Research Article

Multiplicity of Positive Solutions for Semilinear Elliptic Systems

Tsing-San Hsu

Department of Natural Sciences in the Center for General Education, Chang Gung University, Tao-Yuan 333, Taiwan

Correspondence should be addressed to Tsing-San Hsu; tshsu@mail.cgu.edu.tw

We study the effect of the coefficient $h(x)$ of the critical nonlinearity on the number of positive solutions for semilinear elliptic systems. Under suitable assumptions for $f(x)$, $g(x)$, and $h(x)$, we should prove that for sufficiently small $\lambda, \mu > 0$, there are at least $k + 1$ positive solutions of the semilinear elliptic systems $-\Delta u = \lambda f(x)|u|^{q-2}u + (\alpha/(\alpha + \beta))h(x)|u|^{\alpha-2}u|v|^{\beta}$, $-\Delta v = \mu g(x)|v|^{q-2}v + (\beta/(\alpha + \beta))h(x)|u|^{\alpha}|v|^{\beta-2}v$, where $0 \in \Omega \subset \mathbb{R}^N$ is a bounded domain, $\alpha > 1$, $\beta > 1$, and $N/(N - 2) < q < 2 < \alpha + \beta = 2^*$ for $N > 4$.

1. Introduction and Main Results

For $N \geq 3$, $\alpha > 1$, $\beta > 1$, and $1 \leq q < 2 < \alpha + \beta = 2^* = 2N/(N - 2)$, consider the semilinear elliptic systems

$$\begin{cases} -\Delta u = \lambda f(x)|u|^{q-2}u + \dfrac{\alpha}{\alpha + \beta}h(x)|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ -\Delta v = \mu g(x)|v|^{q-2}v + \dfrac{\beta}{\alpha + \beta}h(x)|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{$P_{\lambda,\mu}$}$$

where $\lambda, \mu > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.

Let f, g, and h satisfy the following conditions.

(H1) f, g, and h are positive continuous functions in $\overline{\Omega}$ and $\max_{x\in\overline{\Omega}}h(x) = 1$.

(H2) There exist k points $a_1, a_2, \ldots, a_k \in \Omega$ and some $\sigma \geq N - 2$ such that $h(a_i)$ are strict maxima and satisfy

$$h(a_i) = \max_{x\in\overline{\Omega}}h(x) = 1 \quad \forall 1 \leq i \leq k \tag{1}$$

and $h(x) = h(a_i) + O(|x - a_i|^{\sigma})$ as $x \to a_i$ uniformly in i.

Recent studies [1–10] have investigated the elliptic systems with subcritical or critical exponents and have proved the existence of a ground state solution or the existence of at least two positive solutions for these problems. For the case of

$N > 4, \alpha > 1, \beta > 1$, and $2 < q < \alpha + \beta = 2^* = 2N/(N-2)$, Lin [11] constructs the k compact Palais-Smale sequences that are suitably localized in correspondence of k maximum points of h. Under assumptions $(H1)$-$(H2)$, she has showed that there are at least k positive solutions of the problem $(P_{\lambda,\mu})$ for sufficiently small $\lambda, \mu > 0$. In this paper, we study the problem $(P_{\lambda,\mu})$ and complement the results of [11] to the case $1 \leq q < 2$. Under assumptions $(H1)$-$(H2)$, we should prove that there exist at least $k + 1$ positive solutions of the problem $(P_{\lambda,\mu})$ for sufficiently small $\lambda, \mu > 0$.

Let $E = H_0^1(\Omega) \times H_0^1(\Omega)$ be the space with the standard norm

$$\|(u, v)\|_E = \left(\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)dx\right)^{1/2}. \tag{2}$$

Associated with the problem $(P_{\lambda,\mu})$, we consider the C^1-functional $I_{\lambda,\mu}$, for $(u, v) \in E$,

$$\begin{aligned} I_{\lambda,\mu}(u, v) = {} & \frac{1}{2}\|(u, v)\|_E^2 \\ & - \frac{1}{q}\int_\Omega \left(\lambda f(x)|u|^q + \mu g(x)|v|^q\right)dx \\ & - \frac{1}{2^*}\int_\Omega h(x)|u|^\alpha|v|^\beta dx. \end{aligned} \tag{3}$$

The weak solution $(u, v) \in E$ of the problem $(P_{\lambda,\mu})$ is the critical point of the functional $I_{\lambda,\mu}$; that is, $(u, v) \in E$ satisfies

$$
\int_\Omega \left(\nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2 \right) dx - \lambda \int_\Omega f(x) |u|^{q-2} u \varphi_1 dx
$$
$$
- \mu \int_\Omega g(x) |v|^{q-2} v \varphi_2 dx - \frac{\alpha}{2^*} \int_\Omega h(x) |u|^{\alpha-2} u |v|^\beta \varphi_1 dx
$$
$$
- \frac{\beta}{2^*} \int_\Omega h(x) |u|^\alpha |v|^{\beta-2} v \varphi_2 dx = 0
$$

$$(4)$$

for any $(\varphi_1, \varphi_2) \in E$.

Let $D^{1,2}(\mathbb{R}^N) = \{ u \in L^{2^*}(\mathbb{R}^N) \mid \nabla u \in (L^2(\mathbb{R}^N))^N \}$ with the norm $\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx$, and let S be the best Sobolev constant defined by

$$
S = \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{2/2^*}}
$$
$$
\left(= \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*} dx \right)^{2/2^*}} \right) > 0,
$$

$$(5)$$

and let

$$
S_{\alpha,\beta} = \inf_{u,v \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 \right) dx}{\left(\int_\Omega |u|^\alpha |v|^\beta dx \right)^{2/(\alpha+\beta)}};
$$

$$(6)$$

then, by [1, Theorem 5], we have

$$
S_{\alpha,\beta} = \left(\left(\frac{\alpha}{\beta} \right)^{\beta/(\alpha+\beta)} + \left(\frac{\beta}{\alpha} \right)^{\alpha/(\alpha+\beta)} \right) S,
$$

$$(7)$$

where $\alpha + \beta = 2^*$.

Set

$$
\Lambda_1 = \left(\frac{2-q}{2^*-q} \right)^{2/(2^*-2)} \left(\frac{(2^*-q)\gamma_\infty}{2^*-2} |\Omega|^{(2^*-q)/2^*} \right)^{-2/(2-q)}
$$
$$
\times S^{N/2 + q/(2-q)} > 0,
$$

$$(8)$$

where $\gamma_\infty = \max\{|f|_{L^\infty(\Omega)}, |g|_{L^\infty(\Omega)}\}$.

The main results of this paper are given as follows.

Theorem 1. *Assume that* (H1) *holds. If* $\lambda, \mu > 0$ *satisfy* $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$, *then there exists at least one positive ground state solution of the problem* $(P_{\lambda,\mu})$.

Theorem 2. *Under the assumptions* (H1)-(H2), *and* $N/(N-2) < q < 2$ *and* $N > 4$, *there exists a positive number* $\Lambda^* \in (0, \Lambda_1)$ *such that for* $\lambda, \mu > 0$ *and* $\lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda^*$, *the problem* $(P_{\lambda,\mu})$ *has* $k + 1$ *positive solutions.*

This paper is organized as follows. In Section 2, we consider the Nehari manifold

$$
\mathcal{N}_{\lambda,\mu} = \left\{ (u, v) \in E \setminus \{0\} \mid \left\langle I'_{\lambda,\mu}(u, v), (u, v) \right\rangle = 0 \right\}, \quad (9)
$$

where

$$
\left\langle I'_{\lambda,\mu}(u, v), (u, v) \right\rangle
$$
$$
= \|(u, v)\|_E^2 - \int_\Omega \left(\lambda f(x) |u|^q + \mu g(x) |v|^q \right) dx \quad (10)
$$
$$
- \int_\Omega h(x) |u|^\alpha |v|^\beta dx.
$$

Note that $\mathcal{N}_{\lambda,\mu}$ contains all nontrivial weak solution of the problem $(P_{\lambda,\mu})$. Using the argument of Tarantello [12, 13], we split $\mathcal{N}_{\lambda,\mu}$ into two parts $\mathcal{N}^+_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$ for $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$. In Section 3, we prove Theorem 1. In Section 4, since $I_{\lambda,\mu}$ satisfies the (PS)$_\gamma$-condition for $\gamma \in (-\infty, (1/N)(S_{\alpha,\beta})^{N/2} - C_0(\lambda^{2/(2-q)} + \mu^{2/(2-q)}))$, for sufficiently small λ, μ, and some restriction on q and N, we construct the k compact Palais-Smale sequences which are suitably localized in correspondence with the k maximum points of h and which converge to distinct solutions of the problem $(P_{\lambda,\mu})$ belonging to $\mathcal{N}^-_{\lambda,\mu}$. Hence, we prove Theorem 2 (one is the ground state solution belonging to $\mathcal{N}^+_{\lambda,\mu}$ and the others are in $\mathcal{N}^-_{\lambda,\mu}$).

2. Nehari Manifold

Throughout this paper, (H1) will be assumed. First, we give some notations.

Notations. We make use of the following notations.

$L^p(\Omega), 1 \le p \le \infty$, denote Lebesgue spaces; the norm L^p is denoted by $|\cdot|_{L^p(\Omega)}$ for $1 \le p \le \infty$.

$E = [H_0^1(\Omega)]^2$, endowed with norm $\|z\|_E^2 = \|(u, v)\|_E^2 = |\nabla u|_2^2 + |\nabla v|_2^2$.

The dual space of a Banach space E will be denoted by E^{-1}.

$|z| = |(u, v)| = (|u|, |v|)$ and $tz = t(u, v) = (tu, tv)$ for all $z \in E$ and $t \in \mathbb{R}$.

$z = (u, v)$ is said to be nonnegative in Ω if $u \ge 0$ and $v \ge 0$ in Ω.

$z = (u, v)$ is said to be positive in Ω if $u > 0$ and $v > 0$ in Ω.

$|\Omega|$ is the Lebesgue measure of Ω.

$B_r(a) = \{x \in \mathbb{R}^N \mid |x - a| < r\}$ is a ball in $\mathbb{R}^N$.

$O(\varepsilon^t)$ denotes $|O(\varepsilon^t)|/\varepsilon^t \le C$ as $\varepsilon \to 0$ for $t \ge 0$.

$O_1(\varepsilon^t)$ means that there exist the constants $C_1, C_2 > 0$ such that $C_1 \varepsilon^t \le O_1(\varepsilon^t) \le C_2 \varepsilon^t$ as ε is small.

$o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$.

$\gamma_\infty = \max\{|f|_{L^\infty(\Omega)}, |g|_{L^\infty(\Omega)}\}$.

C, C_i will denote various positive constants, the exact values of which are not important.

Let $K_{\lambda,\mu} : E \to \mathbb{R}$ be the functional defined by

$$K_{\lambda,\mu}(z) = \int_\Omega \left(\lambda f(x) |u|^q + \mu g(x) |v|^q \right) dx \quad \forall z = (u,v) \in E. \tag{11}$$

We know that $I_{\lambda,\mu}$ is not bounded below on E. From the following lemma, we have that $I_{\lambda,\mu}$ is bounded from below on the Nehari manifold $\mathcal{N}_{\lambda,\mu}$ defined in (9).

Lemma 3. *The energy functional $I_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. If $z = (u,v) \in \mathcal{N}_{\lambda,\mu}$, then by (10), the Hölder inequality, and the Sobolev embedding theorem, we get

$$I_{\lambda,\mu}(z) = \frac{2^* - 2}{2^* 2} \|z\|_E^2 - \frac{2^* - q}{2^* q} K_{\lambda,\mu}(z) \tag{12}$$

$$\geq \frac{1}{N} \|z\|_E^2 - \frac{2^* - q}{2^* q} \gamma_\infty S^{-q/2} |\Omega|^{(2^* - q)/2^*}$$

$$\times \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)} \right)^{(2-q)/2} \|z\|_E^q. \tag{13}$$

Hence, we have that $I_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$. $\square$

Define

$$\Phi_{\lambda,\mu}(z) = \left\langle I'_{\lambda,\mu}(z), z \right\rangle. \tag{14}$$

Then, for $z \in \mathcal{N}_{\lambda,\mu}$,

$$\left\langle \Phi'_{\lambda,\mu}(z), z \right\rangle \tag{15}$$

$$= 2\|z\|_E^2 - q K_{\lambda,\mu}(z) - 2^* \int_\Omega h(x) |u|^\alpha v^\beta dx$$

$$= (2 - q) \|z\|_E^2 - (2^* - q) \int_\Omega h(x) |u|^\alpha v^\beta dx \tag{16}$$

$$= (2^* - q) K_{\lambda,\mu}(z) - (2^* - 2) \|z\|_E^2. \tag{17}$$

We apply the method in [12]; let

$$\mathcal{N}_{\lambda,\mu}^+ = \left\{ z \in \mathcal{N}_{\lambda,\mu} \mid \left\langle \Phi'_{\lambda,\mu}(z), z \right\rangle > 0 \right\},$$

$$\mathcal{N}_{\lambda,\mu}^0 = \left\{ z \in \mathcal{N}_{\lambda,\mu} \mid \left\langle \Phi'_{\lambda,\mu}(z), z \right\rangle = 0 \right\}, \tag{18}$$

$$\mathcal{N}_{\lambda,\mu}^- = \left\{ z \in \mathcal{N}_{\lambda,\mu} \mid \left\langle \Phi'_{\lambda,\mu}(z), z \right\rangle < 0 \right\}.$$

By using equality (17), we get that $K_{\lambda,\mu}(z) > 0$ for $z \in \mathcal{N}_{\lambda,\mu}^+$. Moreover, we have the following results.

Lemma 4. *Let Λ_1 be a constant defined as in (8). If $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$, then $\mathcal{N}_{\lambda,\mu}^0 = \emptyset$.*

Proof. Assuming the contrary, there exist $\lambda, \mu > 0$ with $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$ such that $\mathcal{N}_{\lambda,\mu}^0 \neq \emptyset$. Then, by (16) and (17), for $u \in \mathcal{N}_{\lambda,\mu}^0$, we have

$$\|z\|_E^2 = \frac{2^* - q}{2 - q} \int_\Omega h(x) |u|^\alpha |v|^\beta dx = \frac{2^* - q}{2^* - 2} K_{\lambda,\mu}(z). \tag{19}$$

Using ($H1$) and both the Hölder and the Sobolev inequalities, we get

$$\|z\|_E \geq \left(\frac{2 - q}{2^* - q} S^{2^*/2} \right)^{1/(2^* - 2)}, \tag{20}$$

$$\|z\|_E \leq \left(\frac{2^* - q}{2^* - 2} S^{-q/2} |\Omega|^{(2^* - q)/2^*} \gamma_\infty \right)^{1/(2-q)}$$

$$\times \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)} \right)^{1/2}. \tag{21}$$

This implies

$$\lambda^{2/(2-q)} + \mu^{2/(2-q)}$$

$$\geq \left(\frac{2 - q}{2^* - q} \right)^{2/(2^* - 2)} \left(\frac{(2^* - q) \gamma_\infty}{2^* - 2} |\Omega|^{(2^* - q)/2^*} \right)^{-2/(2-q)}$$

$$\times S^{N/2 + q/(2-q)} = \Lambda_1, \tag{22}$$

which is a contradiction. $\square$

For each $z \in E$ with $\int_\Omega h(x) |u|^\alpha |v|^\beta dx > 0$, we write

$$t_{\max} = \left(\frac{(2 - q) \|z\|_E^2}{(2^* - q) \int_\Omega h(x) |u|^\alpha |v|^\beta dx} \right)^{1/(2^* - 2)} > 0. \tag{23}$$

Then, the following lemma holds.

Lemma 5. *Suppose that $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$, and $z \in E$ with $\int_\Omega h(x) |u|^\alpha |v|^\beta dx > 0$. Then, there exist unique $0 < t^+ < t_{\max} < t^-$ such that $t^+ z \in \mathcal{N}_{\lambda,\mu}^+$, $t^- z \in \mathcal{N}_{\lambda,\mu}^-$ and*

$$I_{\lambda,\mu}(t^+ z) = \inf_{0 \leq t \leq t_{\max}} I_{\lambda,\mu}(tz);$$

$$I_{\lambda,\mu}(t^- z) = \sup_{t \geq 0} I_{\lambda,\mu}(tz). \tag{24}$$

Proof. This is similar to the proof of Hsu [14, Lemma 2.7]. $\square$

Applying Lemma 4 ($\mathcal{N}_{\lambda,\mu}^0 = \emptyset$ for $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$), we write $\mathcal{N}_{\lambda,\mu} = \mathcal{N}_{\lambda,\mu}^+ \cup \mathcal{N}_{\lambda,\mu}^-$ and define

$$\theta_{\lambda,\mu} = \inf_{z \in \mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(z);$$

$$\theta_{\lambda,\mu}^+ = \inf_{z \in \mathcal{N}_{\lambda,\mu}^+} I_{\lambda,\mu}(z); \tag{25}$$

$$\theta_{\lambda,\mu}^- = \inf_{z \in \mathcal{N}_{\lambda,\mu}^-} I_{\lambda,\mu}(z).$$

The following lemma shows that the minimizers on $\mathcal{N}_{\lambda,\mu}$ are usual critical points for $I_{\lambda,\mu}$.

Lemma 6. *For the case when $\lambda \in (0, \Lambda_1)$, if z_0 is a local minimizer for $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$, then $I'_{\lambda,\mu}(z_0) = 0$ in E^{-1}.*

Proof. See Brown and Zhang [15, theorem 2.3]. □

Lemma 7. *(i) If $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$ and $z = (u,v) \in \mathcal{N}_{\lambda,\mu}^+$, then one has*

$$K_{\lambda,\mu}(z) > 0, \qquad I_{\lambda,\mu}(z) < 0. \tag{26}$$

In particular, $\theta_{\lambda,\mu} \le \theta_{\lambda,\mu}^+ < 0$.

(ii) If $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < (q/2)^{2/(2-q)}\Lambda_1$ and $z = (u,v) \in \mathcal{N}_{\lambda,\mu}^-$, then one has $u \not\equiv 0$, $v \not\equiv 0$ in Ω,

$$\|z\|_E > \left(\frac{2-q}{2^*-q}\right)^{1/(2^*-2)} S^{N/4}, \tag{27}$$

and $\theta_{\lambda,\mu}^- > d_0$ for some positive constant $d_0 = d_0(\lambda, \mu, q, N, S, \gamma_\infty, |\Omega|)$.

Proof. (i) Let $z = (u,v) \in \mathcal{N}_{\lambda,\mu}^+$. By (16) and (17), we have

$$K_{\lambda,\mu}(z) > \frac{2^*-2}{2^*-q}\|z\|_E^2 > 0,$$
$$\frac{2-q}{2^*-q}\|z\|_E^2 > \int_\Omega h(x)|u|^\alpha|v|^\beta dx. \tag{28}$$

Then,

$$I_{\lambda,\mu}(z) = \left(\frac{1}{2}-\frac{1}{q}\right)\|z\|_E^2 + \left(\frac{1}{q}-\frac{1}{2^*}\right)\int_\Omega h(x)|u|^\alpha|v|^\beta dx$$
$$< \left[\left(\frac{1}{2}-\frac{1}{q}\right) + \left(\frac{1}{q}-\frac{1}{2^*}\right)\frac{2-q}{2^*-q}\right]\|z\|_E^2$$
$$= -\frac{2-q}{qN}\|z\|_E^2 < 0. \tag{29}$$

By the definition of $\theta_{\lambda,\mu}$, $\theta_{\lambda,\mu}^+$, we deduce that $\theta_{\lambda,\mu} \le \theta_{\lambda,\mu}^+ < 0$.

(ii) Let $z \in \mathcal{N}_{\lambda,\mu}^-$; by (16) and the Hölder and the Sobolev inequalities, we get

$$\frac{2-q}{2^*-q}\|z\|_E^2 < \int_\Omega h(x)|u|^\alpha|v|^\beta dx \le S^{-2^*/2}\|z\|_E^{2^*}. \tag{30}$$

This implies

$$\int_\Omega h(x)|u|^\alpha|v|^\beta dx > \frac{2^*-q}{2-q}\|z\|_E$$
$$> \left(\frac{2-q}{2^*-q}\right)^{\frac{1}{2^*-2}} S^{\frac{N}{4}} \quad \forall z \in \mathcal{N}_{\lambda,\mu}^-. \tag{31}$$

By (13) and (31), we obtain that $u \not\equiv 0$, $v \not\equiv 0$ in Ω, and

$$I_{\lambda,\mu}(z)$$
$$\ge \|z\|_E^q\left[\frac{1}{N}\|z\|_E^{2-q} - \left(\frac{2^*-q}{2^*q}\right)\gamma_\infty S^{-q/2}|\Omega|^{(2^*-q)/2^*}\right.$$
$$\left. \times \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right)^{(2-q)/2}\right]$$
$$> \left(\frac{2-q}{2^*-q}\right)^{q/(2^*-2)} S^{qN/4}$$
$$\times \left[\frac{1}{N}\left(\frac{2-q}{2^*-q}\right)^{(2-q)/(2^*-2)} S^{(2-q)N/4}\right.$$
$$- \left(\frac{2^*-q}{2^*q}\right)\gamma_\infty S^{-q/2}|\Omega|^{(2^*-q)/2^*}$$
$$\left. \times\left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right)^{(2-q)/2}\right]. \tag{32}$$

Thus, if $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < (q/2)^{2/(2-q)}\Lambda_1$, for all $z \in \mathcal{N}_{\lambda,\mu}^-$, then

$$I_{\lambda,\mu}(z) \ge d_0(\lambda, \mu, q, N, S, \gamma_\infty, |\Omega|) > 0. \tag{33}$$
□

3. Existence of a Ground State Solution

First of all, we define the Palais-Smale (denote by (PS)) sequences and (PS)-condition in E for $I_{\lambda,\mu}$ as follows.

Definition 8. (i) For $\gamma \in \mathbb{R}$, a sequence $\{z_n\}$ is a $(PS)_\gamma$-sequence in E for $I_{\lambda,\mu}$ if $I_{\lambda,\mu}(z_n) = \gamma + o_n(1)$ and $I'_{\lambda,\mu}(z_n) = o_n(1)$ strongly in E^{-1} as $n \to \infty$.

(ii) $I_{\lambda,\mu}$ satisfies the $(PS)_\gamma$-condition in E if any $(PS)_\gamma$-sequence $\{z_n\}$ in E for $I_{\lambda,\mu}$ contains a convergent subsequence.

Proof of Theorem 1. Using the same argument as in Wu [16, Proposition 9] or Hsu [14, Proposition 3.3], there exists a minimizing sequence $\{z_n\}$ for $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu} + o_n(1), \qquad I'_{\lambda,\mu}(z_n) = o_n(1) \text{ in } E^{-1}. \tag{34}$$

Since $I_{\lambda,\mu}$ is coercive on $\mathcal{N}_{\lambda,\mu}$ (see Lemma 3), we get that $\{z_n\}$ is bounded in E. Then, there exist a subsequence $\{z_n = (u_n, v_n)\}$ and $z_{\lambda,\mu}^1 = (u_{\lambda,\mu}^1, v_{\lambda,\mu}^1) \in E$ such that

$$u_n \rightharpoonup u_{\lambda,\mu}^1, \quad v_n \rightharpoonup v_{\lambda,\mu}^1 \quad \text{weakly in } H_0^1(\Omega),$$
$$u_n \longrightarrow u_{\lambda,\mu}^1, \quad v_n \longrightarrow v_{\lambda,\mu}^1 \quad \text{almost everywhere in } \Omega,$$
$$u_n \longrightarrow u_{\lambda,\mu}^1, \quad v_n \longrightarrow v_{\lambda,\mu}^1 \quad \text{strongly in } L^s(\Omega) \ \forall 1 \le s < 2^*. \tag{35}$$

This implies

$$K_{\lambda,\mu}(z_n) = K_{\lambda,\mu}(z_{\lambda,\mu}^1) + o_n(1) \quad \text{as } n \longrightarrow \infty. \tag{36}$$

First, we claim that $z_{\lambda,\mu}^1$ is a nontrivial solution of $(P_{\lambda,\mu})$. By (34) and (35), it is easy to verify that $z_{\lambda,\mu}^1$ is a weak solution of $(P_{\lambda,\mu})$. From $z_n \in \mathcal{N}_{\lambda,\mu}$ and (12), we deduce that

$$K_{\lambda,\mu}(z_n) = \frac{q(2^*-2)}{2(2^*-q)}\|z_n\|_E^2 - \frac{2^*q}{2^*-q}I_{\lambda,\mu}(z_n). \tag{37}$$

Let $n \to \infty$ in (37); by (34), (36), and $\theta_{\lambda,\mu} < 0$, we get

$$K_{\lambda,\mu}(z_{\lambda,\mu}^1) \geq -\frac{2^*q}{2^*-q}\theta_{\lambda,\mu} > 0. \tag{38}$$

Thus, $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}$ is a nontrivial solution of $(P_{\lambda,\mu})$. Now, we prove that $z_n \to z_{\lambda,\mu}^1$ strongly in E and $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \theta_{\lambda,\mu}$. By (37), if $z \in \mathcal{N}_{\lambda,\mu}$, then

$$I_{\lambda,\mu}(z) = \frac{1}{N}\|z\|_E^2 - \frac{2^*-q}{2^*q}K_{\lambda,\mu}(z). \tag{39}$$

In order to prove that $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \theta_{\lambda,\mu}$, it suffices to recall that $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}$, by (39) and applying Fatou's lemma to get

$$\begin{aligned}
\theta_{\lambda,\mu} &\leq I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \frac{1}{N}\|z_{\lambda,\mu}^1\|_E^2 - \frac{2^*-q}{2^*q}K_{\lambda,\mu}(z_{\lambda,\mu}^1) \\
&\leq \liminf_{n \to \infty}\left(\frac{1}{N}\|z_n\|_E^2 - \frac{2^*-q}{2^*q}K_{\lambda,\mu}(z_n)\right) \\
&\leq \liminf_{n \to \infty}I_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu}.
\end{aligned} \tag{40}$$

This implies that $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \theta_{\lambda,\mu}$ and $\lim_{n \to \infty}\|z_n\|_E^2 = \|z_{\lambda,\mu}^1\|_E^2$. Let $\tilde{z}_n = z_n - z_{\lambda,\mu}^1$; then Brézis-Lieb lemma [17] implies

$$\|\tilde{z}_n\|_E^2 = \|z_n\|_E^2 - \|z_{\lambda,\mu}^1\|_E^2. \tag{41}$$

Therefore, $z_n \to z_{\lambda,\mu}^1$ strongly in E. Since $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = I_{\lambda,\mu}(|z_{\lambda,\mu}^1|) = \theta_{\lambda,\mu}$ and $|z_{\lambda,\mu}^1| \in \mathcal{N}_{\lambda,\mu}^+$, by Lemma 6 we may assume that $z_{\lambda,\mu}^1$ is a nontrivial nonnegative solution of $(P_{\lambda,\mu})$. By an argument of Hsu [18, Lemma 4.2], we can deduce that $u_{\lambda,\mu}^1 \not\equiv 0$ and $v_{\lambda,\mu}^1 \not\equiv 0$ in Ω. Finally, from the maximum principle [19], we deduce that $z_{\lambda,\mu}^1$ is positive in Ω. □

Remark 9. $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^+$ and $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \theta_{\lambda,\mu} = \theta_{\lambda,\mu}^+$.

Proof. We claim that $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^+$. On the contrary, assume that $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^-$ ($\mathcal{N}_{\lambda,\mu}^0 = \emptyset$ for $\lambda^{2/(2-q)} + \mu^{2/(2-q)} \in (0, \Lambda_1)$); then by Lemma 5, there exist unique t_1^+ and t_1^- such that $t_1^+ z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^+$ and $t_1^- z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^-$. In particular, we have $t_1^+ < t_1^- = 1$. Since

$$\frac{d}{dt}I_{\lambda,\mu}(t_1^+ z_{\lambda,\mu}^1) = 0, \qquad \frac{d^2}{dt^2}I_{\lambda,\mu}(t_1^+ z_{\lambda,\mu}^1) > 0, \tag{42}$$

there exists $t_1^+ < \bar{t} \leq t_1^-$ such that $I_{\lambda,\mu}(t_1^+ z_{\lambda,\mu}^1) < I_{\lambda,\mu}(\bar{t} z_{\lambda,\mu}^1)$. By Lemma 5,

$$I_{\lambda,\mu}(t_1^+ z_{\lambda,\mu}^1) < I_{\lambda,\mu}(\bar{t} z_{\lambda,\mu}^1) \leq I_{\lambda,\mu}(t_1^- z_{\lambda,\mu}^1) = I_{\lambda,\mu}(z_{\lambda,\mu}^1), \tag{43}$$

which is a contradiction. Hence, $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^+$ and $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \theta_{\lambda,\mu} = \theta_{\lambda,\mu}^+$. □

4. Existence of $k+1$ Solutions

Throughout this section, $(H1)$-$(H2)$ will be assumed. First of all, we want to show that $I_{\lambda,\mu}$ satisfies the $(PS)_\gamma$-condition in E for $\gamma \in (-\infty, (1/N)(S_{\alpha,\beta})^{N/2} - C_0(\lambda^{2/(2-q)} + \mu^{2/(2-q)}))$, where C_0 is defined in the following lemma.

Lemma 10. *If $\{z_n\} \subset E$ is a $(PS)_\gamma$-sequence for $I_{\lambda,\mu}$ with $z_n \rightharpoonup z$ weakly in E, then $I'_{\lambda,\mu}(z) = 0$ and there exists a constant $C_0 = C_0(q, N, S, \gamma_\infty, |\Omega|) > 0$ such that $I_{\lambda,\mu}(z) \geq -C_0(\lambda^{2/(2-q)} + \mu^{2/(2-q)})$.*

Proof. Let $z_n = (u_n, v_n)$ and $z = (u, v)$. If $\{z_n\}$ is a $(PS)_\gamma$-sequence for $I_{\lambda,\mu}$ with $z_n \rightharpoonup z$ weakly in E, it is easy to check that $I'_{\lambda,\mu}(z) = 0$ in E^{-1}. Then, we get $\langle I'_{\lambda,\mu}(z), z\rangle = 0$; that is, $\int_\Omega h(x)|u|^\alpha|v|^\beta dx = \|z\|_E^2 - K_{\lambda,\mu}(z)$. Thus, by (13), the Hölder, the Young, and the Sobolev inequalities, we have

$$\begin{aligned}
I_{\lambda,\mu}(z) &\geq \frac{1}{N}\|z\|_E^2 - \frac{2^*-q}{2^*q}\gamma_\infty S^{-q/2}|\Omega|^{(2^*-q)/2^*} \\
&\quad \times \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right)^{(2-q)/2}\|z\|_E^q \\
&\geq \frac{1}{N}\|z\|_E^2 - \frac{1}{N}\|z\|_E^2 - C_0\left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right) \\
&= -C_0\left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right),
\end{aligned} \tag{44}$$

where $C_0 = C_0(q, N, S, \gamma_\infty, |\Omega|) > 0$. □

Lemma 11. *If $\{z_n\} \subset E$ is a $(PS)_\gamma$-sequence for $I_{\lambda,\mu}$, then $\{z_n\}$ is bounded in E.*

Proof. See Hsu and Lin [8, Lemma 2.3]. □

Recall that

$$S_{\alpha,\beta} = \inf_{u,v \in H_0^1(\Omega)\setminus\{0\}} \frac{\|(u,v)\|_E^2}{\left(\int_\Omega |u|^\alpha|v|^\beta dx\right)^{2/(\alpha+\beta)}}, \tag{45}$$

and let

$$c^* = \frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2} - C_0\left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right), \tag{46}$$

where $C_0 > 0$ is given in Lemma 10.

Lemma 12. *$I_{\lambda,\mu}$ satisfies the $(PS)_\gamma$-condition in E for $\gamma \in (-\infty, c^*)$.*

Proof. Let $\{z_n\} \subset E$ be a $(PS)_\gamma$-sequence for $I_{\lambda,\mu}$ with $\gamma \in (-\infty, c^*)$. Write $z_n = (u_n, v_n)$. We know from Lemma 11 that $\{z_n\}$ is bounded in E, and then $z_n \rightharpoonup z = (u, v)$ weakly up to a subsequence; z is a critical point of $I_{\lambda,\mu}$. Furthermore, we may assume that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ weakly in $H_0^1(\Omega)$ and $u_n \to u$, $v_n \to v$ strongly in $L^s(\Omega)$ for all $1 \le s < 2^*$, and $u_n \to u$, $v_n \to v$ a.e. on Ω. Hence, we have that $I'_{\lambda,\mu}(z) = 0$ and

$$K_{\lambda,\mu}(z_n) = K_{\lambda,\mu}(z) + o_n(1). \tag{47}$$

Let $\widetilde{u}_n = u_n - u$, $\widetilde{v}_n = v_n - v$ and $\widetilde{z}_n = (\widetilde{u}_n, \widetilde{v}_n)$. Then, we obtain

$$\|\widetilde{z}_n\|_E^2 = \|z_n\|_E^2 - \|z\|_E^2 + o_n(1), \tag{48}$$

and by an argument of Han [20, Lemma 2.1],

$$\int_\Omega h(x)|\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx$$
$$= \int_\Omega h(x)|u_n|^\alpha |v_n|^\beta dx \tag{49}$$
$$- \int_\Omega h(x)|u|^\alpha |v|^\beta dx + o_n(1).$$

Since $I_{\lambda,\mu}(z_n) = \gamma + o_n(1)$, $I'_{\lambda,\mu}(z_n) = o_n(1)$ in E^{-1} and (47)–(49), we deduce that

$$\frac{1}{2}\|\widetilde{z}_n\|_E^2 - \frac{1}{2^*}\int_\Omega h(x)|\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx = \gamma - I_{\lambda,\mu}(z) + o_n(1), \tag{50}$$

$$\|\widetilde{z}_n\|_E^2 - \int_\Omega h(x)|\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx = o_n(1). \tag{51}$$

Hence, we may assume that

$$\|\widetilde{z}_n\|_E^2 \longrightarrow l, \qquad \int_\Omega h(x)|\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx \longrightarrow l. \tag{52}$$

Assume that $l \ne 0$; by the definition of $S_{\alpha,\beta}$, $|h|_{L^\infty(\Omega)} = 1$ and (52), we obtain

$$S_{\alpha,\beta} l^{2/2^*} = S_{\alpha,\beta} \lim_{n \to \infty}\left(\int_\Omega h(x)|\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx\right)^{2/2^*}$$
$$\le |h|_{L^\infty(\Omega)}^{2/2^*} \lim_{n \to \infty}\|\widetilde{z}_n\|^2 = l, \tag{53}$$

which implies that $l \ge (S_{\alpha,\beta})^{N/2}$. In addition, from Lemma 10, (50), and (52), we get

$$\gamma = \left(\frac{1}{2} - \frac{1}{2^*}\right)l + I_{\lambda,\mu}(z)$$
$$\ge \frac{1}{N}(S_{\alpha,\beta})^{N/2} - C_0\left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right), \tag{54}$$

which is a contradiction. Hence, $l = 0$; that is, $z_n \to z$ strongly in E. $\qquad\square$

From assumption (H2), we can choose $r_0 \in (0, 1)$ such that

$$\overline{B_{r_0}(a_i)} \cap \overline{B_{r_0}(a_j)} = \emptyset \quad \text{for } i \ne j, \ 1 \le i, \ j \le k, \tag{55}$$

and $\cup_{i=1}^k \overline{B_{r_0}(a_i)} \subset \Omega$, where $\overline{B_{r_0}(a_i)} = \{x \in \mathbb{R}^N \mid |x - a_i| \le r_0\}$ and $h(a_i) = |h|_\infty = 1$ for $1 \le i \le k$.

Define

$$Q_i(z) = \frac{\int_\Omega \psi_i(x)\left(|\nabla u|^2 + |\nabla v|^2\right)dx}{\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)dx}, \quad z = (u, v) \in E \setminus \{0\}, \tag{56}$$

where $\psi_i(x) = \min\{1, |x - a_i|\}$, $i = 1, 2 \dots, k$.

Then, we have the following separation result.

Lemma 13. *If $Q_i(z) \le r_0/3$ and $Q_j(z) \le r_0/3$ for $z \in E \setminus \{0\}$, then $i = j$.*

Proof. For any $z \in E \setminus \{0\}$ satisfying $Q_i(z) \le r_0/3$ $(1 \le i \le k)$, we get

$$\frac{r_0}{3}\|z\|_E^2 \ge \int_\Omega \psi_i(x)\left(|\nabla u|^2 + |\nabla v|^2\right)dx$$
$$\ge \int_{\Omega \setminus B_{r_0}(a_i)} \psi_i(x)\left(|\nabla u|^2 + |\nabla v|^2\right)dx \tag{57}$$
$$\ge r_0 \int_{\Omega \setminus B_{r_0}(a_i)}\left(|\nabla u|^2 + |\nabla v|^2\right)dx,$$

which implies that

$$\|z\|_E^2 \ge 3\int_{\Omega \setminus B_{r_0}(a_i)}\left(|\nabla u|^2 + |\nabla v|^2\right)dx, \quad 1 \le i \le k. \tag{58}$$

Hence, from (58), we obtain

$$2\|z\|_E^2 \ge 3\left(\int_{\Omega \setminus B_{r_0}(a_i)}\left(|\nabla u|^2 + |\nabla v|^2\right)dx\right.$$
$$\left. + \int_{\Omega \setminus B_{r_0}(a_j)}\left(|\nabla u|^2 + |\nabla v|^2\right)dx\right) \tag{59}$$
$$\ge 3\|z\|_E^2 \quad \text{if } i \ne j,$$

which is a contradiction. $\qquad\square$

For $i = 1, 2, \dots, k$, we set

$$\mathcal{N}_{\lambda,\mu}^i = \left\{u \in \mathcal{N}_{\lambda,\mu}^- \mid Q_i(z) < \frac{r_0}{3}\right\},$$
$$\partial\mathcal{N}_{\lambda,\mu}^i = \left\{u \in \mathcal{N}_{\lambda,\mu}^- \mid Q_i(z) = \frac{r_0}{3}\right\}, \tag{60}$$

and define

$$\theta_{\lambda,\mu}^i = \inf_{\mathcal{N}_{\lambda,\mu}^i} I_{\lambda,\mu}(z), \qquad \widetilde{\theta}_{\lambda,\mu}^i = \inf_{\partial\mathcal{N}_{\lambda,\mu}^i} I_{\lambda,\mu}(z). \tag{61}$$

Recall that the best Sobolev constant S is defined as

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N}|u|^{2^*}dx\right)^{2/2^*}}. \tag{62}$$

It is well known that

$$U(x) = \frac{[N(N-2)]^{(N-2)/4}}{\left[1 + |x|^2\right]^{(N-2)/2}} \tag{63}$$

is a minimizer of S, and $|\nabla U|^2_{L^2(\mathbb{R}^N)} = |U|^{2^*}_{L^{2^*}(\mathbb{R}^N)} = S^{N/2}$. Fix a maximum point a_i of h $(1 \le i \le k)$. Let $\eta_i \in C_0^\infty(\Omega)$ be a cut-off function such that $0 \le \eta_i \le 1$, $|\nabla \eta_i| \le C$, and $\eta_i(x) = 1$ for $|x - a_i| < r_0/2$, $\eta_i(x) = 0$ for $|x - a_i| > r_0$. We define

$$u_\varepsilon^i(x) = \varepsilon^{(2-N)/2} \eta_i(x) U\left(\frac{x - a_i}{\varepsilon}\right) = \frac{c_1 \varepsilon^{(N-2)/2} \eta_i(x)}{\left[\varepsilon^2 + |x - a_i|^2\right]^{(N-2)/2}}, \tag{64}$$

where $c_1 = [N(N-2)]^{(N-2)/4}$ and $\varepsilon > 0$.

From now on, we assume that $N/(N-2) < q < 2$ and $N > 4$.

Lemma 14. *There exist $\varepsilon_0 > 0$, $\Lambda_2 \in (0, (q/2)^{2/(2-q)} \Lambda_1)$, such that for $\varepsilon \in (0, \varepsilon_0)$ and $\lambda^{2/(2-q)} + \mu^{2/(2-q)} \in (0, \Lambda_2)$, one has*

$$\sup_{t \ge 0} I_{\lambda,\mu}\left(t\sqrt{\alpha} u_\varepsilon^i, t\sqrt{\beta} u_\varepsilon^i\right) < c^* \quad \text{uniformly in } i, \tag{65}$$

where c^* is the positive constant given in Lemma 12.

In particular, $0 < \theta_{\lambda,\mu}^- \le \theta_{\lambda,\mu}^i < c^*$ for all $1 \le i \le k$.

Proof. It is well known that (or see Brézis and Nirenberg [21], Cheng and Ma [22, Lemma 3.2], Struwe [23], and Willem [24, Lemma 1.46]) as $\varepsilon \to 0^+$,

$$\left|u_\varepsilon^i\right|^2_{L^{2^*}(\Omega)} = |U|^2_{L^{2^*}(\mathbb{R}^N)} + O\left(\varepsilon^{N-2}\right), \tag{66}$$

$$\left|\nabla u_\varepsilon^i\right|^2_{L^2(\Omega)} |\nabla U|^2_{L^2(\mathbb{R}^N)} + O\left(\varepsilon^{N-2}\right). \tag{67}$$

For $N/(N-2) < q < 2$, $N > 4$ and $\varepsilon < r_0/2$,

$$\left|u_\varepsilon^i\right|^q_{L^q(\Omega)} = \int_{B_{r_0/2}(a_i)} \left[\varepsilon^{(2-N)/2} U\left(\frac{x - a_i}{\varepsilon}\right)\right]^q dx + O\left(\varepsilon^{N-2}\right)$$

$$\ge C\varepsilon^\theta + O\left(\varepsilon^{N-2}\right), \quad \text{where } \theta = N - \frac{(N-2)q}{2}. \tag{68}$$

Set $\overline{z}_\varepsilon^i = (\sqrt{\alpha} u_\varepsilon^i, \sqrt{\beta} u_\varepsilon^i)$. By Lemma 5, there exists $t_\varepsilon^i > 0$ such that $z_\varepsilon^i = t_\varepsilon^i \overline{z}_\varepsilon^i \in \mathcal{N}_{\lambda,\mu}^-$ for $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$. Furthermore,

$$Q_i\left(z_\varepsilon^i\right) = \frac{\int_\Omega \psi_i(x) \left|\nabla u_\varepsilon^i\right|^2 dx}{\int_\Omega \left|\nabla u_\varepsilon^i\right|^2 dx}$$

$$= \frac{\int_{(\Omega - a_i)/\varepsilon} \psi_i(a_i + \varepsilon y) \left|\nabla\left(\eta_i(a_i + \varepsilon y) U(y)\right)\right|^2 dy}{\int_{(\Omega - a_i)/\varepsilon} \left|\nabla\left(\eta_i(a_i + \varepsilon y) U(y)\right)\right|^2 dy}$$

$$\longrightarrow \psi_i(a_i) = 0 \quad \text{as } \varepsilon \longrightarrow 0. \tag{69}$$

Hence, there exists $\overline{\varepsilon}_0 > 0$ for any

$$\varepsilon \in (0, \overline{\varepsilon}_0), \qquad Q_i\left(z_\varepsilon^i\right) < \frac{r_0}{3}, \tag{70}$$

which implies

$$z_\varepsilon^i = t_\varepsilon^i \overline{z}_\varepsilon^i \in \mathcal{N}_{\lambda,\mu}^i \quad \text{for } \varepsilon \in (0, \overline{\varepsilon}_0), \tag{71}$$

and then

$$\theta_{\lambda,\mu}^- \le \theta_{\lambda,\mu}^i \le I_{\lambda,\mu}\left(z_\varepsilon^i\right) \le \sup_{t \ge 0} I_{\lambda,\mu}\left(tt_\varepsilon^i \overline{z}_\varepsilon^i\right) = \sup_{t \ge 0} I_{\lambda,\mu}\left(t\overline{z}_\varepsilon^i\right). \tag{72}$$

First, we consider the functional $I_{0,0} : E \to \mathbb{R}$ defined by

$$I_{0,0}(u, v) = \frac{1}{2}\|(u, v)\|^2_E - \frac{1}{2^*}\int_\Omega h(x) |u|^\alpha |v|^\beta dx. \tag{73}$$

Step I. Show that $\sup_{t \ge 0} I_{0,0}(\overline{z}_\varepsilon^i) \le (1/N)(S_{\alpha,\beta})^{N/2} + O(\varepsilon^{N-2})$.

According to condition $(H2)$, we conclude that

$$\left|\int_\Omega h(x) \left|u_\varepsilon^i(x)\right|^{2^*} dx - \int_\Omega h(a_i) \left|u_\varepsilon^i(x)\right|^{2^*} dx\right|$$

$$\le \int_\Omega |h(x) - h(a_i)| \left|u_\varepsilon^i(x)\right|^{2^*} dx$$

$$= O\left(\int_{B_{r_0}(a_i)} |x - a_i|^\sigma \left|u_\varepsilon^i(x)\right|^{2^*} dx\right)$$

$$= O\left(\varepsilon^\sigma\right). \tag{74}$$

From (66), (74), $h(a_i) = 1$, and $\sigma \ge N - 2$, we can deduce that

$$\left(\int_\Omega h(x) \left|u_\varepsilon^i(x)\right|^{2^*} dx\right)^{2/2^*}$$

$$= \left(\left|u_\varepsilon^i\right|^{2^*}_{L^{2^*}(\Omega)} + O\left(\varepsilon^\sigma\right)\right)^{2/2^*}$$

$$= \left|u_\varepsilon^i\right|^2_{L^{2^*}(\Omega)} + O\left(\varepsilon^\sigma\right)$$

$$= |U|^2_{L^{2^*}(\mathbb{R}^N)} + O\left(\varepsilon^{N-2}\right). \tag{75}$$

Using (67) and (75), then

$$\frac{\left|\nabla u_\varepsilon^i\right|^2_{L^2(\Omega)}}{\left(\int_\Omega h(x) \left|u_\varepsilon^i(x)\right|^{2^*} dx\right)^{2/2^*}}$$

$$= \frac{|\nabla U|^2_{L^2(\mathbb{R}^N)} + O\left(\varepsilon^{N-2}\right)}{|U|^2_{L^{2^*}(\mathbb{R}^N)} + O\left(\varepsilon^{N-2}\right)}$$

$$= S + O\left(\varepsilon^{N-2}\right). \tag{76}$$

Since

$$\sup_{t \ge 0}\left(\frac{A}{2}t^2 - \frac{B}{2^*}t^{2^*}\right)$$

$$= \frac{1}{N}\left(\frac{A}{B^{2/2^*}}\right)^{N/2}, \quad \text{for any } A > 0, \ B > 0, \tag{77}$$

by (7) and (76), we conclude that

$$
\begin{aligned}
&\sup_{t \geq 0} I_{0,0}\left(t\overline{z}_\varepsilon^i\right) \\
&= \frac{1}{N}\left(\frac{(\alpha+\beta)\left|\nabla u_\varepsilon^i\right|_{L^2(\Omega)}^2}{\left(\alpha^{\alpha/2}\beta^{\beta/2}\int_\Omega h(x)\left|u_\varepsilon^i(x)\right|^{2^*}dx\right)^{2/2^*}}\right)^{N/2} \\
&\leq \frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2} + O\left(\varepsilon^{N-2}\right).
\end{aligned}
\tag{78}
$$

Step II. Let C_0 be the positive constant given in Lemma 10. We can choose $\delta_1 > 0$ such that for all $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \delta_1$, we have

$$
c^* = \frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2} - C_0\left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right) > 0.
\tag{79}
$$

Since $I_{\lambda,\mu}$ is continuous in E, $I_{\lambda,\mu}(0) = 0$, and $\{\overline{z}_\varepsilon^i\}$ is uniformly bounded in E for any $0 < \varepsilon < \min\{\overline{\varepsilon}_0, r_0/2\}$ (see (67)), then there exists $t_0 > 0$ (independent of ε) such that for any $0 < \varepsilon < \min\{\overline{\varepsilon}_0, r_0/2\}$,

$$
\begin{aligned}
&\sup_{0 \leq t \leq t_0} I_{\lambda,\mu}\left(t\overline{z}_\varepsilon^i\right) < c^*, \quad \text{uniformly in } i, \\
&\forall 0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \delta_1.
\end{aligned}
\tag{80}
$$

According to condition $(H1)$, $f_{\min} = \min_{x \in \overline{\Omega}} f(x) > 0$ and $g_{\min} = \min_{x \in \overline{\Omega}} g(x) > 0$. Applying the results of Step I and (68), we have that for $N/(N-2) < q < 2$ and $N > 4$,

$$
\begin{aligned}
&\sup_{t \geq t_0} I_{\lambda,\mu}\left(t\overline{z}_\varepsilon^i\right) \\
&= \sup_{t \geq t_0}\left(I_{0,0}\left(t\overline{z}_\varepsilon^i\right) - \frac{t^q}{q}K_{\lambda,\mu}\left(t\overline{z}_\varepsilon^i\right)\right) \\
&\leq \frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2} + O\left(\varepsilon^{N-2}\right) - \frac{t_0^q}{q}m(\lambda+\mu) \\
&\quad \times \int_{B_{r_0/2}(a_i)}\left|u_\varepsilon^i\right|^q dx \\
&\leq \frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2} + O\left(\varepsilon^{N-2}\right) - (\lambda+\mu)O_1\left(\varepsilon^\theta\right),
\end{aligned}
\tag{81}
$$

where $m = \min\{\alpha^{q/2}f_{\min}, \beta^{q/2}g_{\min}\}$ and $\theta = N-((N-2)q)/2$.

Therefore, we can choose $\lambda = O_1(\varepsilon^{\tau_1})$ and $\mu = O_1(\varepsilon^{\tau_2})$ such that

$$
\frac{2-q}{q}\theta < \tau_1, \qquad \tau_2 < (N-2) - \theta.
\tag{82}
$$

This implies that

$$
\begin{aligned}
&\min\{\tau_1, \tau_2\} + \theta < \frac{2}{2-q}\min(\tau_1, \tau_2), \\
&\min\{\tau_1, \tau_2\} + \theta < N-2, \\
&(\lambda+\mu)O_1\left(\varepsilon^\theta\right) = O_1\left(\varepsilon^{\min\{\tau_1,\tau_2\}+\theta}\right), \\
&\lambda^{2/(2-q)} + \mu^{2/(2-q)} = O_1\left(\varepsilon^{2/(2-q)\min\{\tau_1,\tau_2\}}\right).
\end{aligned}
\tag{83}
$$

There exist $\delta_2 > 0$, $\varepsilon_0 \in (0, \min\{\overline{\varepsilon}_0, r_0/2\})$ such that for all $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \delta_2$ and $0 < \varepsilon < \varepsilon_0$, we have

$$
O\left(\varepsilon^{N-2}\right) - (\lambda+\mu)O_1\left(\varepsilon^\theta\right) < -C_0\left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right).
\tag{84}
$$

Thus, we can choose $\Lambda_2 = \min\{(q/2)^{2/(2-q)}\Lambda_1, \delta_1, \delta_2\} > 0$. Then, for all $\lambda^{2/(2-q)} + \mu^{2/(2-q)} \in (0, \Lambda_2)$, there holds

$$
\sup_{t \geq 0} I_{\lambda,\mu}\left(t\overline{z}_\varepsilon^i\right) < c^* \quad \text{uniformly in } i.
\tag{85}
$$

Step III. For $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_2$ and $0 < \varepsilon < \varepsilon_0$, by Lemma 7, (72), and (85), we get

$$
0 < \theta_{\lambda,\mu}^- \leq \theta_{\lambda,\mu}^i \leq I_{\lambda,\mu}\left(t\overline{z}_\varepsilon^i\right) < c^* \quad \forall 1 \leq i \leq k.
\tag{86}
$$

$\square$

To proceed, we need to quote the concentration-compactness principle (see [24, 25]) about the case of systems.

Lemma 15. *Let $\{u_n, v_n\} \subset H_0^1(\Omega) \times H_0^1(\Omega)$ be a sequence such that*

$$
\begin{aligned}
&u_n \rightharpoonup u, \quad v_n \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega); \\
&u_n \longrightarrow u, \quad v_n \longrightarrow v \quad \text{a.e. on } \Omega, \\
&\left|\nabla(u_n-u)\right|^2 + \left|\nabla(v_n-v)\right|^2 \rightharpoonup \widetilde{\mu} \\
&\text{weakly in the sense of measures,} \\
&\left|u_n-u\right|^\alpha\left|v_n-v\right|^\beta \rightharpoonup \widetilde{\nu} \\
&\text{weakly in the sense of measures.}
\end{aligned}
\tag{87}
$$

Then, it follows that

$$
\begin{aligned}
&\limsup_{n \to \infty} \int_{\mathbb{R}^N}\left(\left|\nabla u_n\right|^2 + \left|\nabla v\right|_n^2\right)dx \\
&\quad = \int_{\mathbb{R}^N}\left(\left|\nabla u\right|^2 + \left|\nabla v\right|^2\right)dx + \|\widetilde{\mu}\|, \\
&\limsup_{n \to \infty} \int_{\mathbb{R}^N}\left|u_n\right|^\alpha\left|v_n\right|^\beta dx = \int_{\mathbb{R}^N}\left|u\right|^\alpha\left|v\right|^\beta dx + \|\widetilde{\nu}\|, \\
&\|\widetilde{\nu}\|^{2/(\alpha+\beta)} \leq S_{\alpha,\beta}^{-1}\|\widetilde{\mu}\|.
\end{aligned}
\tag{88}
$$

Moreover, if $u \equiv v \equiv 0$ and $\|\widetilde{\nu}\|^{2/(\alpha+\beta)} = S_{\alpha,\beta}^{-1}\|\widetilde{\mu}\|$, then $\widetilde{\mu}$ and $\widetilde{\nu}$ concentrate at a single point.

Proof. See Han [20, Lemma 2.2]. ☐

Lemma 16. *For any $i \in \{1, 2, \ldots, k\}$, there exist $\widetilde{\Lambda}_i > 0$ such that*

$$\widetilde{\theta}^i_{\lambda,\mu} > \frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2} \quad \forall 0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \widetilde{\Lambda}_i. \quad (89)$$

Proof. Fix $i \in \{1, 2, \ldots, k\}$. Assume the contrary. There then exists a sequence $\{(\lambda_n, \mu_n)\}$ with $(\lambda_n, \mu_n) \to (0, 0)$ as $n \to \infty$ such that $\widetilde{\theta}^i_{\lambda_n,\mu_n} \to c \leq (1/N)(S_{\alpha,\beta})^{N/2}$ as $n \to \infty$. Consequently, there exists a sequence $\{z_n = (u_n, v_n)\} \subset \partial\mathcal{N}^i_{\lambda_n,\mu_n}$ such that as $n \to \infty$,

$$\int_\Omega \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) dx$$

$$= \int_\Omega \left(\lambda_n f(x)|u_n|^q + \mu_n g(x)|v_n|^q\right) dx \quad (90)$$

$$+ \int_\Omega h(x)|u_n|^\alpha |v_n|^\beta dx,$$

$$I_{\lambda_n,\mu_n}(z_n) \longrightarrow c \leq \frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2} \quad \text{as } n \longrightarrow \infty. \quad (91)$$

It then follows easily that $\{z_n\}$ is uniformly bounded in E, and since f and g are continuous on $\overline{\Omega}$, we obtain

$$K_{\lambda_n,\mu_n}(z_n) = \int_\Omega \left(\lambda_n f(x)|u_n|^q + \mu_n g(x)|v_n|^q\right) dx \quad (92)$$

$$= o_n(1) \quad \text{as } n \longrightarrow \infty.$$

From (90), and by the Hölder and the Sobolev inequalities, we can fix $m_0 > 0$ such that

$$\int_\Omega \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) dx \geq m_0,$$

$$\int_\Omega h(x)|u_n|^\alpha |v_n|^\beta dx \geq m_0. \quad (93)$$

Thus, up to a subsequence, we infer that

$$\lim_{n\to\infty} \int_\Omega \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) dx$$

$$= \lim_{n\to\infty} \int_\Omega h(x)|u_n|^\alpha |v_n|^\beta dx = l > 0. \quad (94)$$

Furthermore, by $|h|_{L^\infty(\Omega)} = 1$, we deduce

$$l = \lim_{n\to\infty} \int_\Omega h(x)|u_n|^\alpha |v_n|^\beta dx$$

$$\leq |h|_{L^\infty(\Omega)} \lim_{n\to\infty} \int_\Omega |u_n|^\alpha |v_n|^\beta dx \quad (95)$$

$$\leq S_{\alpha,\beta}^{-2^*/2} \lim_{n\to\infty} \left(\int_\Omega \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) dx\right)^{2^*/2}$$

$$\leq S_{\alpha,\beta}^{-2^*/2} l^{2^*/2},$$

which implies

$$l \geq \left(S_{\alpha,\beta}\right)^{N/2}. \quad (96)$$

On the other hand, we have, as $n \to \infty$,

$$\frac{1}{N}l = \frac{1}{2}\|z_n\|^2 dx - \frac{1}{2^*}\int_\Omega h(x)|u_n|^\alpha |v_n|^\beta dx$$

$$- \frac{1}{q}K_{\lambda_n,\mu_n}(z_n) + o_n(1)$$

$$= I_{\lambda_n,\mu_n}(z_n) + o_n(1)$$

$$\leq \frac{1}{N}\left(S_{\alpha,\beta}\right)^{N/2}. \quad (97)$$

Hence, together with (96), we get

$$l = \left(S_{\alpha,\beta}\right)^{N/2}, \quad (98)$$

and then from (95), we also have

$$\lim_{n\to\infty} \int_\Omega h(x)|u_n|^\alpha |v_n|^\beta dx = \lim_{n\to\infty} \int_\Omega |u_n|^\alpha |v_n|^\beta dx = l. \quad (99)$$

Therefore,

$$\lim_{n\to\infty} \int_\Omega |u_n|^\alpha |v_n|^\beta dx = l. \quad (100)$$

Set $\widetilde{z}_n = (\widetilde{u}_n, \widetilde{v}_n) = z_n/\|z_n\|$; then, we have $\|\widetilde{z}_n\| = 1$. Moreover, by (94),(98), and (100), we get

$$\lim_{n\to\infty} \int_\Omega |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx = \lim_{n\to\infty} \frac{\int_\Omega |u_n|^\alpha |v_n|^\beta dx}{\|z_n\|^{2^*}} = S_{\alpha,\beta}^{-N/(N-2)}. \quad (101)$$

Thus, up to a subsequence, we may assume that

$$\widetilde{u}_n \rightharpoonup u, \quad \widetilde{v}_n \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega);$$

$$\widetilde{u}_n \longrightarrow u, \quad \widetilde{v}_n \longrightarrow v \quad \text{a.e. on } \Omega,$$

$$|\nabla(\widetilde{u}_n - u)|^2 + |\nabla(\widetilde{v}_n - v)|^2 \rightharpoonup \widetilde{\mu}$$

weakly in the sense of measures, $\quad (102)$

$$|\widetilde{u}_n - u|^\alpha |\widetilde{v}_n - v|^\beta \rightharpoonup \widetilde{\nu}$$

weakly in the sense of measures.

Since Ω is bounded, from (101) and **Lemma 15**, we deduce that

$$1 = \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) dx + \|\widetilde{\mu}\|, \quad (103)$$

$$S_{\alpha,\beta}^{-N/(N-2)} = \int_\Omega |u|^\alpha |v|^\beta dx + \|\widetilde{\nu}\|, \quad (104)$$

$$\|\widetilde{\nu}\|^{2/(\alpha+\beta)} \leq S_{\alpha,\beta}^{-1}\|\widetilde{\mu}\|. \quad (105)$$

If $\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx \neq 0$ and $\|\tilde\mu\| \neq 0$, we deduce that

$$
\begin{aligned}
1 &= \left(\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)dx + \|\tilde\mu\|\right)^{(\alpha+\beta)/2} \\
&> \left(\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)dx\right)^{(\alpha+\beta)/2} + \|\tilde\mu\|^{(\alpha+\beta)/2} \\
&\geq S_{\alpha,\beta}^{(\alpha+\beta)/2} \int_\Omega |u|^\alpha |v|^\beta dx + S_{\alpha,\beta}^{(\alpha+\beta)/2} \|\tilde v\| \\
&= S_{\alpha,\beta}^{(\alpha+\beta)/2} \cdot S_{\alpha,\beta}^{-N/(N-2)} \\
&= 1,
\end{aligned}
\tag{106}
$$

which is a contradiction.

Thus, $\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx = 0$ or $\|\tilde\mu\| = 0$. If $\|\tilde\mu\| = 0$, from (103)–(105), we get $\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx = 1$ and $\int_\Omega |u|^\alpha |v|^\beta dx = S_{\alpha,\beta}^{-N/(N-2)}$. Then,

$$
\frac{\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)dx}{\left(\int_\Omega |u|^\alpha |v|^\beta dx\right)^{2/(\alpha+\beta)}} = S_{\alpha,\beta},
\tag{107}
$$

which means that $S_{\alpha,\beta}$ is achieved by (u, v). It is impossible since $S_{\alpha,\beta}$ cannot be achieved on any bounded domain Ω. Hence,

$$
\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)dx = 0, \qquad \|\tilde\mu\| = 1.
\tag{108}
$$

Then, $u \equiv v \equiv 0$ on Ω, and from (103), (104), we easily have $\|\tilde v\|^{2/(\alpha+\beta)} = S_{\alpha,\beta}^{-1} = S_{\alpha,\beta}^{-1}\|\tilde\mu\|$. By Lemma 15, we conclude that $x_0 \in \overline\Omega$ such that

$$
|\nabla \tilde u_n|^2 + |\nabla \tilde v_n|^2 \rightharpoonup \delta_{x_0}
$$

weakly in the sense of measures,

$$
|\tilde u_n|^\alpha |\tilde v_n|^\beta \rightharpoonup S_{\alpha,\beta}^{-N/(N-2)} \delta_{x_0}
\tag{109}
$$

weakly in the sense of measures.

Observe that $Q_i(\tilde z_n) = Q_i(z_n) = r_0/3$;

$$
\begin{aligned}
\frac{r_0}{3} &= \lim_{n\to\infty} Q_i(z_n) \\
&= \lim_{n\to\infty} \frac{\int_\Omega \psi_i(x)\left(|\nabla \tilde u_n|^2 + |\nabla \tilde v_n|^2\right)dx}{\int_\Omega \left(|\nabla \tilde u_n|^2 + |\nabla \tilde v_n|^2\right)dx} = \psi_i(x_0),
\end{aligned}
\tag{110}
$$

which implies that $x_0 \neq a_i$ by the definition of $\psi_i(x)$. On the other hand, from (95) and (101), we get

$$
\begin{aligned}
S_{\alpha,\beta}^{-N/(N-2)} h(x_0) &= \lim_{n\to\infty} \int_\Omega h(x) |\tilde u_n|^\alpha |\tilde v_n|^\beta dx \\
&= \lim_{n\to\infty} \frac{\int_\Omega h(x) |u_n|^\alpha |v_n|^\beta dx}{\|z_n\|^{2^*}} \\
&= \lim_{n\to\infty} \frac{\int_\Omega |u_n|^\alpha |v_n|^\beta dx}{\|z_n\|^{2^*}} \\
&= \lim_{n\to\infty} \int_\Omega |\tilde u_n|^\alpha |\tilde v_n|^\beta dx \\
&= S_{\alpha,\beta}^{-N/(N-2)},
\end{aligned}
\tag{111}
$$

which is impossible, because $h(x)$ is not a constant function by condition $(H2)$. $\qquad\square$

Throughout this section, take $\Lambda^* = \min\{\Lambda_2, \min_{1\leq i\leq k}\widetilde\Lambda_i\}$; Λ_2 and $\widetilde\Lambda_i$ are as in Lemmas 14 and 16. Using the idea of Tarantello [12], we have the following results. For $z = (u, v)$, $\varphi = (\varphi_1, \varphi_2) \in E$, we define

$$
z - \varphi = (u - \varphi_1, v - \varphi_2),
$$

$$
\langle z, \varphi\rangle = \int_\Omega \left(\nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2\right)dx,
$$

$$
G_{\lambda,\mu}(z,\varphi) = \int_\Omega \left(\lambda f(x) |u|^{q-2}u\varphi_1 + \mu g(x) |v|^{q-2}v\varphi_2\right)dx,
$$

$$
\begin{aligned}
H(z,\varphi) = {}&\frac{\alpha}{\alpha+\beta}\int_\Omega h(x)|u|^{\alpha-2}u|v|^\beta\varphi_1 dx \\
&+ \frac{\beta}{\alpha+\beta}\int_\Omega h(x)|u|^\alpha|v|^{\beta-2}v\varphi_2 dx.
\end{aligned}
\tag{112}
$$

Lemma 17. *For each $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda^*$ and $z = (u, v) \in \mathcal{N}_{\lambda,\mu}^i$ $(1 \leq i \leq k)$, there exist $\epsilon > 0$ and a differentiable function $\xi : B_\epsilon(0) \subset E \to \mathbb{R}^+$ such that $\xi(0) = 1$, $\xi(\phi)(z-\phi) \in \mathcal{N}_{\lambda,\mu}^i$ for all $\phi \in B_\epsilon(0)$ and*

$$
\langle \xi'(0), \varphi\rangle = \frac{2\langle z, \varphi\rangle - qG_{\lambda,\mu}(z,\varphi) - 2^* H(z,\varphi)}{(2-q)\|z\|_E^2 - (2^*-q)H(z,z)},
\tag{113}
$$

for all $\varphi = (\varphi_1, \varphi_2) \in E$.

Proof. For $z \in \mathcal{N}_{\lambda,\mu}^i$, define a function $F_z : \mathbb{R} \times E \to \mathbb{R}$ by

$$
\begin{aligned}
F_z(\xi, \phi) &= \left\langle I'_{\lambda,\mu}(\xi(z-\phi)), \xi(z-\phi)\right\rangle \\
&= \xi^2 \|z-\phi\|^2 - \xi^q G_{\lambda,\mu}(z-\phi, z-\phi) \\
&\quad - \xi^{\alpha+\beta} H(z-\phi, z-\phi).
\end{aligned}
\tag{114}
$$

Then, $F_u(1,0) = \langle I'_{\lambda,\mu}(z), z\rangle = 0$ and

$$
\begin{aligned}
\frac{d}{d\xi}F_z(1,0) &= 2\|z\|_E^2 - qG_{\lambda,\mu}(z,z) - (\alpha+\beta)H(z,z) \\
&= (2-q)\|z\|_E^2 - (2^*-q)H(z,z) < 0.
\end{aligned}
\tag{115}
$$

According to the implicit function theorem, there exist $\epsilon > 0$ and a differentiable function $\xi : B_\epsilon(0) \subset E \to \mathbb{R}$ such that $\xi(0) = 1$;

$$\langle \xi'(0), \varphi \rangle = \frac{2\langle z, \varphi \rangle - qG_{\lambda,\mu}(z,\varphi) - 2^* H(z,\varphi)}{(2-q)\|z\|_E^2 - (2^* - q)H(z,z)},$$ (116)

$$F_z(\xi(\varphi), \varphi) = 0 \quad \forall \varphi \in B_\epsilon(0),$$

which is equivalent to

$$\langle I'_{\lambda,\mu}(\xi(\varphi)(z-\varphi)), \xi(\varphi)(z-\varphi) \rangle = 0 \quad \forall \varphi \in B(0;\epsilon);$$ (117)

that is, $\xi(\varphi)(z-\varphi) \in \mathcal{N}_{\lambda,\mu}$ for all $\varphi \in B_\epsilon(0)$. Furthermore, by the continuity of the functions ξ and Q_i, we have that

$$(2-q)\|\xi(\varphi)(z-\varphi)\|^2$$
$$- (2^* - q)H(\xi(\varphi)(z-\varphi), \xi(\varphi)(z-\varphi)) < 0,$$ (118)
$$Q_i(\xi(\varphi)(z-\varphi)) < \frac{r_0}{3}$$

still holds if ϵ is sufficiently small. This implies that $\xi(\varphi)(z-\varphi) \in \mathcal{N}^i_{\lambda,\mu}$. $\qquad\square$

Proposition 18. *If* $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda^*$, *then there exists a* $(PS)_{\theta^i_{\lambda,\mu}}$-*sequence* $\{z^i_n\} \subset \mathcal{N}^i_{\lambda,\mu}$ *in* E *for* $I_{\lambda,\mu}$.

Proof. If $\overline{\mathcal{N}^i_{\lambda,\mu}}$ denotes the closure of $\mathcal{N}^i_{\lambda,\mu}$, at first we note that $\overline{\mathcal{N}^i_{\lambda,\mu}} = \mathcal{N}^i_{\lambda,\mu} \cup \partial\mathcal{N}^i_{\lambda,\mu}$ for all $i = 1, 2, \ldots, k$. It then follows from Lemmas 14 and 16, that

$$\theta^i_{\lambda,\mu} < \widetilde{\theta}^i_{\lambda,\mu} \quad \text{for } i = 1, 2, \ldots, k, \ 0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda^*.$$ (119)

Hence,

$$\theta^i_{\lambda,\mu} = \inf\left\{ I_{\lambda,\mu}(z) \mid z \in \overline{\mathcal{N}^i_{\lambda,\mu}} \right\} \quad \text{for } i = 1, 2, \ldots, k.$$ (120)

Now, we fix $i \in \{1, 2, \ldots, k\}$. Applying the Ekeland variational principle [26], there exists a minimizing sequence $\{z^i_n\} \subset \overline{\mathcal{N}^i_{\lambda,\mu}}$ such that

$$I_{\lambda,\mu}(z^i_n) < \theta^i_{\lambda,\mu} + \frac{1}{n},$$
$$I_{\lambda,\mu}(z^i_n) \le I_{\lambda,\mu}(\varphi) + \frac{1}{n}\|\varphi - z^i_n\|_E \quad \text{for each } \varphi \in \overline{\mathcal{N}^i_{\lambda,\mu}}.$$ (121)

Using (119), we may assume that $z^i_n \in \mathcal{N}^i_{\lambda,\mu}$ for n sufficiently large. Applying Lemma 17 with $z = z^i_n$, we obtain the function $\xi_n : B_{\epsilon_n}(0) \to \mathbb{R}$ for some $\epsilon_n > 0$ such that $\xi_n(\varphi)(z^i_n - \varphi) \in \mathcal{N}^i_{\lambda,\mu}$ for all $\varphi \in B_{\epsilon_n}(0)$. Let $0 < \delta < \epsilon_n$ and $z \in E \setminus \{0\}$; we set

$$\varphi_\delta = \frac{\delta z}{\|z\|_E}$$ (122)

and $z_\delta = \xi_n(\varphi_\delta)(z^i_n - \varphi_\delta)$. Since $z_\delta \in \mathcal{N}^i_{\lambda,\mu}$, we deduce from (121) that

$$I_{\lambda,\mu}(z_\delta) - I_{\lambda,\mu}(z^i_n) \ge -\frac{1}{n}\|z_\delta - z^i_n\|_E.$$ (123)

By the mean-value theorem, we obtain

$$\langle I'_{\lambda,\mu}(z^i_n), (z_\delta - z^i_n) \rangle + o\left(\|z_\delta - z^i_n\|_E\right) \ge -\frac{1}{n}\|z_\delta - z^i_n\|_E.$$ (124)

Therefore,

$$\langle I'_{\lambda,\mu}(z^i_n), -\varphi_\delta \rangle + (\xi_n(\varphi_\delta) - 1)\langle I'_{\lambda,\mu}(z^i_n), (z^i_n - \varphi_\delta) \rangle$$
$$\ge -\frac{1}{n}\|z_\delta - z^i_n\|_E + o\left(\|z_\delta - z^i_n\|_E\right).$$ (125)

Now, we observe that $\xi_n(\varphi_\delta)(z^i_n - \varphi_\delta) \in \mathcal{N}^i_{\lambda,\mu}$, and consequently we get from (125) that

$$-\delta\left\langle I'_{\lambda,\mu}(z^i_n), \frac{z}{\|z\|_E} \right\rangle$$
$$+ \frac{(\xi_n(\varphi_\delta) - 1)}{\xi_n(\varphi_\delta)}\left\langle I'_{\lambda,\mu}(z_\delta), \xi_n(\varphi_\delta)(z^i_n - \varphi_\delta) \right\rangle$$
$$+ (\xi_n(\varphi_\delta) - 1)\left\langle I'_{\lambda,\mu}(z^i_n) - I'_{\lambda,\mu}(z_\delta), (z^i_n - \varphi_\delta) \right\rangle$$
$$\ge -\frac{1}{n}\|z_\delta - z^i_n\|_E + o\left(\|z_\delta - z^i_n\|_E\right).$$ (126)

Then, we write the pervious inequality in the following form:

$$\left\langle I'_{\lambda,\mu}(z^i_n), \frac{z}{\|z\|_E} \right\rangle$$
$$\le \frac{\|z_\delta - z^i_n\|_E}{\delta n} + \frac{o\left(\|z_\delta - z^i_n\|_E\right)}{\delta}$$
$$+ \frac{(\xi_n(\varphi_\delta) - 1)}{\delta}\left\langle I'_{\lambda,\mu}(z^i_n) - I'_{\lambda,\mu}(z_\delta), (z^i_n - \varphi_\delta) \right\rangle.$$ (127)

We can find a constant $C > 0$ independent of δ such that

$$\|z_\delta - z^i_n\| \le \delta + C\left(|\xi_n(\varphi_\delta) - 1|\right),$$
$$\lim_{\delta \to 0} \frac{|\xi_n(\varphi_\delta) - 1|}{\delta} \le \|\xi'_n(0)\| \le C.$$ (128)

For a fixed n, let $\delta \to 0$ in (127). Using the fact that

$$\lim_{\delta \to 0} \|z_\delta - z^i_n\|_E = 0,$$ (129)

we obtain

$$\left\langle I'_{\lambda,\mu}(z^i_n), \frac{z}{\|z\|_E} \right\rangle \le \frac{C}{n}.$$ (130)

This implies

$$I_{\lambda,\mu}\left(z_n^i\right) = \theta_{\lambda,\mu}^i + o_n\left(1\right), \quad I_{\lambda,\mu}'\left(z_n^i\right) = o_n\left(1\right) \quad \text{in } E^{-1}. \tag{131}$$

$\square$

Now, we complete the proof of Theorem 2. By Lemmas 12, 14 and Proposition 18, for all $\lambda^{2/(2-q)} + \mu^{2/(2-q)} \in (0, \Lambda^*)$, there exists a sequence $\{z_n^i\} \subset \mathcal{N}_{\lambda,\mu}^i$ and $z_0^i = (u_0^i, v_0^i) \in E$, $1 \leq i \leq k$, such that

$$I_{\lambda,\mu}\left(z_n^i\right) = \theta_{\lambda,\mu}^i + o_n\left(1\right),$$

$$I_{\lambda,\mu}'\left(z_n^i\right) = o_n\left(1\right) \quad \text{in } E^{-1}, \tag{132}$$

$$z_n^i \longrightarrow z_0^i \quad \text{strongly in } E.$$

Moreover, $\{z_n^i\} \subset \mathcal{N}_{\lambda,\mu}^-$, and by Lemma 7 (ii), we get $z_0^i \in \mathcal{N}_{\lambda,\mu}^-$, $u_0^i \not\equiv 0$, $v_0^i \not\equiv 0$ in Ω,

$$\left\|z_0^i\right\|_E > \left(\frac{2-q}{2^*-q}\right)^{1/(2^*-2)} S^{N/4},$$

$$\theta_{\lambda,\mu}^i \geq \theta_{\lambda,\mu}^- > 0 \quad \text{for } i = 1, 2, \ldots, k. \tag{133}$$

Thus, z_0^i is a nontrivial solution of the problem $\left(P_{\lambda,\mu}\right)$ and $I_{\lambda,\mu}(z_0^i) = \theta_{\lambda,\mu}^i$ for $i = 1, 2, \ldots, k$. Set $u_+ = \max\{u, 0\}$ and $v_+ = \max\{v, 0\}$. Replace the terms $\int_\Omega h(x)|u|^\alpha|v|^\beta dx$ and $\int_\Omega (\lambda f(x)|u|^q + \mu g(x)|v|^q)dx$ of the functional $I_{\lambda,\mu}$ by $\int_\Omega h(x)u_+^\alpha v_+^\beta dx$ and $\int_\Omega (\lambda f(x)u_+^q + \mu g(x)v_+^q)dx$, respectively. It then follows that z_0^i is a nonnegative solution of the problem $\left(P_{\lambda,\mu}\right)$. Applying the maximum principle [19], z_0^i is a positive solution of the problem $\left(P_{\lambda,\mu}\right)$. Since $Q_i(z_0^i) < r_0/3$,

$$z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^+, \qquad z_0^i \in \mathcal{N}_{\lambda,\mu}^i \subset \mathcal{N}_{\lambda,\mu}^- \quad \text{for } i = 1, 2, \ldots, k, \tag{134}$$

where $z_{\lambda,\mu}^1$ is a positive solution of equation $\left(P_{\lambda,\mu}\right)$ as in Theorem 1. From Lemma 13, we conclude that $\mathcal{N}_{\lambda,\mu}^i$ are disjoint for $i = 1, 2, \ldots, k$. This implies that z_0^i $(1 \leq i \leq k)$ and $z_{\lambda,\mu}^1$ are distinct positive solutions of the problem $\left(P_{\lambda,\mu}\right)$.

References

[1] C. O. Alves, D. C. de Morais Filho, and M. A. S. Souto, "On systems of elliptic equations involving subcritical or critical Sobolev exponents," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 42, no. 5, pp. 771–787, 2000.

[2] K. J. Brown and T.-F. Wu, "A semilinear elliptic system involving nonlinear boundary condition and sign-changing weight function," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 2, pp. 1326–1336, 2008.

[3] D. Cao and P. Han, "High energy positive solutions of Neumann problem for an elliptic system of equations with critical nonlinearities," *Calculus of Variations and Partial Differential Equations*, vol. 25, no. 2, pp. 161–185, 2006.

[4] J. Chabrowski and J. Yang, "On the Neumann problem for an elliptic system of equations involving the critical Sobolev exponent," *Colloquium Mathematicum*, vol. 90, no. 1, pp. 19–35, 2001.

[5] C.-M. Chu and C.-L. Tang, "Existence and multiplicity of positive solutions for semilinear elliptic systems with Sobolev critical exponents," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 11, pp. 5118–5130, 2009.

[6] P. Han, "Multiple positive solutions of nonhomogeneous elliptic systems involving critical Sobolev exponents," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 4, pp. 869–886, 2006.

[7] T.-S. Hsu, "Multiple positive solutions for a critical quasilinear elliptic system with concave-convex nonlinearities," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2688–2698, 2009.

[8] T.-S. Hsu and H.-L. Lin, "Multiple positive solutions for a critical elliptic system with concave-convex nonlinearities," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 139, no. 6, pp. 1163–1177, 2009.

[9] Y. Shen and J. Zhang, "Multiplicity of positive solutions for a semilinear p-Laplacian system with Sobolev critical exponent," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 4, pp. 1019–1030, 2011.

[10] T.-F. Wu, "The Nehari manifold for a semilinear elliptic system involving sign-changing weight functions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 6, pp. 1733–1745, 2008.

[11] H.-l. Lin, "Multiple positive solutions for semilinear elliptic systems," *Journal of Mathematical Analysis and Applications*, vol. 391, no. 1, pp. 107–118, 2012.

[12] G. Tarantello, "On nonhomogeneous elliptic equations involving critical Sobolev exponent," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 9, no. 3, pp. 281–304, 1992.

[13] G. Tarantello, "Multiplicity results for an inhomogeneous Neumann problem with critical exponent," *Manuscripta Mathematica*, vol. 81, no. 1-2, pp. 57–78, 1993.

[14] T.-S. Hsu, "Multiplicity results for p-Laplacian with critical nonlinearity of concave-convex type and sign-changing weight functions," *Abstract and Applied Analysis*, vol. 2009, Article ID 652109, 24 pages, 2009.

[15] K. J. Brown and Y. Zhang, "The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function," *Journal of Differential Equations*, vol. 193, no. 2, pp. 481–499, 2003.

[16] T.-F. Wu, "On semilinear elliptic equations involving concave-convex nonlinearities and sign-changing weight function," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 253–270, 2006.

[17] H. Brézis and E. Lieb, "A relation between pointwise convergence of functions and convergence of functionals," *Proceedings of the American Mathematical Society*, vol. 88, no. 3, pp. 486–490, 1983.

[18] T.-S. Hsu, "Existence and multiplicity of positive solutions to a perturbed singular elliptic system deriving from a strongly coupled critical potential," *Boundary Value Problems*, vol. 116, 14 pages, 2012.

[19] J. L. Vázquez, "A strong maximum principle for some quasilinear elliptic equations," *Applied Mathematics and Optimization*, vol. 12, no. 3, pp. 191–202, 1984.

[20] P. Han, "The effect of the domain topology on the number of positive solutions of an elliptic system involving critical Sobolev

exponents," *Houston Journal of Mathematics*, vol. 32, no. 4, pp. 1241–1257, 2006.

[21] H. Brézis and L. Nirenberg, "Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents," *Communications on Pure and Applied Mathematics*, vol. 36, no. 4, pp. 437–477, 1983.

[22] X. Cheng and S. Ma, "Existence of three nontrivial solutions for elliptic systems with critical exponents and weights," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 10, pp. 3537–3548, 2008.

[23] M. Struwe, *Variational Methods*, vol. 34 of *Results in Mathematics and Related Areas (3)*, Springer, Berlin, Germany, 2nd edition, 1996.

[24] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and Their Applications, 24, Birkhäuser, Boston, Mass, USA, 1996.

[25] G. Bianchi, J. Chabrowski, and A. Szulkin, "On symmetric solutions of an elliptic equation with a nonlinearity involving critical Sobolev exponent," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 25, no. 1, pp. 41–59, 1995.

[26] I. Ekeland, "On the variational principle," *Journal of Mathematical Analysis and Applications*, vol. 47, pp. 324–353, 1974.