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Research Article

Stability and Hopf Bifurcation Analysis for a Gause-Type Predator-Prey System with Multiple Delays

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This paper is concerned with a Gause-type predator-prey system with two delays. Firstly, we study the stability and the existence of Hopf bifurcation at the coexistence equilibrium by analyzing the distribution of the roots of the associated characteristic equation. A group of sufficient conditions for the existence of Hopf bifurcation is obtained. Secondly, an explicit formula for determining the stability and the direction of periodic solutions that bifurcate from Hopf bifurcation is derived by using the normal form theory and center manifold argument. Finally, some numerical simulations are carried out to illustrate the main theoretical results.

1. Introduction

Multispecies predator-prey models have been studied by many scholars [1–7]. Guo and Jiang [7] studied the following three-species food-chain system:

$$\begin{aligned}\frac{dx(t)}{dt} &= \alpha x(t)\left(1-\frac{x(t)}{k}\right)-\frac{\beta x(t-\tau)\,y(t)}{1+px(t-\tau)},\\ \frac{dy(t)}{dt} &= y(t)\left(-h+\frac{e\beta x(t)}{1+px(t)}\right)-ry(t)\,z(t),\\ \frac{dz(t)}{dt} &= z(t)\left(-s+mry(t)\right),\end{aligned}\tag{1}$$

where $x(t)$, $y(t)$, and $z(t)$ are the population densities of the prey, the predator and the top predator at time t. The prey grows with intrinsic growth rate α and carrying capacity k in the absence of predation. The predator captures the prey with capture rate β and Holling type II functional response $x/(1+px)$. The top predator captures its prey (the predator) with capture rate r and Holling type I functional response ry. The predator and the top predator contribute to their growth with the conversion rates e and m, respectively. The parameters h and s are the death rates of the predator and the top predator, respectively. All the parameters α, β, e, h, k, m, p, r, s and in system (1) are assumed to be positive. The constant $\tau \geq 0$ represents the time delay due to the gestation of the prey. Guo and Jiang [7] investigated the bifurcation phenomenon and the properties of periodic solutions of system (1).

Predator-prey systems with single delay as system (1) have been investigated extensively [8–12]. However, there are some papers on the bifurcations of a population dynamics with multiple delays [13–16]. Gakkhar and Singh [15] studied the effects of two delays on a delayed predator-prey system with modified Leslie-Gower and Holling type II functional response and established the existence of periodic solutions via Hopf bifurcation with respect to both delays. Motivated by the work of Guo and Jiang [7] and Gakkhar and Singh [15], we consider the following predator-prey system with two delays:

$$\begin{aligned}\frac{dx(t)}{dt} &= \alpha x(t)\left(1-\frac{x(t)}{k}\right)-\frac{\beta x(t)\,y(t)}{1+px(t)},\\ \frac{dy(t)}{dt} &= y(t)\left(-h+\frac{e\beta x(t-\tau_1)}{1+px(t-\tau_1)}\right)-ry(t)\,z(t),\\ \frac{dz(t)}{dt} &= z(t)\left(-s+mry(t-\tau_2)\right),\end{aligned}\tag{2}$$

where τ_1 denotes the time delay due to the gestation of the predator and τ_2 denotes the time delay due to the gestation of the top predator.

This paper is organized as follows. In the next section, we will consider the stability of the positive equilibrium of system (2) and the existence of local Hopf bifurcation at the positive equilibrium. In Section 3, we can determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions from the Hopf bifurcation. Some numerical simulations are also given to illustrate the theoretical prediction in Section 4.

2. Local Stability and Hopf Bifurcation

Because we are only interested in the case in which the species can coexist, then we only consider the positive equilibrium of system (2). It is not difficult to know that if conditions (H_1) : $\beta s < \alpha mr$ and (H_2) : $e\beta x_* > h(1+px_*)$ hold, then system (2) has a unique positive equilibrium $E_*(x_*, y_*, z_*)$, where

$$x_* = \frac{-B+\sqrt{B^2-4AC}}{2}, \qquad y_* = \frac{s}{mr}, \qquad z_* = \frac{1}{r}\left(\frac{e\beta x_*}{1+px_*} - h\right), \tag{3}$$

where

$$A = \alpha mpr, \qquad B = \alpha mr(1-pk), \qquad C = k(\beta s - \alpha mr). \tag{4}$$

Let $x(t) = z_1(t) + x_*$, $y(t) = z_2(t) + y_*$, and $z(t) = z_3(t) + z_*$ and still denote $z_1(t)$, $z_2(t)$, and $z_3(t)$ by $x(t)$, $y(t)$, and $z(t)$ respectively. Then system (2) can be transformed to the following form:

$$\begin{aligned} \frac{dx(t)}{dt} &= a_{11}x(t) + a_{12}y(t) + f_1, \\ \frac{dy(t)}{dt} &= a_{23}z(t) + b_{11}x(t-\tau_1) + f_2, \\ \frac{dz(t)}{dt} &= c_{32}y(t-\tau_2) + f_3, \end{aligned} \tag{5}$$

where

$$a_{11} = \alpha - \frac{2\alpha}{k}x_* - \frac{\beta y_*}{(1+px_*)^2}, \qquad a_{12} = -\frac{\beta x_*}{1+px_*},$$
$$a_{23} = -ry_*, \qquad b_{11} = \frac{\beta e y_*}{(1+px_*)^2}, \qquad c_{32} = mrz_*,$$

$$\begin{aligned} f_1 &= a_{13}x^2(t) + a_{14}x(t)y(t) + a_{15}x^3(t) + a_{16}x^2(t)y(t) + \cdots, \\ f_2 &= a_{24}y(t)z(t) + a_{25}x(t-\tau_1)y(t) + a_{26}x^2(t-\tau_1) + a_{27}x^2(t-\tau_1)y(t) + a_{28}x^3(t-\tau_1) + \cdots, \\ f_3 &= c_{33}y(t-\tau_2)z(t), \end{aligned}$$

$$a_{13} = -\frac{\alpha}{k} + \frac{\beta p y_*}{(1+px_*)^3}, \qquad a_{14} = -\frac{\beta}{(1+px_*)^2},$$
$$a_{15} = -\frac{\beta p^2 y_*}{(1+px_*)^4}, \qquad a_{16} = \frac{\beta p}{(1+px_*)^3},$$
$$a_{24} = -r, \qquad a_{25} = \frac{\beta e}{(1+px_*)^2}, \qquad a_{26} = -\frac{\beta e p y_*}{(1+px_*)^3},$$
$$a_{27} = -\frac{\beta e p}{(1+px_*)^3}, \qquad a_{28} = \frac{\beta e p^2 y_*}{(1+px_*)^4}, \qquad c_{33} = mr. \tag{6}$$

The linearized system of system (5) is

$$\begin{aligned} \frac{dx(t)}{dt} &= a_{11}x(t) + a_{12}y(t), \\ \frac{dy(t)}{dt} &= a_{23}z(t) + b_{11}x(t-\tau_1), \\ \frac{dz(t)}{dt} &= c_{32}y(t-\tau_2). \end{aligned} \tag{7}$$

Then the associated characteristic equation of system (7) at the origin is of the form

$$\lambda^3 + p_2\lambda^2 + m_1\lambda e^{-\lambda\tau_1} + (n_\lambda + n_0)e^{-\lambda\tau_2} = 0, \tag{8}$$

where

$$p_2 = -a_{11}, \qquad m_1 = -a_{12}b_{11}, \qquad n_1 = -a_{23}c_{32}, \qquad n_0 = a_{11}a_{23}c_{32}. \tag{9}$$

Case 1. One has $\tau_1 = \tau_2 = \tau = 0$.

Equation (8) becomes

$$\lambda^3 + p_2\lambda^2 + (m_1 + m_1)\lambda + n_0 = 0. \tag{10}$$

Obviously, if conditions (H_{11}) : $p_2 > 0$ and (H_{12}) : $p_2(m_1 + n_1) > n_0$ hold, then all the roots of (10) must have negative real parts. Then, we can conclude that the positive equilibrium $E_*(x_*, y_*, z_*)$ is locally asymptotically stable in the absence of delay.

Case 2. One has $\tau_1 > 0, \tau_2 = 0$.

Equation (8) becomes

$$\lambda^3 + p_2\lambda^2 + n_1\lambda + n_0 + m_1\lambda e^{-\lambda\tau_1} = 0. \tag{11}$$

Letting $\lambda = i\omega_1$ $(\omega_1 > 0)$ be a root of (11), then we have

$$m_1\omega_1 \sin\tau_1\omega_1 = p_2\omega_1^2 - n_0,$$
$$m_1\omega_1 \cos\tau_1\omega_1 = \omega_1^3 - n_1\omega_1. \quad (12)$$

It follows that

$$\omega_1^6 + c_{22}\omega_1^4 + c_{21}\omega_1^2 + c_{20} = 0, \quad (13)$$

where

$$c_{22} = p_2^2 - 2n_1, \qquad c_{21} = n_1^2 - m_1^2 - 2n_0 p_2, \qquad c_{20} = n_0^2. \quad (14)$$

Letting $\omega_1^2 = v_1$, then (13) becomes

$$v_1^3 + c_{22}v_1^2 + c_{21}v_1 + c_{20} = 0. \quad (15)$$

Obviously, $c_{20} \geq 0$. Thus, we assume that (15) has at least one positive solution. Without loss of generality, we assume that it has three positive roots, which are denoted as v_{11}, v_{12}, and v_{13}. Then (13) has three positive roots $\omega_{1k} = \sqrt{v_{1k}}$, $k = 1, 2, 3$.

From (12), we can get

$$\tau_{1k}^{(j)} = \frac{1}{\omega_{1k}} \arccos\frac{\omega_{1k}^2 - n_1}{m_1} + \frac{2j\pi}{\omega_{1k}}, \quad j = 0, 1, \ldots, k = 1, 2, 3. \quad (16)$$

Then, we denote

$$\tau_{10} = \min\left\{\tau_{1k}^{(0)}\right\}, \quad k = 1, 2, 3, \ \omega_{10} = \omega_{1k_0}. \quad (17)$$

Next, we verify the transversality condition. Differentiating the two sides of (11) with respect to τ_1 and noticing that λ is a function of τ_1, we can get

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = -\frac{3\lambda^2 + 2p_2\lambda + n_1}{\lambda\left(\lambda^3 + p_2\lambda^2 + n_1\lambda + n_0\right)} + \frac{1}{\lambda^2} - \frac{\tau_1}{\lambda}. \quad (18)$$

Therefore

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\lambda=i\omega_{10}} = \frac{3\omega_{10}^4 + 2\left(p_2^2 - 2n_1\right)\omega_{10}^2 + n_1^2 - 2n_0p_2}{\left(\omega_{10}^3 - n_1\omega_{10}\right)^2 + \left(n_0 - p_2\omega_{10}^2\right)^2} - \frac{m_1^2}{m_1^2\omega_{10}^2}. \quad (19)$$

From (13), we have

$$\left(\omega_{10}^3 - n_1\omega_{10}\right)^2 + \left(n_0 - p_2\omega_{10}^2\right)^2 = m_1^2\omega_{10}^2. \quad (20)$$

Therefore

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\lambda=i\omega_{10}} = \frac{f_1'\left(v_{1*}\right)}{\left(\omega_{10}^3 - n_1\omega_{10}\right)^2 + \left(n_0 - p_2\omega_{10}^2\right)^2}, \quad (21)$$

with

$$f_1\left(v_1\right) = v_1^3 + c_{22}v_1^2 + c_{21}v_1 + c_{20}, \quad v_{1*} = \omega_{10}^2. \quad (22)$$

Obviously, if (H_{22}) : $f_1'(v_{1*}) \neq 0$, then $\operatorname{Re}[d\lambda/d\tau_1]^{-1}_{\lambda=i\omega_{10}} \neq 0$. Thus, if condition (H_{22}) holds, the transversality condition is satisfied. In conclusion, we have the following results.

Theorem 1. *Suppose that conditions (H_{21})-(H_{22}) hold. The positive equilibrium $E_*(x_*, y_*, z_*)$ of system (2) is asymptotically stable for $\tau_1 \in [0, \tau_{10})$ and unstable when $\tau_1 > \tau_{10}$. And system (2) undergoes a Hopf bifurcation at $E_*(x_*, y_*, z_*)$ when $\tau_1 = \tau_{10}$.*

Case 3. One has $\tau_1 = 0, \tau_2 > 0$.

Equation (8) becomes

$$\lambda^3 + p_2\lambda^2 + m_1\lambda + \left(n_1\lambda + n_0\right)e^{-\lambda\tau_2} = 0. \quad (23)$$

Let $\lambda = i\omega_2$ $(\omega_2 > 0)$ be a root of (23), then we have

$$n_1\omega_2 \sin\tau_2\omega_2 + n_0\cos\tau_2\omega_2 = p_2\omega_2^2,$$
$$n_1\omega_2 \cos\tau_2\omega_2 - n_0\sin\tau_2\omega_2 = \omega_2^3 - m_1\omega_2, \quad (24)$$

which follows that

$$\omega_2^6 + c_{32}\omega_2^4 + c_{31}\omega_2^2 + c_{30} = 0, \quad (25)$$

with

$$c_{32} = p_2^2 - 2m_1, \qquad c_{31} = m_1^2 - n_1^2, \qquad c_{30} = -n_0^2. \quad (26)$$

Let omega$_2^2 = v_2$, then (24) becomes

$$v_2^3 + c_{32}v_2^2 + c_{31}v_2 + c_{30} = 0. \quad (27)$$

Similar as in Case 2, we assume that (H_{31}) : (27) has at least one positive solution. Without loss of generality, we assume that it has three positive roots, which are denoted by v_{21}, v_{22} and v_{23}. Then (25) has three positive roots $\omega_{2k} = \sqrt{v_{2k}}$, $k = 1, 2, 3$.

From (24), we get

$$\tau_{2k}^{(j)} = \frac{1}{\omega_{2k}} \arccos\frac{n_1\omega_{2k}^4 + \left(n_0p_2 - m_1n_1\right)\omega_{2k}^2}{n_1^2\omega_{2k}^2 + n_0^2} + \frac{2j\pi}{\omega_{2k}}, \quad j = 0, 1, 2, \ldots, \quad k = 1, 2, 3. \quad (28)$$

Then, we denote

$$\tau_{20} = \min\left\{\tau_{2k}^{(0)}\right\}, \quad k = 1, 2, 3, \ \omega_{20} = \omega_{2k_0}. \quad (29)$$

Similar as in Case 2, we know that if condition (H_{32}) : $f_2'(v_{2*}) \neq 0$ holds, where

$$f_2\left(v_2\right) = v_2^3 + c_{32}v_2^2 + c_{31}v_2 + c_{30}, v_{2*} = \omega_{20}^2, \quad (30)$$

then, $\operatorname{Re}[d\lambda/d\tau_2]^{-1}_{\lambda=i\omega_{20}} \neq 0$. Namely, if condition (H_{32}) holds, the transversality condition is satisfied. Therefore, we have the following results. Therefore, we have the following theorem.

Theorem 2. *Suppose that conditions (H_{31})-(H_{32}) hold. The positive equilibrium $E_*(x_*, y_*, z_*)$ of system (2) is asymptotically stable for $\tau_2 \in [0, \tau_{20})$ and unstable when $\tau_2 > \tau_{20}$. And system (2) undergoes a Hopf bifurcation at $E_*(x_*, y_*, z_*)$ when $\tau_2 = \tau_{20}$.*

Case 4. One has $\tau_1 > 0, \tau_2 \in [0, \tau_{20})$.

It is considered that with (8), τ_2 in its stable interval and τ_1 is considered as a parameter.

Let $\lambda = i\omega$ $(\omega > 0)$ be the root of (8). Separating real and imaginary parts leads to

$$\begin{aligned} m_1\omega\sin\tau_1\omega &= p_2\omega^2 - n_0\cos\tau_2\omega - n_1\omega\sin\tau_2\omega, \\ m_1\omega\cos\tau_1\omega &= \omega^3 + n_0\sin\tau_2\omega - n_1\omega\cos\tau_2\omega. \end{aligned} \tag{31}$$

Eliminating τ_1 leads to

$$c_{40}(\omega) + c_{41}\cos\tau_2\omega + c_{42}\sin\tau_2\omega = 0, \tag{32}$$

where

$$\begin{aligned} c_{40}(\omega) &= \omega^6 + p_2^2\omega^4 + \left(n_1^2 + m_1^2\right)\omega^2 + n_0^2, \\ c_{41}(\omega) &= -2\left(n_1\omega^4 + n_0 p_2\omega^2\right), \\ c_{42}(\omega) &= 2\left(n_0 - n_1 p_2\right)\omega^3. \end{aligned} \tag{33}$$

Suppose that (H_{41}): (32) has finite positive roots. If condition (H_{41}) holds, we denote the roots of (32) by $\omega_1, \omega_2, \ldots, \omega_n$. For every fixed ω_i $(i = 1, 2, \ldots, n)$, there exists a sequence $\{\tau_{1k}^{(j)} \mid k = 1, 2, \ldots, n, j = 0, 1, 2, \ldots\}$ satisfying (32).

From (31), we can get

$$\tau_{1k}^{(j)} = \frac{1}{\omega_k}\arccos\frac{\omega_k^3 + n_0\sin\tau_2\omega_k - n_1\omega_k\cos\tau_2\omega_k}{m_1\omega_k} + \frac{2j\pi}{\omega_k}, \quad k = 1, 2, \ldots, \quad n, j = 0, 1, 2, \ldots. \tag{34}$$

Let

$$\tau_1^* = \tau_{1k}^{(0)} = \min\left\{\tau_{1k}^{(0)}, k = 1, 2, \ldots, n\right\}, \tag{35}$$

when $\tau_1 = \tau_1^*$ (8) has a pair of purely imaginary roots $\pm i\omega_1^*$ for $\tau_2 \in [0, \tau_{20})$.

To verify the transversality condition of Hopf bifurcation, differentiating (8) with respect to τ_1 and substituting $\tau_1 = \tau_1^*$, we can get

$$\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau_1}\right)\right]_{\lambda = i\omega_1^*}^{-1} = \frac{P_R Q_R + P_I Q_I}{Q_R^2 + Q_I^2}, \tag{36}$$

where

$$\begin{aligned} P_R &= \left(n_1 - n_0\tau_2\right)\cos\tau_2\omega_1^* - n_1\omega_1^*\tau_2\sin\tau_2\omega_1^* \\ &\quad - 3\left(\omega_1^*\right)^2 + m_1\cos\tau_1^*\omega_1^*, \\ P_I &= \left(n_1 - n_0\tau_2\right)\sin\tau_2\omega_1^* - n_1\omega_1^*\tau_2\cos\tau_2\omega_1^* \\ &\quad + 2p_2\omega_1^* - m_1\sin\tau_1^*\omega_1^*, \\ Q_R &= -m_1\left(\omega_1^*\right)^2\cos\tau_1^*\omega_1^*, \qquad Q_I = m_1\left(\omega_1^*\right)^2\sin\tau_1^*\omega_1^*. \end{aligned} \tag{37}$$

Clearly, if condition (H_{42}) : $P_R Q_R + P_I Q_I \neq 0$ holds, then $[\mathrm{Re}(d\lambda/d\tau)]_{\lambda=i\omega_1^*}^{-1} \neq 0$. Namely, if condition (H_{42}) holds, the transversality condition is satisfied. Therefore, we have the following results. Thus, we have the following theorem.

Theorem 3. *Suppose that conditions (H_{41})-(H_{42}) hold and $\tau_2 \in [0, \tau_{20})$. The positive equilibrium $E_*(x_*, y_*, z_*)$ of system (2) is asymptotically stable for $\tau_1 \in [0, \tau_{1*})$ and unstable when $\tau_1 > \tau_{1*}$. And system (2) undergoes a Hopf bifurcation at $E_*(x_*, y_*, z_*)$ when $\tau_1 = \tau_{1*}$.*

Case 5. One has $\tau_2 > 0, \tau_1 \in [0, \tau_{10})$.

We consider (8) with τ_1 in its stable interval, regarding τ_2 as a parameter.

Let $\lambda = i\omega(\omega > 0)$ be a root of (8). Then we get

$$\begin{aligned} n_1\omega\sin\tau_2\omega + n_0\cos\tau_2\omega &= p_2\omega^2 - m_1\omega\sin\tau_1\omega, \\ n_1\omega\cos\tau_2\omega - n_0\sin\tau_2\omega &= \omega^3 - m_1\omega\cos\tau_1\omega. \end{aligned} \tag{38}$$

It follows that

$$c_{50}(\omega) + c_{51}(\omega)\cos\tau_1\omega + c_{52}(\omega)\sin\tau_1\omega = 0, \tag{39}$$

where

$$\begin{aligned} c_{50}(\omega) &= \omega^6 + p_2^2\omega^4 + \left(m_1^2 - n_1^2\right)\omega^2 - n_0^2, \\ c_{51}(\omega) &= -2m_1\omega^4, \qquad c_{52}(\omega) = -2m_1 p_2\omega^3. \end{aligned} \tag{40}$$

Similar as in Case 4, we suppose that (H_{51}) : (39) has finite positive roots. And we denote the roots of (39) by $\omega_1, \omega_2, \ldots, \omega_n$. The corresponding critical value of τ_2 is

$$\tau_{2k}^{(j)} = \frac{n_1\omega_k^2 + n_0 p_2\omega_k^2 - m_1 n_1\omega_k^2\cos\tau_1\omega_k - m_1 n_0\omega_k\sin\tau_1\omega_k}{n_2\omega_k^2 + n_0^2}, \quad k = 1, 2, \ldots, n, j = 0, 1, 2, \ldots. \tag{41}$$

Let

$$\tau_2^* = \tau_{2k}^{(0)} = \min\left\{\tau_{2k}^{(0)}, k = 1, 2, \ldots, n\right\}, \tag{42}$$

when $\tau_2 = \tau_2^*$ (8) has a pair of purely imaginary roots $\pm i\omega_2^*$ for $\tau_1 \in [0, \tau_{10})$.

Similar as in Case 4, we give the following assumption $(H_{52}) : P_R' Q_R' + P_I' Q_I' \neq 0$, where

$$\begin{aligned} P_R' &= m_1\cos\tau_1\omega_2^* - \tau_1 m_1\omega_2^*\sin\tau_1\omega_2^* - 3\left(\omega_2^*\right)^2 + n_1\cos\tau_2^*\omega_2^*, \\ P_I' &= -m_1\sin\tau_1\omega_2^* - \tau_1 m_1\omega_2^*\cos\tau_1\omega_2^* + 2p_2\omega_2^* - n_1\sin\tau_2^*\omega_2^*, \\ Q_R' &= n_0\omega_2^*\sin\tau_2^*\omega_2^* - n_1\left(\omega_2^*\right)^2\cos\tau_2^*\omega_2^*, \\ Q_I' &= n_0\omega_2^*\cos\tau_2^*\omega_2^* + n_1\left(\omega_2^*\right)^2\sin\tau_2^*\omega_2^*. \end{aligned} \tag{43}$$

Therefore, if condition (H_{52}) : $P_R' Q_R' + P_I' Q_I' \neq 0$ holds, then we can get $\mathrm{Re}[d\lambda/d\tau_2]_{\lambda=i\omega_2^*}^{-1}$. That is, the transversality condition is satisfied. Hence, we have the following theorem.

Theorem 4. *Suppose that conditions* (H_{51})-(H_{52}) *hold and* $\tau_1 \in [0, \tau_{10})$. *The positive equilibrium* $E_*(x_*, y_*, z_*)$ *of system* (2) *is asymptotically stable for* $\tau_2 \in [0, \tau_2^*)$ *and unstable when* $\tau_2 > \tau_2^*$. *And system* (2) *undergoes a Hopf bifurcation at* $E_*(x_*, y_*, z_*)$ *when* $\tau_2 = \tau_2^*$.

3. Direction and Stability of the Hopf Bifurcation

In this section, we will employ the normal form method and center manifold theorem introduced by Hassard et al. [17] to determine the direction of Hopf bifurcation and stability of the bifurcated periodic solutions of system (2) with respect to τ_1 for $\tau_2 \in (0, \tau_{20})$. Without loss of generality, we assume that $\tau_2' < \tau_1^*$, where $\tau_2' \in (0, \tau_{20})$.

Let $\tau_1 = \tau_1^* + \mu, \mu \in R$. Then $\mu = 0$ is the Hopf bifurcation value of system (2). Rescaling the time delay $t \to (t/\tau_1)$, then system (2) can be rewritten as

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \quad (44)$$

where

$$L_\mu \phi = (\tau_{1*} + \mu)\left(A\phi(0) + C\phi\left(-\frac{\tau_2'}{\tau_1^*}\right) + B\phi(-1)\right),$$
$$F(\mu, \phi) = (\tau_1^* + \mu)(F_1, F_2, F_3)^T \quad (45)$$

With

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ b_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{32} & 0 \end{pmatrix},$$

$$F_1 = a_{13}\phi_1^2(0) + a_{14}\phi_1(0)\phi_2(0) + a_{15}\phi_1^3(0)\phi_1(0) + a_{18}\phi_1^2(0)\phi_2(0) + \cdots, \quad (46)$$
$$F_2 = a_{24}\phi_2(0)\phi_3(0) + a_{25}\phi_1(-1)\phi_2(0) + a_{26}\phi_1^2(-1) + a_{27}\phi_1^2(-1)\phi_2(0) + a_{28}\phi_1^3(-1)\cdots,$$
$$F_3 = c_{33}\phi_2\left(-\frac{\tau_1'}{\tau_1^*}\right)\phi_3(0).$$

By Riesz representation theorem, there exists a 3×3 matrix function $\eta(\theta, \mu) : [-1, 0] \to R^3$ whose elements are of bounded variation, such that

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \quad \phi \in C\left([-1, 0], R^3\right). \quad (47)$$

In fact, we can choose

$$\eta(\theta, \mu) = \begin{cases} (\tau_1^* + \mu)(A + B + C), & \theta = 0, \\ (\tau_1^* + \mu)(B + C), & \theta \in \left[-\dfrac{\tau_2'}{\tau_1^*}, 0\right), \\ (\tau_1^* + \mu)C, & \theta \in \left(-1, -\dfrac{\tau_2'}{\tau_1^*}\right), \\ 0, & \theta = -1. \end{cases} \quad (48)$$

For $\phi \in C([-1, 0], R^3)$, we define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \displaystyle\int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), & \theta = 0, \end{cases}$$
$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu, \phi), & \theta = 0. \end{cases} \quad (49)$$

Then system (44) can be transformed into the following operator equation:

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \quad (50)$$

The adjoint operator A^* of A is defined by

$$A^*(\mu)\varphi = \begin{cases} -\dfrac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \displaystyle\int_{-1}^{0} d\eta^T(s, \mu)\varphi(-s), & s = 0, \end{cases} \quad (51)$$

associated with a bilinear form

$$\langle \varphi, \phi \rangle = \overline{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{\varphi}(\xi - \theta)\, d\eta(\theta)\phi(\xi)\, d\xi, \quad (52)$$

where $\eta(\theta) = \eta(\theta, 0)$.

From the above discussion, we know that $\pm i\tau_1^*\omega_1^*$ are the eigenvalues of $A(0)$ and they are also eigenvalues of $A^*(0)$. We assume that $\rho(\theta) = (1, \rho_2, \rho_3)^T e^{i\tau_1^*\omega_1^*\theta}$ is the eigenvector belonging to the eigenvalue $+i\tau_1^*\omega_1^*$ and $\rho^*(\theta) = D(1, \rho_2^*, \rho_3^*)e^{i\tau_1^*\omega_1^* s}$ is the eigenvector belonging to the eigenvalue $-i\tau_1^*\omega_1^*$. Then, by a simple computation, we can obtain

$$\rho_2 = \frac{i\omega_1^* - a_{11}}{a_{12}}, \qquad \rho_3 = \frac{i\omega_1^*(i\omega_1^* - a_{11}) - b_{11}e^{-i\omega_1^*\omega_1^*}}{a_{12}a_{23}},$$
$$\rho_2^* = -\frac{i\omega_1^* + a_{11}}{b_{11}e^{i\tau_1^*\omega_1^*}}, \qquad \rho_3^* = \frac{i\omega_1^*(i\omega_1^* + a_{11})e^{-i\tau_1^*\omega_1^*} - a_{12}b_{11}}{b_{11}c_{32}e^{i\tau_2'\omega_1^*}},$$
$$\overline{D} = \Big[1 + \rho_2\rho_2^* + \rho_3\rho_3^* + b_{11}\overline{\rho}_2^*\tau_1^* e^{-i\tau_1^*\omega_1^*} + c_{32}\rho_2\overline{\rho}_3^*\tau_2' e^{-i\tau_2'\omega_1^*}\Big]^{-1}. \quad (53)$$

Then we have $\langle \rho^*, \rho \rangle = 1$.

Next, we get the coefficients used to determine the important quantities of the periodic solution by using a computation process similar to that in [18]:

$$\begin{aligned} g_{20} = {} & \frac{2\tau_1^*}{\overline{D}} \Big[a_{13} + a_{14}\rho^{(2)}(0) \\ & + \overline{\rho}_2^* \left(a_{24}\rho^{(2)}(0)\rho^{(3)}(0) + a_{25}\rho^{(1)}(-1)\rho^{(2)}(0) + a_{26}\left(\rho^{(1)}(-1)\right)^2 \right) \\ & + \overline{\rho}_3^* c_{33}\rho^{(2)}\left(-\frac{\tau_2'}{\tau_1^*}\right)\rho^{(3)}(0) \Big], \\ g_{11} = {} & \frac{\tau_1^*}{\overline{D}} \Big[2a_{13} + a_{14}\left(\rho^{(2)}(0) + \overline{\rho}^{(2)}(0)\right) \\ & + \overline{\rho}_2^* \Big(a_{24}\left(\rho^{(2)}(0)\overline{\rho}^{(3)}(0) + \overline{\rho}^{(2)}(0)\rho^{(3)}(0)\right) + a_{25}\left(\rho^{(1)}(-1)\overline{\rho}^{(2)}(0) + \overline{\rho}^{(1)}(-1)\rho^{(2)}(0)\right) \\ & + 2a_{26}\rho^{(1)}(-1)\overline{\rho}^{(1)}(-1) \Big) \\ & + c_{33}\overline{\rho}_3^* \left(\rho^{(2)}\left(-\frac{\tau_2'}{\tau_1^*}\right)\overline{\rho}^{(3)}(0) + \overline{\rho}^{(2)}\left(-\frac{\tau_2'}{\tau_1^*}\right)\rho^{(3)}(0) \right) \Big], \\ g_{02} = {} & \frac{2\tau_1^*}{\overline{D}} \Big[a_{13} + a_{14}\overline{\rho}^{(2)}(0) \\ & + \overline{\rho}_2^* \left(a_{24}\overline{\rho}^{(2)}(0)\overline{\rho}^{(3)}(0) + a_{25}\overline{\rho}^{(1)}(-1)\overline{\rho}^{(2)}(0) + a_{26}\left(\overline{\rho}^{(1)}(-1)\right)^2 \right) \\ & + \overline{\rho}_3^* c_{33}\overline{\rho}^{(2)}\left(-\frac{\tau_2'}{\tau_1^*}\right)\overline{\rho}^{(3)}(0) \Big], \\ g_{21} = {} & \frac{2\tau_1^*}{\overline{D}} \Big[a_{13}\left(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right) \\ & + a_{14}\left(W_{11}^{(1)}(0)\rho^{(2)}(0) + \frac{1}{2}W_{20}^{(1)}(0)\overline{\rho}^{(2)}(0) + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0) \right) + 3a_{15} \\ & + a_{16}\left(\overline{\rho}^{(2)}(0) + 2\rho^{(2)}(0)\right) \\ & + \overline{\rho}_2^* \Big(a_{24}\Big(W_{11}^{(2)}(0)\rho^{(3)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\overline{\rho}^{(3)}(0) + W_{11}^{(3)}(0)\rho^{(2)}(0) + \frac{1}{2}W_{20}^{(3)}(0)\overline{\rho}^{(2)}(0) \Big) \\ & + a_{25}\Big(W_{11}^{(1)}(-1)\rho^{(2)}(0) + \frac{1}{2}W_{20}^{(1)}(-1)\overline{\rho}^{(2)}(0) + W_{11}^{(2)}(0)\rho^{(2)}(-1) + \frac{1}{2}W_{20}^{(2)}(0)\overline{\rho}^{(1)}(-1) \Big) \\ & + a_{26}\left(2W_{11}^{(1)}(-1)\rho^{(1)}(-1) + W_{20}^{(1)}(-1)\overline{\rho}^{(1)}(-1) \right) \\ & + a_{27}\left(\left(\rho^{(1)}(-1)\right)^2 + 2\rho^{(1)}(-1)\rho^{(2)}(0)\overline{\rho}^{(1)}(-1) \right) \\ & + 3a_{28}\left(\rho^{(1)}(-1)\right)^2\overline{\rho}^{(1)}(-1) \Big) \\ & + c_{33}\overline{\rho}_3^* \Big(W_{11}^{(2)}\left(-\frac{\tau_2'}{\tau_1^*}\right)\rho^{(3)}(0) + \frac{1}{2}W_{20}^{(2)}\left(-\frac{\tau_2'}{\tau_1^*}\right)\overline{\rho}^{(3)}(0) \\ & + W_{11}^{(3)}(0)\rho^{(2)}\left(-\frac{\tau_2'}{\tau_1^*}\right) + \frac{1}{2}W_{20}^{(3)}(0)\overline{\rho}^{(2)}\left(-\frac{\tau_2'}{\tau_1^*}\right) \Big) \Big], \end{aligned} \tag{54}$$

with

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}q(0)}{\tau_1^*\omega_1^*}e^{i\tau_1^*\omega_1^*\theta} + \frac{i\overline{g}_{02}\overline{q}(0)}{3\tau_1^*\omega_1^*}e^{-i\tau_1^*\omega_1^*\theta} + E_1 e^{2i\tau_1^*\omega_1^*\theta}, \\ W_{11}(\theta) &= -\frac{ig_{11}q(0)}{\tau_1^*\omega_1^*}e^{i\tau_1^*\omega_1^*\theta} + \frac{i\overline{g}_{11}\overline{q}(0)}{\tau_1^*\omega_1^*}e^{-i\tau_1^*\omega_1^*\theta} + E_2, \end{aligned} \tag{55}$$

where E_1 and E_2 can be computed as the following equations, respectively

$$\begin{aligned} & \begin{pmatrix} 2i\omega_1^* - a_{11} & -a_{12} & 0 \\ -b_{11}e^{-2i\tau_1^*\omega_1^*} & 2i\omega_1^* & -a_{23} \\ 0 & -c_{32}e^{-2i\tau_1^*\omega_1^*} & 2i\omega_1^* \end{pmatrix} E_1 = 2\begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ E_1^{(3)} \end{pmatrix}, \\ & \begin{pmatrix} a_{11} & a_{12} & 0 \\ b_{11} & a_{22} & 0 \\ 0 & c_{32} & 0 \end{pmatrix} E_2 = -\begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \end{pmatrix}, \end{aligned} \tag{56}$$

with

$$E_1^{(1)} = a_{13} + a_{14}\rho^{(2)}(0),$$
$$E_1^{(2)} = a_{24}\rho^{(2)}(0)\rho^{(3)}(0) + a_{25}\rho^{(1)}(-1)\rho^{(2)}(0) + a_{26}\left(\rho^{(1)}(-1))^2\right),$$
$$E_1^{(3)} = c_{33}\rho^{(2)}\left(-\frac{\tau_2'}{\tau_1^*}\right)\rho^{(3)}(0),$$
$$E_2^{(1)} = 2a_{13} + a_{14}\left(\rho^{(2)}(0) + \overline{\rho}^{(2)}(0)\right),$$
$$E_2^{(2)} = a_{24}\left(\rho^{(2)}(0)\overline{\rho}^{(3)}(0) + \overline{\rho}^{(2)}(0)\rho^{(3)}(0)\right) + a_{25}\left(\rho^{(1)}(-1)\overline{\rho}^{(2)}(0) + \overline{\rho}^{(1)}(-1)\rho^{(2)}(0)\right) + 2a_{26}\rho^{(1)}(-1)\overline{\rho}^{(1)}(-1),$$
$$E_2^{(3)} = c_{33}\left(\rho^{(2)}\left(-\frac{\tau_2'}{\tau_1^*}\right)\overline{\rho}^{(3)}(0) + \overline{\rho}^{(2)}\left(-\frac{\tau_2'}{\tau_1^*}\right)\rho^{(3)}(0)\right). \tag{57}$$

Thus, we can calculate the following values:

$$C_1(0) = \frac{i}{2\tau_1^*\omega_1^*}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},$$
$$\mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_1^*)\}},$$
$$\beta_2 = 2\operatorname{Re}\{C_1(0)\},$$
$$T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2\operatorname{Im}\{\lambda'(\tau_1^*)\}}{\tau_1^*\omega_1^*}. \tag{58}$$

Based on the discussion above, we can obtain the following results.

Theorem 5. *If $\mu_2 > 0 (\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical); if $\beta_2 < 0$ ($\beta_2 > 0$), the bifurcating periodic solutions are stable (unstable); if $T_2 > 0$ ($T_2 < 0$), the period of the bifurcating periodic solutions increases (decreases).*

4. Numerical Simulation and Discussion

In this section, we present some numerical simulations to illustrate the analytical results obtained in the previous sections. Let $\alpha = 3$, $k = 2$, $\beta = 2$, $p = 0.3$, $h = 0.2$, $e = 0.4$, $r = 1$, $s = 0.5$, and $m = 0.6$. Then we have the following particular case of system (2):

$$\frac{dx(t)}{dt} = 3x(t)\left(1 - \frac{x(t)}{2}\right) - \frac{2x(t)y(t)}{1+0.3x(t)},$$
$$\frac{dy(t)}{dt} = y(t)\left(-0.2 + \frac{0.8x(t-\tau_1)}{1+0.3x(t-\tau_1)}\right) - y(t)z(t),$$
$$\frac{dz(t)}{dt} = z(t)\left(-0.5 + 0.6y(t-\tau_2)\right), \tag{59}$$

which has a positive equilibrium $E_*(1.1809, 0.8333, 0.4976)$.

For $\tau_1 > 0, \tau_2 = 0$, we have $\omega_{10} = 0.7318$, $\tau_{10} = 1.4887$. From Theorem 1, we know that the positive equilibrium $E_*(1.1809, 0.8333, 0.4976)$ is asymptotically stable for $\tau_1 \in [0, 1.4887)$. As can be seen from Figure 1, if $\tau_1 = 1.38 \in [0, 1.4887)$, $E_*(1.1809, 0.8333, 0.4976)$ is asymptotically stable. However, if $\tau_1 = 1.498 > \tau_{10} = 1.4887$, then $E_*(1.1809, 0.8333, 0.4976)$ is unstable and system (59) undergoes a Hopf bifurcation at $E_*(1.1809, 0.8333, 0.4976)$, and a family of periodic solutions bifurcate from the positive equilibrium $E_*(1.1809, 0.8333, 0.4976)$. This property can be illustrated by Figure 2. For $\tau_2 > 0, \tau_1 = 0$, by a simple computation, we can easily get $\omega_{20} = 0.4737$, $\tau_{20} = 1.7990$. The corresponding waveform and the phase plots are shown in Figures 3 and 4.

For $\tau_1 > 0$ and $\tau_2 = 0.8 \in [0, \tau_{20})$, we get $\omega_1^* = 0.6698$, $\tau_1^* = 0.8095$. That is, when τ_1 increases from zero to the critical value τ_1^*, the positive equilibrium $E_*(1.1809, 0.8333, 0.4976)$ is asymptotically stable; then it will lose stability, and a Hopf bifurcation occurs once $\tau_1 > \tau_1^* = 0.8095$. This property can be illustrated by Figures 5 and 6. Further, we get $C_1(0) = -11.2213 + 16.3520i$, $[d\lambda/d\tau_1]_{\lambda=i\omega_1^*} = 4.1212 - 2.5252i$. Then we have $\mu_2 = 2.7228$, $\beta_2 = -22.4426$, $T_2 = -17.4776$. Therefore, from Theorem 5, we can know that the Hopf bifurcation is supercritical and the bifurcating periodic solutions are stable.

At last, for $\tau_2 > 0$ and $\tau_1 = 0.5 \in (0, \tau_{10})$, we obtain $\omega_2^* = 0.5525$, $\tau_2^* = 1.4818$. The corresponding waveform and the phase plots are shown in Figures 7 and 8.

Guo and Jiang [7] have obtained that the three species in system (2) with only one time delay can coexist, however, we get that the species could also coexist with some available time delays of the predator and the top predator. This is valuable from the view of ecology. As the future work, we shall consider the following more general and more complicated system with multiple delays:

$$\frac{dx(t)}{dt} = \alpha x(t)\left(1 - \frac{x(t-\tau_1)}{k}\right) - \frac{\beta x(t)y(t)}{1+px(t)},$$
$$\frac{dy(t)}{dt} = y(t)\left(-h + \frac{e\beta x(t-\tau_2)}{1+px(t-1\tau_2)}\right) - ry(t)z(t), \tag{60}$$
$$\frac{dz(t)}{dt} = z(t)\left(-s + mry(t-\tau_3)\right),$$

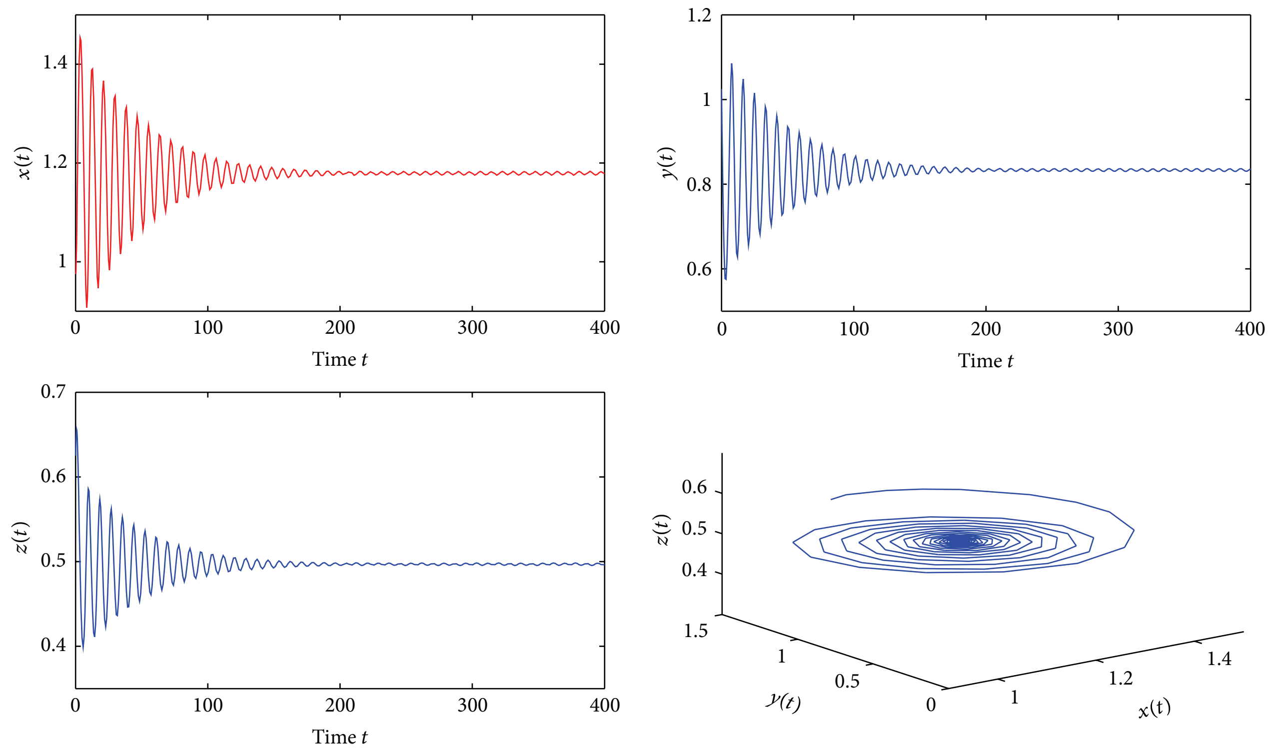

Figure 1: E_* is asymptotically stable for $\tau_1 = 1.38 < \tau_{10} = 1.4887$ with initial values 0.975, 1.025, and 0.625.

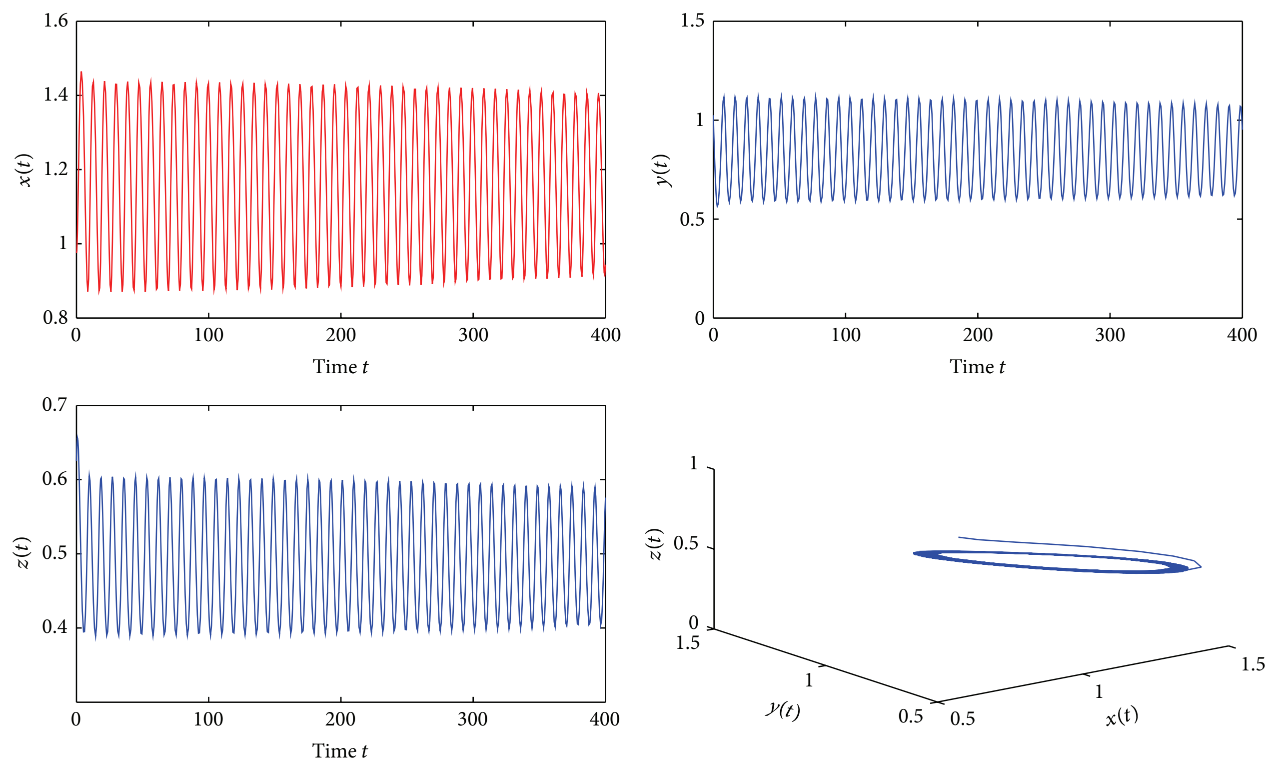

Figure 2: E_* is unstable for $\tau_1 = 1.498 > \tau_{10} = 1.4887$ with initial values 0.975, 1.025, and 0.625.

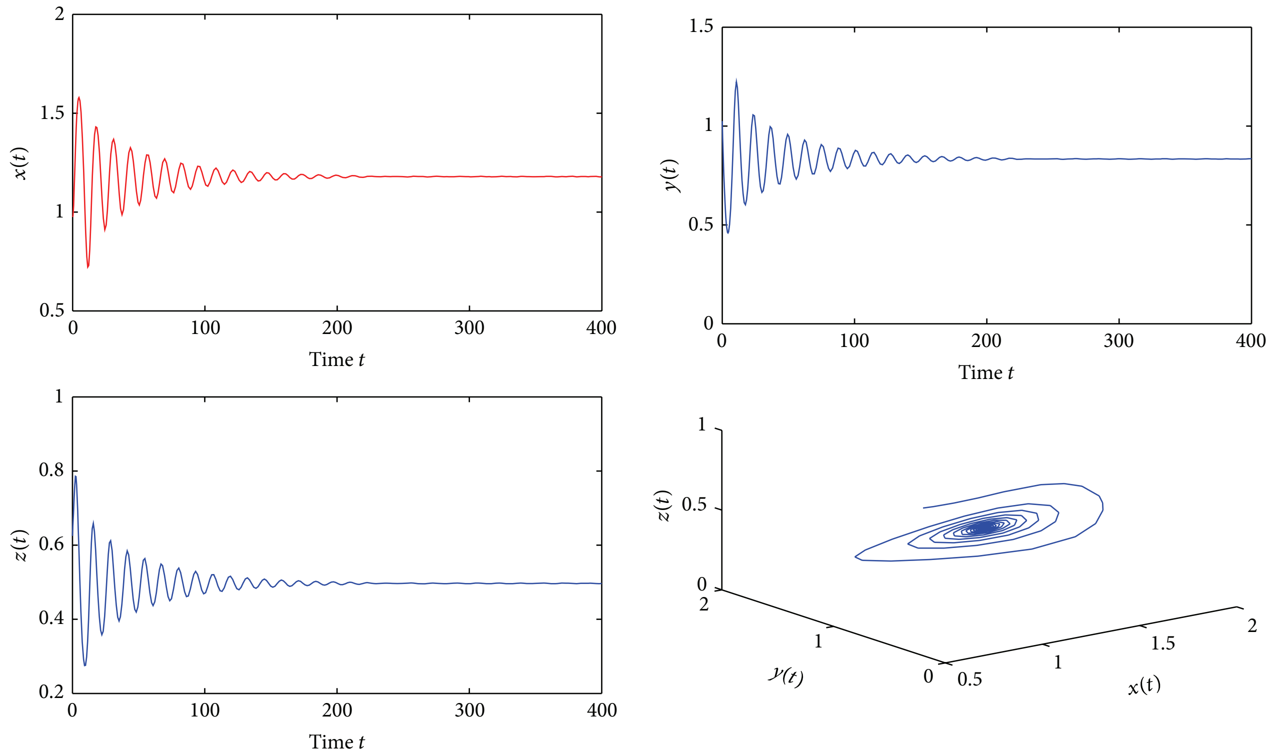

FIGURE 3: E_* is asymptotically stable for $\tau_2 = 1.60 < \tau_{20} = 1.7990$ with initial values 0.975, 1.025, and 0.625.

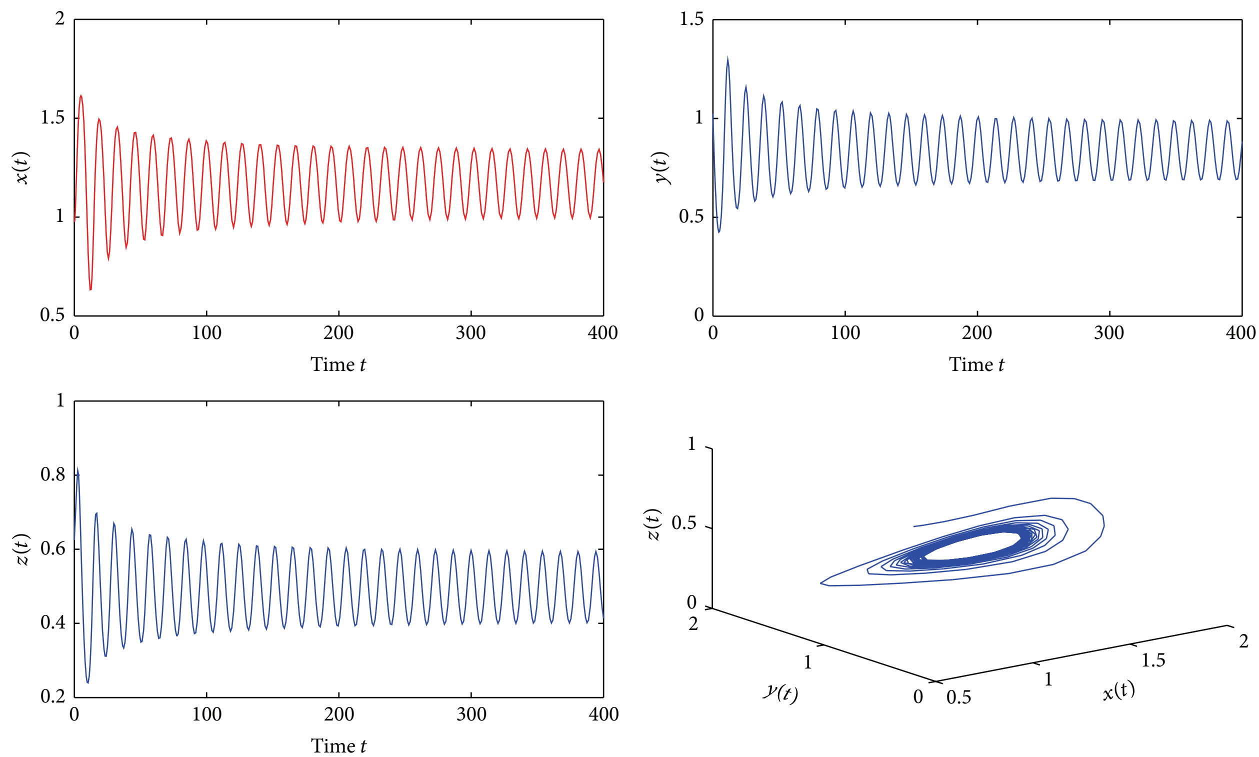

FIGURE 4: E_* is unstable for $\tau_2 = 1.96 > \tau_{20} = 1.7990$ with initial values 0.975, 1.025, and 0.625.

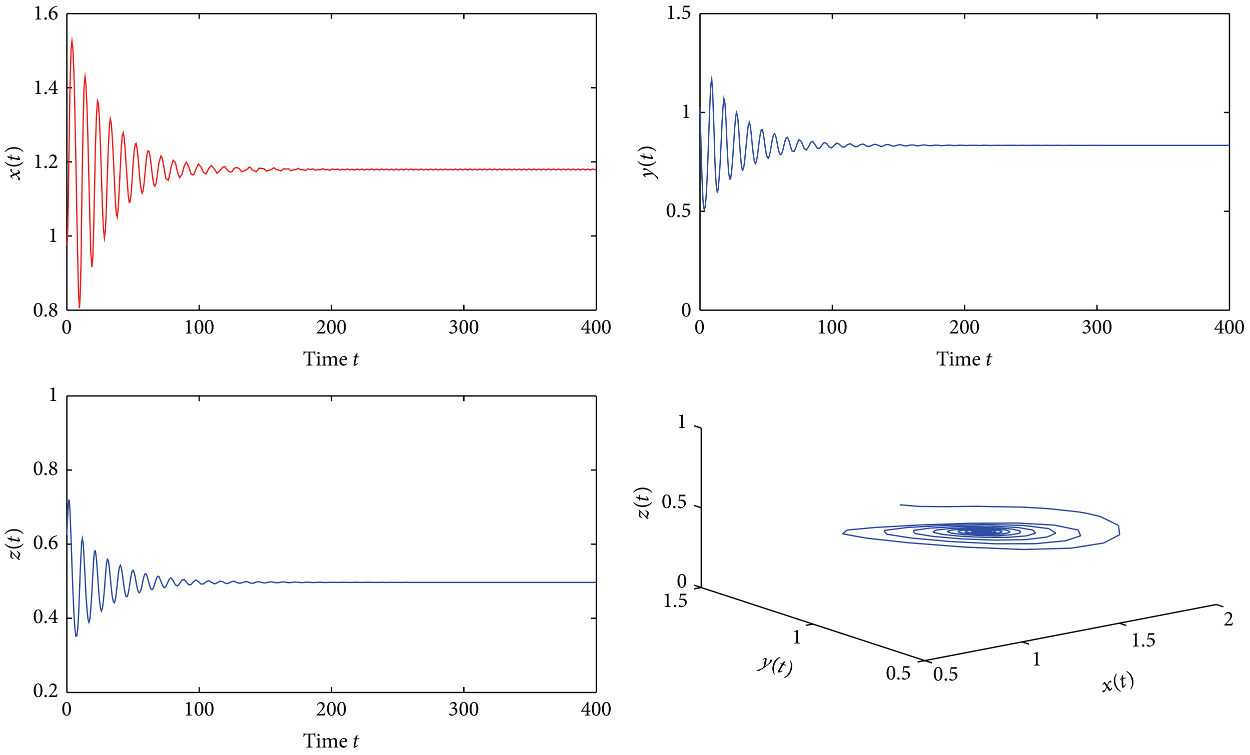

FIGURE 5: E_* is asymptotically stable for $\tau_1 = 0.75 < \tau_1^* = 0.8095$ and $\tau_2 = 0.8$ with initial values 0.975, 1.025, and 0.625.

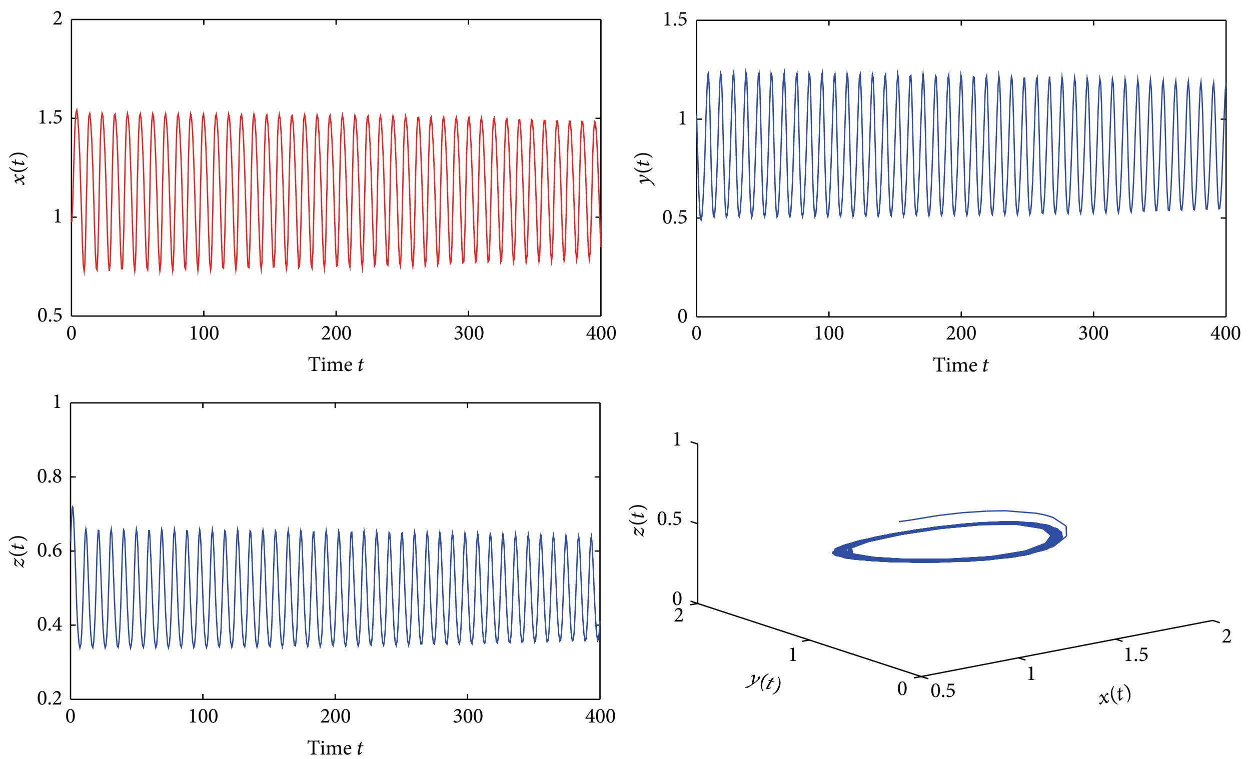

FIGURE 6: E_* is unstable for $\tau_1 = 0.936 > \tau_1^* = 0.8095$ and $\tau_2 = 0.8$ with initial values 0.975, 1.025, and 0.625.

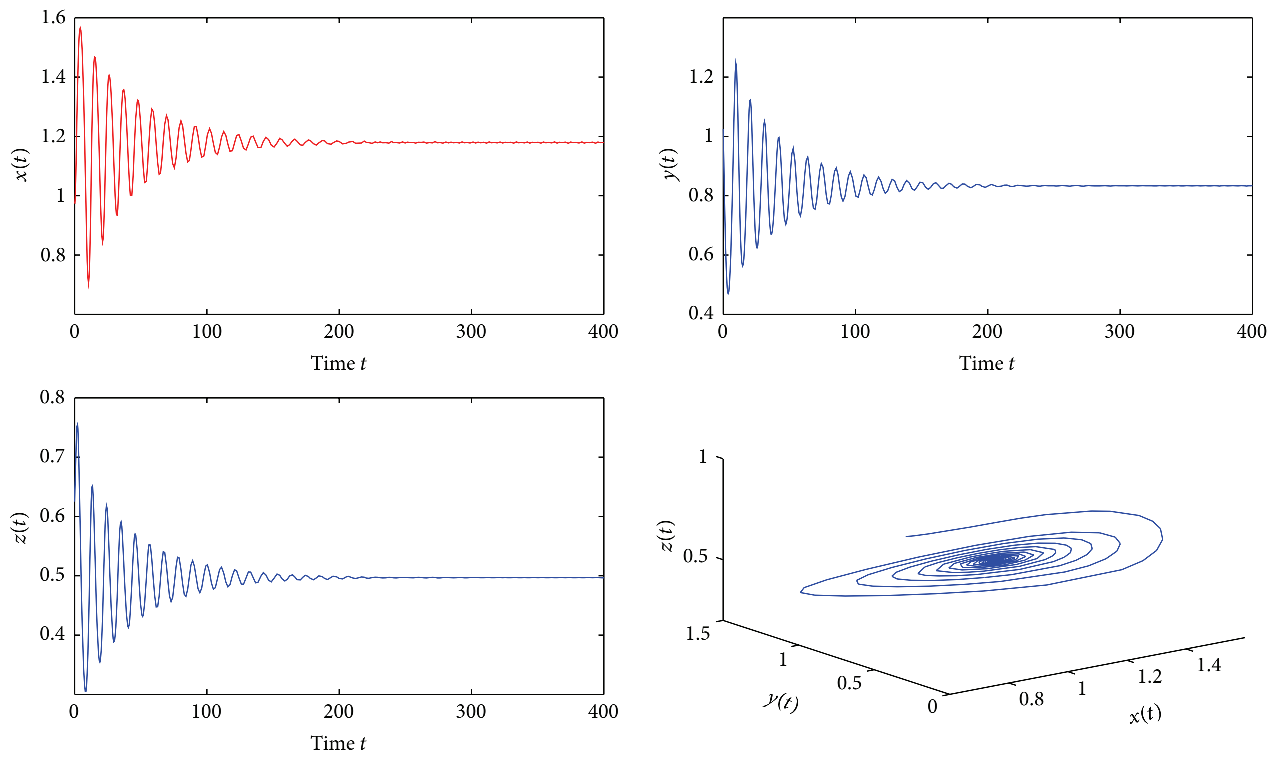

FIGURE 7: E_* is asymptotically stable for $\tau_2 = 1.25 < \tau_2^* = 1.4818$ and $\tau_1 = 0.5$ with initial values 0.975, 1.025, and 0.625.

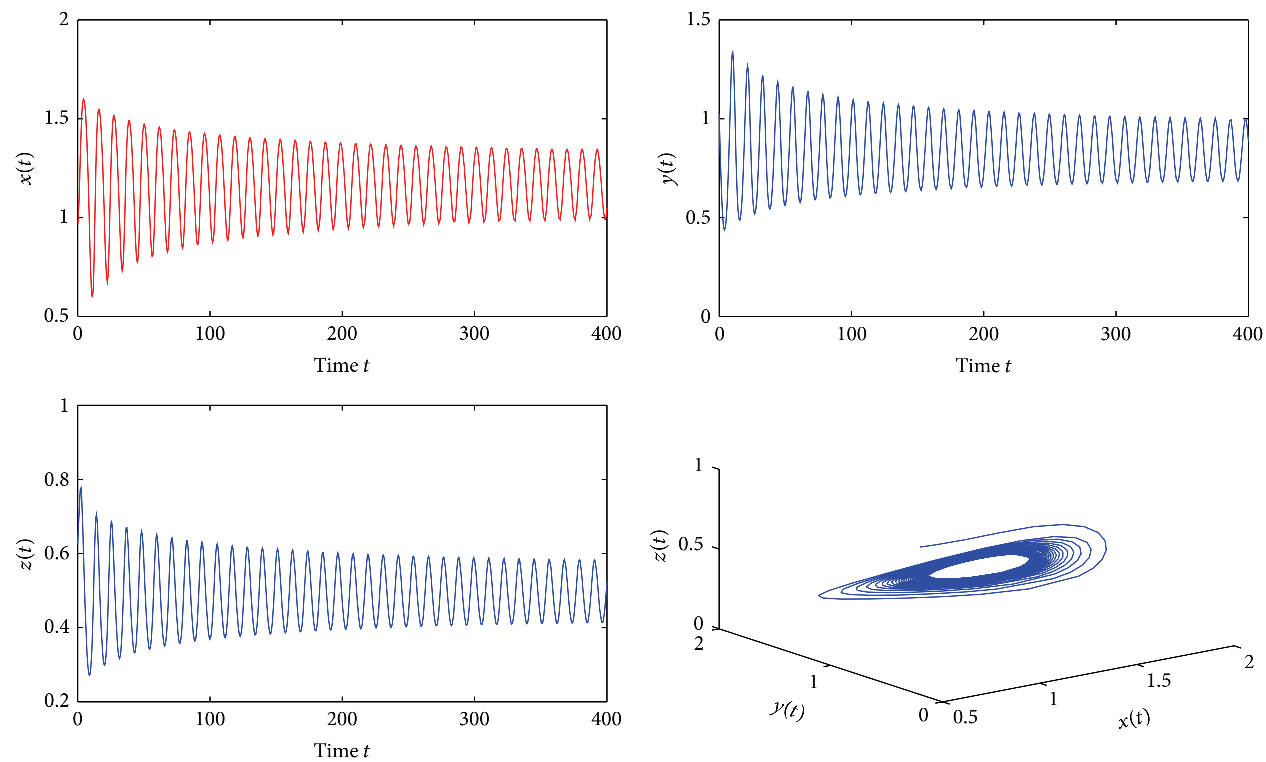

FIGURE 8: E_* is unstable for $\tau_2 = 1.50 > \tau_2^* = 1.4818$ and $\tau_1 = 0.5$ with initial values 0.975, 1.025, and 0.625.

where τ_1 is feedback delay of the prey and τ_2, τ_3 are the time delays due to the gestation of the predator and the top predator, respectively.

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