\delta-1}}{(p-1)^{\alpha+\delta}(s-a)^{\alpha+\delta}}. \quad (180)$$

By (180) and

$$\int_{s-((s-a)/p)}^{s}(s-t)^{\delta-1}dt = \int_0^{(s-a)/p}u^{\delta-1}du = \frac{(s-a)^\delta}{\delta p^\delta}, \quad (181)$$

we have

$$\int_{s-((s-a)/p)}^{s}|h_s(t)|\,\|A(t)-A(s)\|_X dt \le \frac{p^\alpha\|A\|_{\alpha,\delta}}{(p-1)^{\alpha+\delta}\delta(s-a)^\alpha}. \quad (182)$$

Similarly, since $t-s>0$ for all $t \in (s, s+(s-a)/p)$, we get

$$|h_s(t)|\,\|A(t)-A(s)\|_X = h_s(t)\|A(s+t-s)-A(s)\|_X \le \frac{\|A\|_{\alpha,\delta}(t-s)^{\delta-1}}{(s-a)^{\alpha+\delta}} \quad (183)$$

from (28). Also, by

$$\int_s^{s+((s-a)/p)}(t-s)^{\delta-1}dt = \int_0^{(s-a)/p}u^{\delta-1}du = \frac{(s-a)^\delta}{\delta p^\delta} \quad (184)$$

and (183), we derive

$$\int_s^{s+((s-a)/p)}|h_s(t)|\,\|A(t)-A(s)\|_X dt \le \frac{\|A\|_{\alpha,\delta}}{\delta p^\delta(s-a)^\alpha}. \quad (185)$$

Thus, by taking $I = \int_{s-(s-a)/p}^{s+(s-a)/p} h_s(t)A(t)dt$,

$$\|I\| \le \int_{s-((s-a)/p)}^{s}|h_s(t)|\,\|A(t)-A(s)\|_X dt + \int_s^{s+((s-a)/p)}|h_s(t)|\,\|A(t)-A(s)\|_X dt \le \left(\frac{p^\alpha}{(p-1)^{\alpha+\delta}\delta}+\frac{1}{\delta p^\delta}\right)\frac{\|A\|_{\alpha,\delta}}{(s-a)^\alpha} \quad (186)$$

from (182) and (185).

From the inequality $t-a > s+((s-a)/p)-a = (p+1)(s-a)/p$ which is satisfied for all $t \in (s+(s-a)/p, b)$, we get

$t-s = t-a-(s-a) > t-a-p(t-a)/(p+1) = (t-a)/(p+1) > 0$, and, thus, $1/(t-s) < (p+1)/(t-a)$. Hence,

$$\left\|h_s(t)A(t)\right\|_X \leq \frac{\|A\|_{\alpha,\delta}}{(t-a)^{\alpha}|t-s|} \leq \frac{(p+1)\|A\|_{\alpha,\delta}}{(t-a)^{\alpha+1}}. \quad (187)$$

From

$$\int_{s+((s-a)/p)}^{b} \frac{dt}{(t-a)^{\alpha+1}} = \frac{p^{\alpha}}{\alpha(p+1)^{\alpha}(s-a)^{\alpha}} - \frac{1}{\alpha(b-a)^{\alpha}} \leq \frac{p^{\alpha}}{\alpha(p+1)^{\alpha}(s-a)^{\alpha}} \quad (188)$$

and (187), it is obtained that

$$\left\|\int_{s+((s-a)/p)}^{b} h_s(t)A(t)\,dt\right\|_X \leq \int_{s+((s-a)/p)}^{b} \left\|h_s(t)A(t)\right\|_X dt \leq \frac{(p+1)p^{\alpha}\|A\|_{\alpha,\delta}}{\alpha(p+1)^{\alpha}(s-a)^{\alpha}}. \quad (189)$$

By (178), (186), and (189), we have

$$\begin{aligned}\|\Gamma(A)(s)\|_X &\leq \left\|\int_a^{s-((s-a)/p)} h_s(t)A(t)\,dt\right\|_X \\ &+ \left\|\int_{s-((s-a)/p)}^{s+((s-a)/p)} h_s(t)A(t)\,\mathrm{d}t\right\|_X \\ &+ \left\|\int_{s+((s-a)/p)}^{b} h_s(t)A(t)\,dt\right\|_X \\ &\leq \frac{M_1(\alpha,\delta,p)\|A\|_{\alpha,\delta}}{(s-a)^{\alpha}}\end{aligned} \quad (190)$$

such that

$$M_1(\alpha,\delta,p) = \frac{p^{\alpha}(p-1)^{1-\alpha}}{1-\alpha} + \frac{p^{\alpha}}{(p-1)^{\alpha+\delta}\delta} + \frac{1}{\delta p^{\delta}} + \frac{(p+1)p^{\alpha}}{\alpha(p+1)^{\alpha}}. \quad (191)$$

Furthermore, since

$$\begin{aligned}&\frac{1}{(t-s-\Delta s)(t-s)} = \frac{1}{\Delta s}\left(\frac{1}{t-s-\Delta s} - \frac{1}{t-s}\right), \\ &0 < \Delta s < \frac{s-a}{p},\end{aligned} \quad (192)$$

we can write

$$\begin{aligned}&\Gamma(A)(s+\Delta s) - \Gamma(A)(s) \\ &= \int_a^b h_{s+\Delta s}(t)A(t)\,dt - \int_a^b h_s(t)A(t)\,dt \\ &= \Delta s\int_a^{s-((s-a)/p)} h_{s+\Delta s}(t)h_s(t)A(t)\,dt \\ &+ \int_{s-((s-a)/p)}^{s+\Delta s+((s-a)/p)} \left[h_{s+\Delta s}(t)A(t) - h_s(t)A(t)\right]dt \\ &+ \Delta s\int_{s+\Delta s+((s-a)/p)}^{b} h_{s+\Delta s}(t)h_s(t)A(t)\,dt.\end{aligned} \quad (193)$$

Thus,

$$\begin{aligned}&\left\|h_{s+\Delta s}(t)h_s(t)A(t)\right\|_X \\ &\leq \frac{\|A\|_{\alpha,\delta}}{(t-a)^{\alpha}|t-s-\Delta s||t-s|} \\ &\leq \frac{p^2\|A\|_{\alpha,\delta}}{(s-a)^2(t-a)^{\alpha}},\end{aligned} \quad (194)$$

since $|t-s-\Delta s| > |t-s| > (s-a)/p > 0$ and $1/|t-s-\Delta s| < 1/|t-s| < p/(s-a)$ for all $t \in (a, s-(s-a)/p)$. Besides, by

$$\begin{aligned}\int_a^{s-((s-a)/p)} \frac{dt}{(t-a)^{\alpha}} &= \int_0^{((p-1)(s-a))/p} u^{-\alpha}du \\ &= \frac{(p-1)^{1-\alpha}}{(1-\alpha)p^{1-\alpha}(s-a)^{\alpha-1}}, \\ \Delta s = (\Delta s)^{\delta}&(\Delta s)^{1-\delta} \leq (\Delta s)^{\delta}(s-a)^{1-\delta}\end{aligned} \quad (195)$$

and (194), we get

$$\begin{aligned}&\left\|\Delta s\int_a^{s-((s-a)/p)} h_{s+\Delta s}(t)h_s(t)A(t)\,dt\right\|_X \\ &\leq \frac{p^{1+\alpha}(p-1)^{1-\alpha}\|A\|_{\alpha,\delta}(\Delta s)^{\delta}}{(1-\alpha)(s-a)^{\alpha+\delta}}.\end{aligned} \quad (196)$$

Since $t-a > t-s > t-s-\Delta s > 0$ for all $t \in (s+\Delta s+(s-a)/p, b)$, the inequality $1/(t-a) < 1/(t-s) < 1/(t-s-\Delta s)$ holds, and, so,

$$\begin{aligned}&\left\|h_{s+\Delta s}(t)h_s(t)A(t)\right\|_X \\ &\leq \frac{\|A\|_{\alpha,\delta}}{(t-a)^{\alpha}(t-s-\Delta s)(t-s)} \\ &\leq \frac{\|A\|_{\alpha,\delta}}{(t-s-\Delta s)^{2+\alpha}}.\end{aligned} \quad (197)$$

By

$$\begin{aligned}\int_{s+\Delta s+((s-a)/p)}^{b}\frac{dt}{(t-s-\Delta s)^{2+\alpha}} &= \int_{(s-a)/p}^{b-s-\Delta s}u^{-2-\alpha}du\\ &= \frac{p^{1+\alpha}}{(1+\alpha)(s-a)^{1+\alpha}}-\frac{1}{(1+\alpha)(b-s-\Delta s)^{1+\alpha}}\\ &\le \frac{p^{1+\alpha}}{(1+\alpha)(s-a)^{1+\alpha}}\end{aligned} \tag{198}$$

and (195), we obtain

$$\begin{aligned}&\left\|\Delta s\int_{s+\Delta s+((s-a)/p)}^{b}h_{s+\Delta s}(t)\,h_s(t)\,A(t)\,dt\right\|_X\\ &\quad\le \frac{p^{1+\alpha}\|A\|_{\alpha,\delta}(\Delta s)^{\delta}}{(1+\alpha)(s-a)^{\alpha+\delta}}\end{aligned} \tag{199}$$

from (197). Then,

$$\begin{aligned}&\int_{s-((s-a)/p)}^{s+\Delta s+((s-a)/p)}\left[h_{s+\Delta s}(t)\,A(t)-h_s(t)\,A(t)\right]dt\\ &\quad=\int_{s-((s-a)/p)}^{s+\Delta s+((s-a)/p)}h_{s+\Delta s}(t)\left[A(t)-A(s+\Delta s)\right]dt\\ &\quad-\int_{s-((s-a)/p)}^{s+\Delta s+((s-a)/p)}h_s(t)\left[A(t)-A(s)\right]dt\\ &\quad+A(s+\Delta s)\int_{s-((s-a)/p)}^{s+\Delta s+((s-a)/p)}h_{s+\Delta s}(t)\,dt\\ &\quad-A(s)\int_{s-((s-a)/p)}^{s+\Delta s+((s-a)/p)}h_s(t)\,dt.\end{aligned} \tag{200}$$

Also, we have

$$\begin{aligned}&\int_{s-((s-a)/p)}^{s+\Delta s+((s-a)/p)}h_{s+\Delta s}(t)\,dt\\ &\quad=\int_{s-((s-a)/p)}^{s+\Delta s-((s-a)/p)}h_{s+\Delta s}(t)\,dt\\ &\quad+\int_{s+\Delta s-((s-a)/p)}^{s+\Delta s+((s-a)/p)}h_{s+\Delta s}(t)\,dt\\ &\quad=\int_{s-((s-a)/p)}^{s+\Delta s-((s-a)/p)}h_{s+\Delta s}(t)\,dt,\end{aligned} \tag{201}$$

since $\int_{s+\Delta s-(s-a)/p}^{s+\Delta s+(s-a)/p}dt/(t-s-\Delta s)=\int_{u-(s-a)/p}^{u+(s-a)/p}dt/(t-u)=0$. Therefore, we get by (195) that

$$\begin{aligned}&\left|\int_{s-((s-a)/p)}^{s+\Delta s+((s-a)/p)}h_{s+\Delta s}(t)\,dt\right|\le\int_{s-((s-a)/p)}^{s+\Delta s-((s-a)/p)}\left|h_{s+\Delta s}(t)\right|dt\\ &\quad=\int_{s-((s-a)/p)}^{s+\Delta s-((s-a)/p)}-h_{s+\Delta s}(t)\,dt\\ &\quad\le\frac{p}{s-a}\int_{s-((s-a)/p)}^{s+\Delta s-((s-a)/p)}dt\le\frac{p\Delta s}{s-a}\le\frac{p(\Delta s)^{\delta}}{(s-a)^{\delta}},\end{aligned} \tag{202}$$

since $s+\Delta s-t>(s-a)/p>0$ which implies that $1/(s+\Delta s-t)<p/(s-a)$ for all $t\in(s-(s-a)/p,s+\Delta s-(s-a)/p)$. So,

$$\begin{aligned}&\left\|A(s+\Delta s)\int_{s-((s-a)/p)}^{s+\Delta s+((s-a)/p)}h_{s+\Delta s}(t)\,dt\right\|_X\\ &\quad\le\frac{p\|A\|_{\alpha,\delta}(\Delta s)^{\delta}}{(s+\Delta s-a)^{\alpha}(s-a)^{\delta}}\le\frac{p\|A\|_{\alpha,\delta}(\Delta s)^{\delta}}{(s-a)^{\alpha+\delta}}.\end{aligned} \tag{203}$$

Since $\int_{s-((s-a)/p)}^{s+((s-a)/p)}dt/(t-s)=0$ and $t-s>(s-a)/p>0$ which yields $1/(t-s)<p/(s-a)$ for all $t\in(s+(s-a)/p,s+\Delta s+(s-a)/p)$, we obtain

$$\begin{aligned}&\int_{s-((s-a)/p)}^{s+\Delta s+((s-a)/p)}h_s(t)\,dt\\ &\quad=\int_{s-((s-a)/p)}^{s+((s-a)/p)}h_s(t)\,dt+\int_{s+((s-a)/p)}^{s+\Delta s+((s-a)/p)}h_s(t)\,dt\\ &\quad=\int_{s+((s-a)/p)}^{s+\Delta s+((s-a)/p)}h_s(t)\,dt\\ &\quad\le\frac{p}{s-a}\int_{s+((s-a)/p)}^{s+\Delta s+((s-a)/p)}dt=\frac{p\Delta s}{s-a}\le\frac{p(\Delta s)^{\delta}}{(s-a)^{\delta}}\end{aligned} \tag{204}$$

from (195). Hence,

$$\begin{aligned}&\left\|-A(s)\int_{s-((s-a)/p)}^{s+\Delta s+((s-a)/p)}h_s(t)\,dt\right\|_X\\ &\quad=\|A(s)\|_X\int_{s-((s-a)/p)}^{s+\Delta s+((s-a)/p)}h_s(t)\,dt\le\frac{p\|A\|_{\alpha,\delta}(\Delta s)^{\delta}}{(s-a)^{\alpha+\delta}}.\end{aligned} \tag{205}$$

By (192),

$$\begin{aligned}&\int_{s-((s-a)/p)}^{s+\Delta s+((s-a)/p)} h_{s+\Delta s}(t)\,[A(t)-A(s+\Delta s)]\,dt\\&\quad-\int_{s-((s-a)/p)}^{s+\Delta s+((s-a)/p)} h_s(t)\,[A(t)-A(s)]\,dt\\&\quad=\Delta s\int_{s-((s-a)/p)}^{s-(\Delta s/2)} h_{s+\Delta s}(t)\,h_s(t)\,[A(t)-A(s)]\,dt\\&\quad-\int_{s-((s-a)/p)}^{s-(\Delta s/2)} h_{s+\Delta s}(t)\,[A(s+\Delta s)-A(s)]\,dt\\&\quad+\int_{s-(\Delta s/2)}^{s+(3\Delta s/2)} h_{s+\Delta s}(t)\,[A(t)-A(s+\Delta s)]\,dt\\&\quad-\int_{s-(\Delta s/2)}^{s+(3\Delta s/2)} h_s(t)\,[A(t)-A(s)]\,dt\\&\quad+\Delta s\int_{s+(3\Delta s/2)}^{s+\Delta s+((s-a)/p)} h_{s+\Delta s}(t)\,h_s(t)\,[A(t)-A(s+\Delta s)]\,dt\\&\quad-\int_{s+(3\Delta s/2)}^{s+\Delta s+((s-a)/p)} h_s(t)\,[A(s+\Delta s)-A(s)]\,dt.\end{aligned} \tag{206}$$

Since $s-t+\Delta s>s-t>0$ and $t-a>(p-1)(s-a)/p>0$ which imply that $1/(s-t+\Delta s)<1/(s-t)$ and $1/(t-a)<p/(p-1)(s-a)$ for all $t\in(s-((s-a)/p),s-(\Delta s/2))$, it is found that

$$\begin{aligned}&\left\|h_{s+\Delta s}(t)\,h_s(t)\,[A(t)-A(s)]\right\|_X\\&\quad=\left\|h_{s+\Delta s}(t)\,h_s(t)\,[A(t+s-t)-A(t)]\right\|_X\\&\quad\le\frac{\|A\|_{\alpha,\delta}(s-t)^{\delta-2}}{(t-a)^{\alpha+\delta}}\le\frac{p^{\alpha+\delta}\|A\|_{\alpha,\delta}(s-t)^{\delta-2}}{(p-1)^{\alpha+\delta}(s-a)^{\alpha+\delta}}.\end{aligned} \tag{207}$$

By

$$\begin{aligned}\int_{s-((s-a)/p)}^{s-(\Delta s/2)}(s-t)^{\delta-2}dt&=\int_{\Delta s/2}^{(s-a)/p}u^{\delta-2}du\\&=\frac{(\Delta s)^{\delta-1}}{(1-\delta)\,2^{\delta-1}}-\frac{(s-a)^{\delta-1}}{(1-\delta)\,p^{\delta-1}}\\&\le\frac{(\Delta s)^{\delta-1}}{(1-\delta)\,2^{\delta-1}},\end{aligned} \tag{208}$$

and (207),

$$\begin{aligned}&\left\|\Delta s\int_{s-((s-a)/p)}^{s-(\Delta s/2)} h_{s+\Delta s}(t)\,h_s(t)\,[A(t)-A(s)]\,dt\right\|_X\\&\quad\le\frac{p^{\alpha+\delta}\|A\|_{\alpha,\delta}(\Delta s)^{\delta}}{(p-1)^{\alpha+\delta}\,(1-\delta)\,2^{\delta-1}(s-a)^{\alpha+\delta}}.\end{aligned} \tag{209}$$

Since

$$\begin{aligned}&\int_{s-(\Delta s/2)}^{s+(3\Delta s/2)} h_{s+\Delta s}(t)\,[A(t)-A(s+\Delta s)]\,dt\\&\quad=\int_{s-(\Delta s/2)}^{s+\Delta s} h_{s+\Delta s}(t)\,[A(t)-A(s+\Delta s)]\,dt\\&\quad+\int_{s+\Delta s}^{s+(3\Delta s/2)} h_{s+\Delta s}(t)\,[A(t)-A(s+\Delta s)]\,dt\end{aligned} \tag{210}$$

and $s+\Delta s-t>0$, $t-a>s-a-(\Delta s/2)>s-a-((s-a)/2p)=(2p-1)(s-a)/2p$ which yields $1/(t-a)<2p/(2p-1)(s-a)$ for all $t\in(s-(\Delta s/2),s+\Delta s)$, we get

$$\begin{aligned}&\left\|h_{s+\Delta s}(t)\,[A(t)-A(s+\Delta s)]\right\|_X\\&\quad=h_{s+\Delta s}(t)\,\|A(t+s+\Delta s-t)-A(t)\|_X\\&\quad\le\frac{\|A\|_{\alpha,\delta}(s+\Delta s-t)^{\delta-1}}{(t-a)^{\alpha+\delta}}\\&\quad\le\frac{(2p)^{\alpha+\delta}\|A\|_{\alpha,\delta}(s+\Delta s-t)^{\delta-1}}{(2p-1)^{\alpha+\delta}(s-a)^{\alpha+\delta}}.\end{aligned} \tag{211}$$

By

$$\int_{s-(\Delta s/2)}^{s+\Delta s}(s+\Delta s-t)^{\delta-1}dt=\int_0^{3\Delta s/2}u^{\delta-1}du=\frac{3^{\delta}(\Delta s)^{\delta}}{\delta 2^{\delta}} \tag{212}$$

and (211),

$$\begin{aligned}&\left\|\int_{s-(\Delta s/2)}^{s+\Delta s} h_{s+\Delta s}(t)\,[A(t)-A(s+\Delta s)]\,dt\right\|_X\\&\quad\le\frac{(2p)^{\alpha+\delta}3^{\delta}\|A\|_{\alpha,\delta}(\Delta s)^{\delta}}{(2p-1)^{\alpha+\delta}\delta 2^{\delta}(s-a)^{\alpha+\delta}}.\end{aligned} \tag{213}$$

Since $t-s-\Delta s>0$ for all $t\in(s+\Delta s,s+(3\Delta s/2))$, we derive

$$\begin{aligned}&\left\|h_{s+\Delta s}(t)\,[A(t)-A(s+\Delta s)]\right\|_X\\&\quad=\left\|h_{s+\Delta s}(t)\,[A(s+\Delta s+t-s-\Delta s)-A(s+\Delta s)]\right\|_X\\&\quad\le\frac{\|A\|_{\alpha,\delta}(t-s-\Delta s)^{\delta-1}}{(s+\Delta s-a)^{\alpha+\delta}}\le\frac{\|A\|_{\alpha,\delta}(t-s-\Delta s)^{\delta-1}}{(s-a)^{\alpha+\delta}}.\end{aligned} \tag{214}$$

By

$$\int_{s+\Delta s}^{s+(3\Delta s/2)}(t-s-\Delta s)^{\delta-1}dt=\int_0^{\Delta s/2}u^{\delta-1}du=\frac{(\Delta s)^{\delta}}{\delta 2^{\delta}} \tag{215}$$

and (214),

$$\begin{aligned}&\left\|\int_{s+\Delta s}^{s+(3\Delta s/2)} h_{s+\Delta s}(t)\,[A(t)-A(s+\Delta s)]\,dt\right\|_X\\&\quad\le\frac{\|A\|_{\alpha,\delta}(\Delta s)^{\delta}}{\delta 2^{\delta}(s-a)^{\alpha+\delta}}.\end{aligned} \tag{216}$$

So, we derive

$$\left\| \int_{s-(\Delta s/2)}^{s+(3\Delta s/2)} h_{s+\Delta s}(t)\left[A(t) - A(s+\Delta s)\right] dt \right\|_X \leq \left(\frac{(2p)^{\alpha+\delta} 3^{\delta}}{(2p-1)^{\alpha+\delta}\delta 2^{\delta}} + \frac{1}{\delta 2^{\delta}} \right) \frac{\|A\|_{\alpha,\delta}(\Delta s)^{\delta}}{(s-a)^{\alpha+\delta}} \tag{217}$$

from (213) and (216).

$$\begin{aligned} \int_{s-(\Delta s/2)}^{s+(3\Delta s/2)} h_s(t)\left[A(t) - A(s)\right] dt &= \int_{s-(\Delta s/2)}^{s} h_s(t)\left[A(t) - A(s)\right] dt \\ &+ \int_{s}^{s+(3\Delta s/2)} h_s(t)\left[A(t) - A(s)\right] dt. \end{aligned} \tag{218}$$

Since $t - s < 0$ and $t - a > (2p-1)(s-a)/2p$, which implies $1/(t-a) < 2p/(2p-1)(s-a)$ for all $t \in (s - (\Delta s/2), s)$,

$$\begin{aligned} \left\| h_s(t)\left[A(t) - A(s)\right] \right\|_X &= -h_s(\mathrm{t}) \left\| A(s) - A(t) \right\|_X \\ &= -h_s(t) \left\| A(t+s-t) - A(t) \right\|_X \\ &\leq \frac{\|A\|_{\alpha,\delta}(s-t)^{\delta-1}}{(t-a)^{\alpha+\delta}} \\ &\leq \frac{(2p)^{\alpha+\delta}\|A\|_{\alpha,\delta}(s-t)^{\delta-1}}{(2p-1)^{\alpha+\delta}(s-a)^{\alpha+\delta}}. \end{aligned} \tag{219}$$

Since $t - s > 0$ for all $t \in (s, s + (3\Delta s/2))$,

$$\begin{aligned} \left\| h_s(t)\left[A(t) - A(s)\right] \right\|_X = h_s(t) \left\| A(s+t-s) - A(s) \right\|_X \\ \leq \frac{\|A\|_{\alpha,\delta}(t-s)^{\delta-1}}{(s-a)^{\alpha+\delta}}. \end{aligned} \tag{220}$$

By

$$\begin{aligned} &\int_{s-(\Delta s/2)}^{s} (s-t)^{\delta-1} dt = \int_0^{\Delta s/2} u^{\delta-1} du = \frac{(\Delta s)^{\delta}}{\delta 2^{\delta}}, \\ &\int_{s}^{s+(3\Delta s/2)} (t-s)^{\delta-1} dt = \int_0^{3\Delta s/2} u^{\delta-1} du = \frac{3^{\delta}(\Delta s)^{\delta}}{\delta 2^{\delta}}, \end{aligned} \tag{221}$$

equations (219) and (220), it is obtained that

$$\begin{aligned} &\left\| \int_{s-(\Delta s/2)}^{s} h_s(t)\left[A(t) - A(s)\right] dt \right\|_X \leq \frac{(2p)^{\alpha+\delta}\|A\|_{\alpha,\delta}(\Delta s)^{\delta}}{(2p-1)^{\alpha+\delta}\delta 2^{\delta}(s-a)^{\alpha+\delta}}, \\ &\left\| \int_{s}^{s+(3\Delta s/2)} h_s(t)\left[A(t) - A(s)\right] dt \right\|_X \leq \frac{3^{\delta}\|A\|_{\alpha,\delta}(\Delta s)^{\delta}}{\delta 2^{\delta}(s-a)^{\alpha+\delta}}. \end{aligned} \tag{222}$$

By (222),

$$\left\| \int_{s-(\Delta s/2)}^{s+(3\Delta s/2)} h_s(t)\left[A(t) - A(s)\right] dt \right\|_X \leq \left(\frac{(2p)^{\alpha+\delta}}{(2p-1)^{\alpha+\delta}\delta 2^{\delta}} + \frac{3^{\delta}}{\delta 2^{\delta}} \right) \frac{\|A\|_{\alpha,\delta}(\Delta s)^{\delta}}{(s-a)^{\alpha+\delta}}. \tag{223}$$

Since $t-s > t-s-\Delta s > 0$ and $s+\Delta s-a > s-a > 0$ which yield $1/(t-s) < 1/(t-s-\Delta s)$ and $1/(s+\Delta s-a) < 1/(s-a)$,

$$\begin{aligned} &\left\| h_{s+\Delta s}(t)\, h_s(t)\left[A(t) - A(s+\Delta s)\right] \right\|_X \\ &= \left\| h_{s+\Delta s}(t)\, h_s(t)\left[A(s+\Delta s+t-s-\Delta s) - A(s+\Delta s)\right] \right\|_X \\ &\leq \frac{\|A\|_{\alpha,\delta}(t-s-\Delta s)^{\delta-2}}{(s+\Delta s-a)^{\alpha+\delta}} \leq \frac{\|A\|_{\alpha,\delta}(t-s-\Delta s)^{\delta-2}}{(s-a)^{\alpha+\delta}}. \end{aligned} \tag{224}$$

By

$$\begin{aligned} \int_{s+(3\Delta s/2)}^{s+\Delta s+((s-a)/p)} (t-s-\Delta s)^{\delta-2} dt &= \int_{\Delta s/2}^{(s-a)/p} u^{\delta-2} du \\ &= \frac{(\Delta s)^{\delta-1}}{(1-\delta)\, 2^{\delta-1}} - \frac{(s-a)^{\delta-1}}{(1-\delta)\, p^{\delta-1}} \\ &\leq \frac{(\Delta s)^{\delta-1}}{(1-\delta)\, 2^{\delta-1}} \end{aligned} \tag{225}$$

and (224),

$$\begin{aligned}
&\left\| \Delta s \int_{s+(3\Delta s/2)}^{s+\Delta s+((s-a)/p)} h_{s+\Delta s}(t) h_s(t) [A(t) - A(s+\Delta s)] dt \right\|_X \\
&\quad \leq \frac{\|A\|_{\alpha,\delta} (\Delta s)^\delta}{(1-\delta) 2^{\delta-1} (s-a)^{\alpha+\delta}}, \\
&- \int_{s-((s-a)/p)}^{s-(\Delta s/2)} h_{s+\Delta s}(t) [A(s+\Delta s) - A(s)] dt \\
&\quad - \int_{s+(3\Delta s/2)}^{s+\Delta s+((s-a)/p)} h_s(t) [A(s+\Delta s) - A(s)] dt \\
&\quad = [A(s) - A(s+\Delta s)] \left(\int_{-((s-a)/p)-\Delta s}^{-3\Delta s/2} \frac{du}{u} - \int_{3\Delta s/2}^{\Delta s+((s-a)/p)} \frac{du}{u} \right) \\
&\quad = [A(s) - A(s+\Delta s)] \left(\int_{3\Delta s/2}^{\Delta s+((s-a)/p)} \frac{du}{u} - \int_{3\Delta s/2}^{\Delta s+((s-a)/p)} \frac{du}{u} \right) \\
&\quad = 0.
\end{aligned} \tag{226}$$

Thus, by taking

$$\begin{aligned}
M_2(\alpha,\delta,p) = 2p &+ \frac{p^{\alpha+\delta}}{(p-1)^{\alpha+\delta} (1-\delta) 2^{\delta-1}} \\
&+ \frac{(2p)^{\alpha+\delta} 3^\delta}{(2p-1)^{\alpha+\delta} \delta 2^\delta} + \frac{1}{\delta 2^\delta} \\
&+ \frac{(2p)^{\alpha+\delta}}{(2p-1)^{\alpha+\delta} \delta 2^\delta} + \frac{3^\delta}{\delta 2^\delta} \\
&+ \frac{1}{(1-\delta) 2^{\delta-1}},
\end{aligned} \tag{227}$$

we obtain

$$\begin{aligned}
&\left\| \int_{s-((s-a)/p)}^{s+\Delta s+((s-a)/p)} [h_{s+\Delta s}(t) A(t) - h_s(t) A(t)] dt \right\|_X \\
&\quad \leq \frac{M_2(\alpha,\delta,p) \|A\|_{\alpha,\delta} (\Delta s)^\delta}{(s-a)^{\alpha+\delta}}
\end{aligned} \tag{228}$$

from (203), (205), (209), (217), (223), (226). By (196), (199), and (228), we find

$$\begin{aligned}
&\|\Gamma(A)(s+\Delta s) - \Gamma(A)(s)\|_X \\
&\quad \leq \frac{(M_2(\alpha,\delta,p) + M_3(\alpha,p)) \|A\|_{\alpha,\delta} (\Delta s)^\delta}{(s-a)^{\alpha+\delta}}
\end{aligned} \tag{229}$$

such that

$$M_3(\alpha,p) = \frac{p^{1+\alpha} \left[(1+\alpha)(p-1)^{1-\alpha} + 1 - \alpha \right]}{(1-\alpha)(1+\alpha)}. \tag{230}$$

By (190) and (229),

$$\begin{aligned}
\|\Gamma(A)(s)\|_X &\leq \frac{M(\alpha,\delta,p) \|A\|_{\alpha,\delta}}{(s-a)^\alpha}, \\
\|\Gamma(A)(s+\Delta s) - \Gamma(A)(s)\|_X &\leq \frac{M(\alpha,\delta,p) \|A\|_{\alpha,\delta} (\Delta s)^\delta}{(s-a)^{\alpha+\delta}}
\end{aligned} \tag{231}$$

with

$$M(\alpha,\delta,p) = \max \{ M_1(\alpha,\delta,p), M_2(\alpha,\delta,p) + M_3(\alpha,p) \} \tag{232}$$

which terminates the proof. □

Hereafter, we assume unless stated otherwise that $M(\alpha,\delta,p)$ is defined by (232).

Theorem 24. *The operator T defined by*

$$T(A)(s) = p.v. \int_a^b \frac{A(t)}{t-s} dt; \quad A \in H_{\alpha,\delta}((a,b),X),\ s \in I_p \tag{233}$$

is the singular integral operator in the space $B(H_{\alpha,\delta}((a,b),X), H_{\alpha,\delta}(I_p,X))$, and $\|T\|$ satisfies the inequality $\|T\| \leq 2p^\delta M(\alpha,\delta,p)$.

Proof. If $0 \leq \Delta s < (s-a)/p$, we get by Lemma 23 that

$$\|T(A)(s)\|_X \leq \frac{M(\alpha,\delta,p) \|A\|_{\alpha,\delta}}{(s-a)^\alpha}, \tag{234}$$

$$\|T(A)(s+\Delta s) - T(A)(s)\|_X \leq \frac{M(\alpha,\delta,p) \|A\|_{\alpha,\delta} (\Delta s)^\delta}{(s-a)^{\alpha+\delta}} \tag{235}$$

hold for all $A \in H_{\alpha,\delta}((a,b),X)$, $s, s+\Delta s \in I_p$. Furthermore, if $\Delta s \geq (s-a)/p > 0$, we obtain by (234) and

$$\begin{aligned}
\frac{1}{s+\Delta s-a} &\leq \frac{1}{s-a}, \\
\frac{1}{(\Delta s)^\delta} &\leq \frac{p^\delta}{(s-a)^\delta}
\end{aligned} \tag{236}$$

that

$$\begin{aligned}
\|\Delta T(A)\|_X &\leq \|T(A)(s+\Delta s)\|_X + \|T(A)(s)\|_X \\
&\leq \frac{M(\alpha,\delta,p) \|A\|_{\alpha,\delta}}{(s+\Delta s-a)^\alpha} + \frac{M(\alpha,\delta,p) \|A\|_{\alpha,\delta}}{(s-a)^\alpha} \\
&\leq \frac{2M(\alpha,\delta,p) \|A\|_{\alpha,\delta} (\Delta s)^\delta}{(s-a)^\alpha (\Delta s)^\delta} \\
&\leq \frac{2p^\delta M(\alpha,\delta,p) \|A\|_{\alpha,\delta} (\Delta s)^\delta}{(s-a)^{\alpha+\delta}},
\end{aligned} \tag{237}$$

which holds for all $s, s + \Delta s \in I_p$, where $\Delta T(A) = T(A)(s + \Delta s) - T(A)(s)$. Thus, by (235) and (237),

$$\|T(A)(s+\Delta s) - T(A)(s)\|_X \leq \frac{2p^\delta M(\alpha,\delta,p)\|A\|_{\alpha,\delta}(\Delta s)^\delta}{(s-a)^{\alpha+\delta}} \tag{238}$$

and by (234),

$$\|T(A)(s)\|_X \leq \frac{2p^\delta M(\alpha,\delta,p)\|A\|_{\alpha,\delta}}{(s-a)^\alpha}. \tag{239}$$

Since $T(A) \in H_{\alpha,\delta}(I_p, X)$ and

$$\|T(A)\|_{\alpha,\delta} \leq 2p^\delta M(\alpha,\delta,p)\|A\|_{\alpha,\delta} \tag{240}$$

from (238) and (239), we get

$$\|T\| = \sup_{\|A\|_{\alpha,\delta}=1} \|T(A)\|_{\alpha,\delta} \leq 2p^\delta M(\alpha,\delta,p). \tag{241}$$

Also, T is linear. So, $T \in B(H_{\alpha,\delta}((a,b),X), H_{\alpha,\delta}(I_p,X))$. This also concludes the proof. □

Theorem 25. *The operator T defined by*

$$T(A)(s) = p.v.\int_a^b K(s,t)A(t)\,dt, \qquad A \in H_{\alpha,\delta}((a,b),X),\ s \in (a,b) \tag{242}$$

is the singular integral operator in the space $B(H_{\alpha,\delta}((a,b),X), L_q((a,b),X))$ such that

$$\|T\| \leq 2p^\delta M(\alpha,\delta,p)[M(\alpha q)]^{1/q}, \tag{243}$$

where $1 \leq q < \infty$, $\alpha < 1/q$ and $K : (a,b)\times(a,b) \to \mathbb{R}$ is given by

$$K(s,t) = \begin{cases} \dfrac{1}{t-s}, & s \in I_p,\ t \in (a,b),\ s \neq t \\ 0, & s \notin I_p,\ t \in (a,b). \end{cases} \tag{244}$$

Proof. Equation (242) may be rewritten as

$$T(A)(s) = \begin{cases} p.v.\displaystyle\int_a^b \frac{A(t)}{t-s}dt, & s \in I_p \\ 0, & s \notin I_p. \end{cases} \tag{245}$$

From (239) and (245),

$$\|T(A)(s)\|_X \leq \frac{2p^\delta M(\alpha,\delta,p)\|A\|_{\alpha,\delta}}{(s-a)^\alpha} \tag{246}$$

for all $s \in (a,b)$. Thus, we have by (246) that

$$\begin{aligned}\int_a^b \|T(A)(s)\|_X^q ds &\leq \left(2p^\delta M(\alpha,\delta,p)\|A\|_{\alpha,\delta}\right)^q \int_a^b \frac{ds}{(s-a)^{\alpha q}} \\ &= \left(2p^\delta M(\alpha,\delta,p)\|A\|_{\alpha,\delta}\right)^q M(\alpha q).\end{aligned} \tag{247}$$

So, it is obvious by (247) that the inequality

$$\|T(A)\|_{L_q((a,b),X)} \leq 2p^\delta M(\alpha,\delta,p)\|A\|_{\alpha,\delta}[M(\alpha q)]^{1/q} \tag{248}$$

holds. From (248), we obtain $\|T\| \leq 2p^\delta M(\alpha,\delta,p)[M(\alpha q)]^{1/q}$ and since T is linear, $T \in B(H_{\alpha,\delta}((a,b),X), L_q((a,b),X))$, and this completes the proof of theorem. □

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