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Research Article

General Split Feasibility Problems in Hilbert Spaces

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Introducing a general split feasibility problem in the setting of infinite-dimensional Hilbert spaces, we prove that the sequence generated by the purposed new algorithm converges strongly to a solution of the general split feasibility problem. Our results extend and improve some recent known results.

1. Introduction

Let H and K be infinite-dimensional real Hilbert spaces, and let $A : H \to K$ be a bounded linear operator. Let $\{C_i\}_{i=1}^{p}$ and $\{Q_i\}_{i=1}^{r}$ be the families of nonempty closed convex subsets of H and K, respectively.

(a) The *convex feasibility problem* (CFP) is formulated as the problem of finding a point $x^\star$ with the property:

$$x^\star \in \bigcap_{i=1}^{p} C_i. \tag{1}$$

(b) The *split feasibility problem* (SEP) is formulated as the problem of finding a point $x^\star$ with the property:

$$x^\star \in C, \qquad Ax^\star \in Q, \tag{2}$$

where C and Q are nonempty closed convex subsets of H and K, respectively.

(c) The *multiple-set split feasibility problem* (MSSFP) is formulated as the problem of finding a point $x^\star$ with the property:

$$x^\star \in \bigcap_{i=1}^{p} C_i, \qquad Ax^\star \in \bigcap_{i=1}^{r} Q_i. \tag{3}$$

Note that (MSSFP) reduces to (SEP) if we take $p = r = 1$.

There is a considerable investigation on CFP in view of its applications in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning [1]. The split feasibility problem SFP in the setting of finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [2] for modelling inverse problems which arise from phase retrievals and in medical image reconstruction [3]. Since then, a lot of work has been done on finding a solution of SFP and MSSFP; see, for example, [2–25]. Recently, it is found that the SFP can also be applied to study the intensity-modulated radiation therapy; see, for example, [6, 16] and the references therein. Very recently, Xu [8] considered the SFP in the setting of infinite-dimensional Hilbert spaces.

The original algorithm given in [2] involves the computation of the inverse A^{-1} provided it exists. In [8], Xu studied some algorithm and its convergence. In particular, he applied Mann's algorithm to the SFP and purposed an algorithm which is proved to be weakly convergent to a solution of the SFP. He also established the strong convergence result, which shows that the minimum-norm solution can be obtained. In [7], Wang and Xu purposed the following cyclic algorithm to solve MSSFP:

$$x_{n+1} = P_{C[n]} \left(x_n + \gamma A^* \left(P_{Q[n]} - I \right) A x_n \right), \tag{4}$$

where $[n] := n\,(\mathrm{mod}\, p)$, (mod function take values in $\{1, 2, \ldots, p\}$), and $\gamma \in (0, 2/\|A\|^2)$. They show that the sequence $\{x_n\}$ convergence weakly to a solution of MSSFP provided the solution exists. To study strong convergence to a solution of MSSFP, first we introduce a general form of the split feasibility problem for infinite families as follows.

(d) *General split feasibility problem* (GSFP) is to find a point x^* such that

$$x^\star \in \bigcap_{i=1}^{\infty} C_i, \qquad Ax^\star \in \bigcap_{i=1}^{\infty} Q_i. \tag{5}$$

We denote by Ω the solution set of GSFP.

In this paper, using viscosity iterative method defined by Moudafi [21], we propose an algorithm for finding the solutions of the general split feasibility problem in a Hilbert space. We establish the strong convergence of the proposed algorithm to a solution of GSFP.

2. Preliminaries

Throughout the paper, we denote by H a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $\{x_n\}$ be a sequence in H and $x \in H$. Weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$, and strong convergence by $x_n \to x$. Let C be a closed and a convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$. This point satisfies

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{6}$$

The operator P_C is called the metric projection or the nearest point mapping of H onto C. The metric projection P_C is characterized by the fact that $P_C(x) \in C$ and

$$\langle y - P_C(x), x - P_C(x) \rangle \le 0, \quad \forall x \in H, \ y \in C. \tag{7}$$

Recall that a mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{8}$$

It is well known that P_C is a nonexpansive mapping. It is also known that H satisfies Opial's condition, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{9}$$

holds for every $y \in H$ with $y \ne x$.

Lemma 1. *Let H be a Hilbert space. Then, for all $x, y \in H$*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle. \tag{10}$$

Lemma 2 (see [22]). *Let H be a Hilbert space, and let $\{x_n\}$ be a sequence in H. Then, for any given sequence $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ and for any positive integer i, j with $i < j$,*

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \le \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j \|x_i - x_j\|^2. \tag{11}$$

Lemma 3 (see [23]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n) a_n + \gamma_n \delta_n + \beta_n, \quad n \ge 0, \tag{12}$$

where $\{\gamma_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ satisfy the following conditions:

(i) $\gamma_n \subset [0,1]$, $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$,

(iii) $\beta_n \ge 0$ *for all* $n \ge 0$ *with* $\sum_{n=0}^{\infty} \beta_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 4 (see [24]). *Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{\tau(n)\} \subset \mathbb{N}$ such that $\tau(n) \to \infty$, and the following properties are satisfied by all (sufficiently large) numbers $n \in \mathbb{N}$:*

$$t_{\tau(n)} \le t_{\tau(n)+1}, \qquad t_n \le t_{\tau(n)+1}. \tag{13}$$

In fact

$$\tau(n) = \max\{k \le n : t_k < t_{k+1}\}. \tag{14}$$

Lemma 5 (demiclosedness principle [25]). *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(T) \ne \emptyset$. *Then, T is demiclosed on C, that is, if $y_n \rightharpoonup z \in C$, and $(y_n - Ty_n) \to y$, then $(I - T)z = y$.*

3. Main Result

In the following result, we propose an algorithm and prove that the sequence generated by the proposed method converges strongly to a solution of the GSFP.

Theorem 6. *Let H and K be real Hilbert spaces, and let $A : H \to K$ be a bounded linear operator. Let $\{C_i\}_{i=1}^{\infty}$ and $\{Q_i\}_{i=1}^{\infty}$ be the families of nonempty closed convex subsets of H and K, respectively. Assume that GSFP* (5) *has a nonempty solution set Ω. Suppose that f is a self k-contraction mapping of H, and let $\{x_n\}$ be a sequence generated by $x_0 \in H$ as*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i} \left(I - \lambda_{n,i} A^* \left(I - P_{Q_i} \right) A \right) x_n, \quad n \ge 0, \tag{15}$$

where $\alpha_n + \beta_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1$. If the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,i}\}$, and $\{\lambda_{n,i}\}$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) *for each* $i \in \mathbb{N}$, $\liminf_n \alpha_n \gamma_{n,i} > 0$,

(iii) *for each* $i \in \mathbb{N}$, $\{\lambda_{n,i}\} \subset (0, 2/\|A\|^2)$ *and* $0 < \liminf_{n\to\infty} \lambda_{n,i} \le \limsup_{n\to\infty} \lambda_{n,i} < 2/\|A\|^2$,

then, the sequence $\{x_n\}$ converges strongly to $x^\star \in \Omega$, where $x^\star = P_\Omega f(x^\star)$.

Proof. First, we show that $\{x_n\}$ is bounded. In fact, let $z \in \Omega$. Since $\{\lambda_{n,i}\} \subset (0, 2/\|A\|^2)$, the operators $P_{C_i}(I - \lambda_{n,i}A^*(I - P_{Q_i})A)$ are nonexpansive, and hence we have

$$\begin{aligned}
&\|x_{n+1} - z\| \\
&\quad = \left\|\alpha_n x_n + \beta_n f(x_n) + \sum_{i=1}^{\infty}\gamma_{n,i} P_{C_i}\left(I - \lambda_{n,i}A^*\left(I - P_{Q_i}\right)A\right)x_n - z\right\| \\
&\quad \le \alpha_n \|x_n - z\| + \beta_n \|f(x_n) - z\| + \sum_{i=1}^{\infty}\gamma_{n,i}\left\|P_{C_i}\left(I - \lambda_{n,i}A^*\left(I - P_{Q_i}\right)A\right)x_n - z\right\| \\
&\quad \le \alpha_n \|x_n - z\| + \beta_n \|f(x_n) - z\| + \sum_{i=1}^{\infty}\gamma_{n,i}\|x_n - z\| \\
&\quad \le (1 - \beta_n)\|x_n - z\| + \beta_n \|f(x_n) - z\| \\
&\quad \le (1 - \beta_n)\|x_n - z\| + \beta_n \|f(x_n) - f(z)\| + \beta_n \|f(z) - z\| \\
&\quad \le (1 - \beta_n)\|x_n - z\| + \beta_n k \|x_n - z\| + \beta_n \|f(z) - z\| \\
&\quad \le (1 - (1 - k))\beta_n \|x_n - z\| + (1 - k)\frac{\beta_n}{1 - k}\|f(z) - z\| \\
&\quad \le \max\left\{\|x_n - z\|, \frac{1}{1 - k}\|f(z) - z\|\right\} \\
&\quad\ \vdots \\
&\quad \le \max\left\{\|x_0 - z\|, \frac{1}{1 - k}\|f(z) - z\|\right\},
\end{aligned} \tag{16}$$

which implies that $\{x_n\}$ is bounded, and we also obtain that $\{f(x_n)\}$ is bounded. Next, we show that for each $i \in \mathbb{N}$,

$$\lim_{n\to\infty}\left\|x_n - P_{C_i}\left(I - \lambda_{n,i}A^*\left(I - P_{Q_i}\right)A\right)x_n\right\| = 0. \tag{17}$$

By using Lemma 2, for every $z \in \Omega$ and $i \in \mathbb{N}$, we have that

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\quad = \left\|\alpha_n x_n + \beta_n f(x_n) + \sum_{j=1}^{\infty}\gamma_{n,j} P_{C_j}\left(I - \lambda_{n,j}A^*\left(I - P_{Q_j}\right)A\right)x_n - z\right\|^2 \\
&\quad \le \alpha_n\|x_n - z\|^2 + \beta_n\|f(x_n) - z\|^2 + \sum_{j=1}^{\infty}\gamma_{n,j}\left\|P_{C_j}\left(I - \lambda_{n,j}A^*\left(I - P_{Q_j}\right)A\right)x_n - z\right\|^2 \\
&\qquad - \alpha_n\gamma_{n,i}\left\|P_{C_i}\left(I - \lambda_{n,i}A^*\left(I - P_{Q_i}\right)A\right)x_n - x_n\right\|^2 \\
&\quad \le \alpha_n\|x_n - z\|^2 + \beta_n\|f(x_n) - z\|^2 + \sum_{j=1}^{\infty}\gamma_{n,j}\|x_n - z\|^2 \\
&\qquad - \alpha_n\gamma_{n,i}\left\|P_{C_i}\left(I - \lambda_{n,i}A^*\left(I - P_{Q_i}\right)A\right)x_n - x_n\right\|^2 \\
&\quad \le (1 - \beta_n)\|x_n - z\|^2 + \beta_n\|f(x_n) - z\|^2 \\
&\qquad - \alpha_n\gamma_{n,i}\left\|P_{C_i}\left(I - \lambda_{n,i}A^*\left(I - P_{Q_i}\right)A\right)x_n - x_n\right\|^2.
\end{aligned} \tag{18}$$

Hence, for each $i \in \mathbb{N}$, we have

$$\begin{aligned}
&\alpha_n\gamma_{n,i}\left\|P_{C_i}\left(I - \lambda_{n,i}A^*\left(I - P_{Q_i}\right)A\right)x_n - x_n\right\|^2 \\
&\quad \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \beta_n\|f(x_n) - z\|^2.
\end{aligned} \tag{19}$$

Next, we show that there exists a unique $x^\star \in \Omega$ such that $x^\star = P_\Omega f(x^\star)$. We observe that for each $n \ge 0$, $x^\star \in \Omega$ solves the GSFP (5) if and only if $x^\star$ solves the fixed point equation

$$x^\star = P_{C_i}\left(I - \lambda_{n,i}A^*\left(I - P_{Q_i}\right)A\right)x^\star, \quad i \in \mathbb{N}, \tag{20}$$

that is, the solution sets of fixed point equation (20) and GSFP (5) are the same (see for details [8]). Note that if $\{\lambda_{n,i}\} \subset (0, 2/\|A\|^2)$, then the operators $P_{C_i}(I - \lambda_{n,i}A^*(I - P_{Q_i})A)$ are nonexpansive. Since the fixed point set of nonexpansive operators is closed and convex, the projection onto the solution set Ω is well defined whenever $\Omega \neq \emptyset$. We observe that $P_\Omega(f)$ is a contraction of H into itself. Indeed, since P_Ω is nonexpansive,

$$\|P_\Omega(f)(x) - P_\Omega(f)(y)\| \le \|f(x) - f(y)\| \le k\|x - y\|. \tag{21}$$

Hence, there exists a unique element $x^\star \in \Omega$ such that $x^\star = P_\Omega f(x^\star)$.

In order to prove that $x_n \to x^\star$ as $n \to \infty$, we consider two possible cases.

Case 1. Assume that $\{\|x_n - x^\star\|\}$ is a monotone sequence. In other words, for n_0 large enough, $\{\|x_n - x^\star\|\}_{n\ge n_0}$ is either nondecreasing or nonincreasing. Since $\|x_n - x^\star\|$ is bounded we have $\|x_n - x^\star\|$ is convergent. Since $\lim_{n\to\infty}\beta_n = 0$ and $\{f(x_n)\}$ is bounded, from (19) we get that

$$\lim_{n\to\infty}\alpha_n\gamma_{n,i}\left\|P_{C_i}\left(I - \lambda_{n,i}A^*\left(I - P_{Q_i}\right)A\right)x_n - x_n\right\|^2 = 0. \tag{22}$$

By assuming that $\liminf_n \alpha_n \gamma_{n,i} > 0$, we obtain

$$\lim_{n\to\infty} \left\| P_{C_i}\left(I - \lambda_{n,i} A^* \left(I - P_{Q_i}\right) A\right) x_n - x_n \right\| = 0, \quad \forall i \in \mathbb{N}. \tag{23}$$

Now, we show that

$$\limsup_{n\to\infty} \langle f(x^\star) - x^\star, x_n - x^\star \rangle \le 0. \tag{24}$$

To show this inequality, we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\begin{aligned}\lim_{k\to\infty} &\langle f(x^\star) - x^\star, x_{n_k} - x^\star \rangle \\ &= \limsup_{n\to\infty} \langle f(x^\star) - x^\star, x_n - x^\star \rangle.\end{aligned} \tag{25}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup w$ and $\lambda_{n,i} \to \lambda_i \in (0, 2/\|A\|^2)$ for each $i \in \mathbb{N}$. From (23), we have

$$\begin{aligned}&\left\| P_{C_i}\left(I - \lambda_i A^* \left(I - P_{Q_i}\right) A\right) x_n - x_n \right\| \\ &\le \left\| P_{C_i}\left(I - \lambda_i A^* \left(I - P_{Q_i}\right) A\right) x_n \right. \\ &\qquad \left. - P_{C_i}\left(I - \lambda_{n,i} A^* \left(I - P_{Q_i}\right) A\right) x_n \right\| \\ &\quad + \left\| P_{C_i}\left(I - \lambda_{n,i} A^* \left(I - P_{Q_i}\right) A\right) x_n - x_n \right\| \\ &\le \left\| \left(I - \lambda_i A^* \left(I - P_{Q_i}\right) A\right) x_n \right. \\ &\qquad \left. - \left(I - \lambda_{n,i} A^* \left(I - P_{Q_i}\right) A\right) x_n \right\| \\ &\quad + \left\| P_{C_i}\left(I - \lambda_{n,i} A^* \left(I - P_{Q_i}\right) A\right) x_n - x_n \right\| \\ &\le \left|\lambda_i - \lambda_{n,i}\right| \left\| A^* \left(I - P_{Q_i}\right) A x_n \right\| \\ &\quad + \left\| P_{C_i}\left(I - \lambda_{n,i} A^* \left(I - P_{Q_i}\right) A\right) x_n - x_n \right\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{26}$$

Notice that for each $i \in \mathbb{N}$, $P_{C_i}(I - \lambda_i A^*(I - P_{Q_i})A)$ is nonexpansive. Thus, from Lemma 5, we have $w \in \Omega$. Therefore, it follows that

$$\begin{aligned}\limsup_{n\to\infty} &\langle f(x^\star) - x^\star, x_n - x^\star \rangle \\ &= \lim_{k\to\infty} \left\langle f(x^\star) - x^\star, x_{n_k} - x^\star \right\rangle \\ &= \left\langle f(x^\star) - x^\star, w - x^\star \right\rangle \le 0.\end{aligned} \tag{27}$$

Finally, we show that $x_n \to P_\Omega f(x^\star)$. Applying Lemma 1, we have that

$$\begin{aligned}&\|x_{n+1} - x^\star\|^2 \\ &= \left\| \alpha_n (x_n - x^\star) \right. \\ &\qquad \left. + \sum_{i=1}^{\infty} \gamma_{n,i} \left(P_{C_i}\left(I - \lambda_{n,i} A^* \left(I - P_{Q_i}\right) A\right) x_n - x^\star\right) \right\|^2 \\ &\quad + 2\beta_n \langle f(x_n) - x^\star, x_{n+1} - x^\star \rangle \\ &\le (1 - \beta_n)^2 \|x_n - x^\star\|^2 \\ &\quad + 2\beta_n \langle f(x_n) - f(x^\star), x_{n+1} - x^\star \rangle \\ &\quad + 2\beta_n \langle f(x^\star) - x^\star, x_{n+1} - x^\star \rangle \\ &\le (1 - \beta_n)^2 \|x_n - x^\star\|^2 \\ &\quad + 2\beta_n k \|x_n - x^\star\| \|x_{n+1} - x^\star\| \\ &\quad + 2\beta_n \langle f(x^\star) - x^\star, x_{n+1} - x^\star \rangle \\ &\le (1 - \beta_n)^2 \|x_n - x^\star\|^2 \\ &\quad + \beta_n k \left\{\|x_n - x^\star\|^2 + \|x_{n+1} - x^\star\|^2\right\} \\ &\quad + 2\beta_n \langle f(x^\star) - x^\star, x_{n+1} - x^\star \rangle.\end{aligned} \tag{28}$$

This implies that

$$\begin{aligned}&\|x_{n+1} - x^\star\|^2 \\ &\le \frac{(1 - \beta_n)^2 + \beta_n k}{1 - \beta_n k} \|x_n - x^\star\|^2 \\ &\quad + \frac{2\beta_n}{1 - \beta_n k} \langle f(x^\star) - x^\star, x_{n+1} - x^\star \rangle \\ &= \frac{1 - 2\beta_n + \beta_n k}{1 - \beta_n k} \|x_n - x^\star\|^2 \\ &\quad + \frac{\beta_n^2}{1 - \beta_n k} \|x_n - x^\star\|^2 \\ &\quad + \frac{2\beta_n}{1 - \beta_n k} \langle f(z) - x^\star, x_{n+1} - x^\star \rangle \\ &\le \left(1 - \frac{2(1-k)\beta_n}{1 - \beta_n k}\right) \|x_n - x^\star\|^2 \\ &\quad + \frac{2(1-k)\beta_n}{1 - \beta_n k} \left\{ \frac{\beta_n M}{2(1-k)} \right. \\ &\qquad\qquad \left. + \frac{1}{1-k} \langle f(x^\star) - x^\star, x_{n+1} - x^\star \rangle \right\} \\ &\le (1 - \eta_n) \|x_n - x^\star\|^2 + \eta_n \delta_n,\end{aligned} \tag{29}$$

where

$$\delta_n = \frac{\beta_n M}{2(1-k)} + \frac{1}{1-k}\left\langle f(x^\star) - x^\star, x_{n+1} - x^\star \right\rangle, \quad (30)$$

$M = \sup\{\|x_n - x^\star\|^2 : n \geq 0\}$ and $\eta_n = 2(1-k)\beta_n/(1-\beta_n k)$. It is easy to see that $\eta_n \rightarrow 0$, $\sum_{n=1}^{\infty} \eta_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Hence, by Lemma 3, the sequence $\{x_n\}$ converges strongly to $x^\star = P_\Omega f(x^\star)$.

Case 2. Assume that $\{\|x_n - x^\star\|\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \in \mathbb{N}; k \leq n : \|x_k - x^\star\| < \|x_{k+1} - x^\star\|\}. \quad (31)$$

Clearly, $\tau(n)$ is a nondecreasing sequence such that $\tau(n) \rightarrow \infty$ as $n \rightarrow \infty$ and for all $n \geq n_0$,

$$\|x_{\tau(n)} - x^\star\| < \|x_{\tau(n)+1} - x^\star\|. \quad (32)$$

From (19), we obtain that

$$\lim_{n\to\infty} \left\| P_{C_i}\left(I - \lambda_{\tau(n),i} A^* \left(I - P_{Q_i}\right) A\right) x_{\tau(n)} - x_{\tau(n)} \right\| = 0. \quad (33)$$

Following an argument similar to that in Case 1, we have

$$\limsup_{n\to\infty} \left\langle f(x^\star) - x^\star, x_{\tau(n)+1} - x^\star \right\rangle \leq 0. \quad (34)$$

And by similar argument, we have

$$\begin{aligned} &\|x_{\tau(n)+1} - x^\star\|^2 \\ &\quad \leq \left(1 - \eta_{\tau(n)}\right) \|x_{\tau(n)} - x^\star\|^2 + \eta_{\tau(n)} \delta_{\tau(n)}, \end{aligned} \quad (35)$$

where $\eta_{\tau(n)} \rightarrow 0$, $\sum_{n=1}^{\infty} \eta_{\tau(n)} = \infty$ and $\limsup_{n\to\infty} \delta_{\tau(n)} \leq 0$. Hence, by Lemma 3, we obtain $\lim_{n\to\infty} \|x_{\tau(n)} - x^\star\| = 0$ and $\lim_{n\to\infty} \|x_{\tau(n)+1} - x^\star\| = 0$. Now, from Lemma 4, we have

$$\begin{aligned} 0 &\leq \|x_n - x^\star\| \\ &\leq \max\{\|x_{\tau(n)} - x^\star\|, \|x_n - x^\star\|\} \\ &\leq \|x_{\tau(n)+1} - x^\star\|. \end{aligned} \quad (36)$$

Therefore, $\{x_n\}$ converges strongly to $x^\star = P_\Omega f(x^\star)$. □

For finite collections we have the following consequence of Theorem 6.

Theorem 7. *Let H and K be real Hilbert spaces, and let $A : H \rightarrow K$ be a bounded linear operator. Let $\{C_i\}_{i=1}^{p}$ be a family of nonempty closed convex subsets in H, and let $\{Q_i\}_{i=1}^{p}$ be a family of nonempty closed convex subsets in K. Assume that MSSFP has a nonempty solution set Ω. Let u be an arbitrary element in H, and let $\{x_n\}$ be a sequence generated by $x_0 \in H$ and*

$$\begin{aligned} x_{n+1} = {} & \alpha_n x_n + \beta_n u \\ & + \sum_{i=1}^{p} \gamma_{n,i} P_{C_i}\left(I - \lambda_{n,i} A^* \left(I - P_{Q_i}\right) A\right) x_n, \quad n \geq 0, \end{aligned} \quad (37)$$

where $\alpha_n + \beta_n + \sum_{i=1}^{p} \gamma_{n,i} = 1$. If the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,i}\}$, and $\{\lambda_{n,i}\}$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) *for all* $i \in \{1, 2, \ldots, p\}$, $\liminf_n \alpha_n \gamma_{n,i} > 0$,

(iii) *for all* $i \in \{1, 2, \ldots, p\}$, $\{\lambda_{n,i}\} \subset (0, 2/\|A\|^2)$ *and*

$$0 < \liminf_{n\to\infty} \lambda_{n,i} \leq \limsup_{n\to\infty} \lambda_{n,i} < \frac{2}{\|A\|^2}, \quad (38)$$

then the sequence $\{x_n\}$ converges strongly to $x^\star \in \Omega$, where $x^\star = P_\Omega u$.

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