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Research Article

On the Geometry of the Movements of Particles in a Hamilton Space

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We studied on the differential geometry of the Hamilton space including trajectories of the motion of particles exposed to gravitational fields and the cotangent bundle.

1. Introduction

As known, a Hamilton space is constructed as a differentiable manifold and a real valuable function defined on its cotangent bundle. The second order partial differentiation (Hessian) of this real valuable function with respect to momentum coordinate (p_i) determines a metric tensor on the cotangent bundle. However, Hessian of the Hamiltonian with respect to momentum coordinate determines a metric tensor on the manifold. This metric tensor is considered by Miron [1]. Recently, many studies have been done on the metrics defined on the cotangent bundles, and most of these studies are on two distinguished metrics. One of these metrics is the Riemann extension of the torsion-free affine connection [2–4] and the other one is the diagonal lift in cotangent bundle [1, 5]. Willmore [3] showed that a torsion-free affine connection on a manifold determines canonically a pseudo-Riemannian metric on the cotangent bundle. Furthermore, he expressed this pseudo-Riemannian metric as the Riemann extension of the affine connection. Akbulut et al. [5] defined a diagonal lift of a Riemannian metric of a manifold to its cotangent bundle, and they studied the differential geometry of the cotangent bundle with respect to this Riemann metric. Oproiu [6] studied the differential geometry of tangent bundle of a Lagrange manifold when this tangent bundle is endowed with pseudo-Riemannian metric obtained from fundamental tensor field by a method similar to the obtaining of the complete lift of a pseudo-Riemannian metric on a differentiable manifold. Ayhan [7, 8] obtained the images on the cotangent bundle of the some tensor fields (i.e., functions, vector fields, and 1-forms, and tensor fields with types (1,1), (0,2) and (2,0)) on the tangent bundle of a Lagrange manifold which are obtained by vertical, complete, and horizontal lifts under the Legendre transformation.

In this paper, it is proved that the trajectories of particles exposed to gravitational fields are geodesics and the Hamilton function as represented of the total energy of system is constant along these trajectories. We studied the differential geometry of the cotangent bundle T^*M of the Hamilton space M including the trajectories of particles exposed to gravitational fields. We obtained that the pseudo-Riemannian metric G on T^*M corresponds to pseudo-Riemannian metric g^C on TM with respect to Legendre transformation, and we showed that G is the Riemann extension of the Levi-Civita connection. Moreover we considered an almost product structure P is defined on T^*M. By means of P and G, an almost symplectic structure θ on T^*M is defined. Finally we obtained that the coefficients of the Levi-Civita connection $\dot{\nabla}$ and Riemann curvature tensor K of (T^*M, G) and we found the condition under whichT^*M is locally flat.

In this study, all the manifolds and the geometric objects are assumed to be C^∞, and we use the Einstein summation convention.

2. The Movement in a Hamilton Space

The fundamental physical concept is that a gravitational field is identical to geometry of the Hamilton space. This geometry is determined by Hamiltonian

$$H\left(x^i, p^i\right) = \frac{1}{2} g^{ik}(x)\, p_i p_k, \tag{1}$$

where $g^{ik}(x)$ is a tensor with type $(2, 0)$ given by $g^{ik} g_{kj} = \delta^i_j$. $g_{kj}(x)$ is local components of a (pseudo)Riemann metric tensor [9]. At the same time, the second order partial differentiation (Hessian) of Hamiltonian given by (1) with respect to momentum coordinate (p_i) is equal to the following tensor type of $(2, 0)$:

$$g^{ik}(x) = \frac{\partial^2 H}{\partial p_i \partial p_k}. \tag{2}$$

The Hamilton space M, called a Hamilton mechanic system by mechanists, consists of n-dimensional differentiable manifold M and regular Hamiltonian H given by (2) providing $\det[g^{ik}] \neq 0$ [1, 10]. The motion of every particle in the Hamilton space depending on time is represented as a curve $\gamma : I \subset R \rightarrow M$. For any time t, the position coordinates of every particle in the Hamilton space are given by $x^i \circ \gamma(t)$, $i = 1, \ldots, n$, or briefly x^i, $i = 1, \ldots, n$, and, respectively, the velocity and momentum coordinates are given by $y^i = dx^i/dt$, $i = 1, \ldots, n$, and $p_i = g_{ij} y^j$, $i = 1, \ldots, n$.

The movement equation of any particle in the Hamilton space from the position $(x^1(t_1), \ldots, x^n(t_1))$ to $(x^1(t_2), \ldots, x^n(t_2))$ is determined by the canonic Hamilton equation which is defined by

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}; \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}. \tag{3}$$

The solution curves of the differential equation system in (3) are one-parameter group of diffeomorphisms of the Hamilton space [11]. The Hamiltonian H is fixed on family of one-parameter curves, which defines a conservation law. As any particle is moving on any curve in the Hamilton space, the total energy H of the system is the same on every point of curve. In other words, the Hamiltonian is not changed with respect to variable t and the total energy H must be constant as the particles move. Since the tangent vector field of a curve $C : t \rightarrow (x^i(t), p_i(t))$, $i = 1, \ldots, n$, satisfies the canonic Hamilton equation in (3), this tangent vector field of C is called the Hamilton vector field. Integral curves of the Hamilton vector field correspond to geodesics in the Hamilton space M [12]. In Section 3, we proved that the integral curves of the Hamilton vector field on T^*M correspond to geodesics on M and also the value of Hamiltonian H does not change on geodesics of M for the Hamiltonian of the gravitational fields given by (1).

The Hamilton space M is an n-dimensional differentiable manifold with (U, x^i), $i = 1, \ldots, n$, the local chart and T^*M is $2n$-dimensional its cotangent bundle with $(\pi^{-1}(U), x^i, p_i)$, $i = 1, \ldots, n$, the local chart, where $\pi : T^*M \rightarrow M$ is canonical projection, $x^i = x^i o \pi$ and p_i are the vector space coordinates of an element from $\pi^{-1}(U)$ with respect to the local frame $(dx^1, \ldots, dx^n)$ of T^*M defined by the local chart (U, x^i). In classical mechanics, T^*M and TM are called momentum phase space and velocity phase space, respectively. The tangent bundle of T^*M has an integrable vector subbundle $VT^*M = \operatorname{Ker} \pi_*$ called the vertical distribution on T^*M. A nonlinear connection on T^*M is defined by the horizontal distribution by HT^*M and HT^*M is complementary to VT^*M in TT^*M. Thus $TT^*M = VT^*M \oplus HT^*M$. The system of the local vector fields $(\partial/\partial p_1, \ldots, \partial/\partial p_n)$ is a local frame in VT^*M and the system of the local vector fields $(\delta/\delta x^1, \ldots, \delta/\delta x^n)$ is a local frame in HT^*M.

The Legendre transformation φ is a diffeomorphism between the open set of $\widetilde{U} \subset TM$ and the open set of $U \subset T^*M$. Let $\{\delta/\delta x^i, \partial/\partial y^i\}$, $\{dx^i, \delta y^i\}$ be an adapted frame (coframe) on TM and $\{\delta/\delta x^i, \partial/\partial p_i\}$, $\{dx^i, \delta p_i\}$ be an adapted frame (coframe) on T^*M. Then the differential geometric objects on T^*M can be expressed in terms of those of TM by using the Legendre transformation as follows:

$$\begin{gathered}\left(\varphi^{-1}\right)^*\left(dx^i\right) = dx^i,\\ \left(\varphi^{-1}\right)^*\left(\delta y^i\right) = g^{ij}\delta p_j, \qquad \left(\varphi^{-1}\right)^*\left(g_{ij}\right) = g_{ij}.\end{gathered} \tag{4}$$

3. The Integral Curves and Metrics

In this section, we studied the relation between the integral curves of the Hamilton vector field on T^*M and the geodesics on M. Then we obtained the pseudo-Riemann metric G on T^*M by using two different methods. In addition, we defined an almost symplectic structure θ on T^*M by using G and an almost product structure P. Finally, the fact that the total energy H is constant for each stage of the system as the system with n-particles moves with the effect of the gravitational field is reexpressed in terms of differential geometric objects on the cotangent bundle T^*M of the Hamilton space M.

Theorem 1. *Let H be the Hamiltonian given by* (1). *Let C be a curve in T^*M, γ be a projection of C to M; that is, $\pi \circ C = \gamma$, and let w be a 1-form associated with the tangent vector of curve $\gamma(t)$.*

(i) *If the curve C is an integral curve of the Hamilton vector field V, the curve γ is geodesic.*

(ii) *The Hamiltonian of the gravitational field H is constant along the geodesic of the Hamilton space.*

Proof. (i) Let V be the Hamilton vector field. V has local expression with respect to induced coordinate system on T^*M

$$V = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}. \tag{5}$$

The tangent vector field of C in T^*M has local coordinate expression

$$C_*\left(\frac{d}{dt}\right) = \frac{dx^i(t)}{dt}\frac{\partial}{\partial x^i} + \frac{dp_i(t)}{dt}\frac{\partial}{\partial p_i} \tag{6}$$

with respect to induced coordinate system of T^*M. If the curve C is an integral curve of the Hamilton vector field V, the equation $C_*(d/dt) = V_{C(t)}$ holds. From this equation, we obtain the following canonic Hamilton equations:

$$\frac{dx^i(t)}{dt} = \frac{\partial H}{\partial p_i} = g^{ij} p_j, \qquad \frac{dp_i(t)}{dt} = -\frac{\partial H}{\partial x^i}. \tag{7}$$

The right part of the above equations is expressed by

$$\frac{d}{dt}\left(g_{ij}\frac{\partial H}{\partial p_j}\right) + \frac{\partial H}{\partial x^i} = 0. \tag{8}$$

Using the composite function differentiation, we get

$$\frac{dp_k}{dt} + \frac{\partial}{\partial x^k}\left(g_{ij}\frac{\partial H}{\partial p_j}\right)\frac{dx^k}{dt} + \frac{\partial H}{\partial x^i} = 0, \tag{9}$$

and by

$$\frac{dx^k}{dt} = g^{ka} p_a, \tag{10}$$

we get

$$\frac{dp_k}{dt} + \frac{\partial}{\partial x^k}\left(g_{ij}\frac{\partial H}{\partial p_j}\right) g^{ka} p_a + \frac{\partial H}{\partial x^i} = 0. \tag{11}$$

Next, transvecting by g_{ka}, we get

$$g_{ka}\frac{dp_k}{dt} + S_j(x,p) = 0, \tag{12}$$

where

$$S_j(x,p) = \frac{\partial}{\partial x^k}\left(g_{ij}\frac{\partial H}{\partial p_j}\right) p_a + g_{ka}\frac{\partial H}{\partial x^i}. \tag{13}$$

We get a nonlinear connection on T^*M defined by

$$N_{jk} = S^a_j = \frac{1}{2} g_{ka}\frac{\partial^2 H}{\partial x^j \partial p_a}, \tag{14}$$

where

$$N_{jk} = S^a_j = \frac{1}{2}\frac{\partial S_j}{\partial p_a}. \tag{15}$$

Then, we obtain

$$\frac{\partial N_{jk}}{\partial p_b} = \frac{1}{2} g_{ka}\frac{\partial g^{ab}}{\partial x^j}, \tag{16}$$

and since the following equation is satisfied:

$$\frac{\partial g^{ab}}{\partial x^j} = \frac{\partial}{\partial x^j} g\left(dx^a, dx^b\right), \tag{17}$$

we get

$$\frac{\partial N_{jk}}{\partial p_b} = -\Gamma^b_{jk}. \tag{18}$$

Subsequently we obtained that $N_{jk} = -\Gamma^b_{jk} p_b$ and $S_j(x,p) = -\Gamma^b_{jk} p_b p_a$. If we substitute the above equation into (12), we get

$$g_{ka}\frac{dp_k}{dt} - \Gamma^b_{jk} p_b p_a = 0, \tag{19}$$

and transvecting by g^{ka}, we get

$$\frac{dp_k}{dt} - \Gamma^b_{jk} p_b g^{ka} p_a = 0. \tag{20}$$

Thus,

$$\frac{dp_k}{dt} - \Gamma^b_{jk} p_b \frac{dx^k}{dt} = 0. \tag{21}$$

Then we get

$$\nabla_{(dx^k/dt)(\partial/\partial x^k)} p_j dx^j = 0. \tag{22}$$

Since $w = p_i dx^i$ is a 1-form associated with the tangent vector, $\dot{\gamma} = d\gamma(t)/dt$ of curves $\gamma(t)$, and Riemann connection ∇ satisfies the following property:

$$\nabla_{\dot{\gamma}}\left(w\left(\dot{\gamma}\right)\right) = 2g\left(\dot{\gamma}, \nabla_{\dot{\gamma}}\dot{\gamma}\right) \quad \text{for } w\left(\dot{\gamma}\right) = g\left(\dot{\gamma}, \dot{\gamma}\right), \tag{23}$$

we get

$$g\left(\dot{\gamma}, \nabla_{\dot{\gamma}}\dot{\gamma}\right) = 0. \tag{24}$$

Therefore it can be seen that straightforward the curve $\gamma(t)$ is a geodesic curve.

(ii) It is sufficient to show the Hamiltonian H is not changed with respect to variable t in order to prove the theorem. We calculate

$$\frac{dH}{dt} = \frac{\partial H}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial H}{\partial p_i}\frac{dp_i}{dt}. \tag{25}$$

If we take into account (3), we obtain $dH/dt = 0$. Thus H is not dependent on value t. □

Therefore, we obtain that the trajectories of particles exposed to gravitational fields are geodesics and the Hamilton function represented of the total energy of system is constant along these trajectories.

We consider differential geometric objects on the cotangent bundle T^*M of the Hamilton space. Let us start by obtaining a metric on T^*M. A pseudo-Riemann metric G on the cotangent bundle T^*M of the Hamilton space is obtained by using two different ways. Firstly, the pseudo-Riemann metric on the cotangent bundle T^*M is obtained as we were inspired by the paper of Willmore [3] as follows.

Theorem 2. *The Levi-Civita connection ∇ on M determines canonically a pseudo-Riemannian metric on T^*M.*

Proof. Let P be a point on T^*M such that $C(0) = P$. Let X be a tangent vector to C at P. The image of the curve $C(t)$ under the bundle projection map π is a curve $\gamma(t)$ on M, passing through $p = \pi(P) \in U$. The curve $C(t)$ can be regarded as a field of covariant vectors $\omega(t)$ defined along the curve $\gamma(t)$. The covariant derivative $(\nabla_{d\pi(X)}\omega(t))_{t=0}$ is a covector at p which can be evaluated on the projected tangent vector $d\pi(X)$. This defines a quadratic differential form Q on $T(T^*M)$. From this we obtain a bilinear form G on T^*M at P by the usual formula

$$G(X,Y) = Q(X+Y, X+Y) - Q(X,X) - Q(Y,Y), \quad (26)$$

where X and Y are tangent vectors to T^*M at P. We shall consider that G corresponds to Riemann extension of ∇ on M. We choose a local coordinate system (x^i), $i = 1, \dots, n$, valid in some neighborhood U around p. Then a local coordinate system for $\pi^{-1}(U)$ is (x^i, p_j), where $\omega = p_j dx^j$. The curve may be expressed locally by $t \to (x^i(t), p_j(t))$ and the corresponding curve γ is $t \to (x^i(t))$. The vector X at P is given by $(\dot{x}^i(0), \dot{p}_i(0))$ and its projection $d\pi(X)$ by $\dot{x}^i(0)$. Then, at when $t = 0$, we have

$$\begin{aligned} \nabla_{d\pi(X)}\omega(t) &= \dot{x}^i(0)\left(\nabla_i p_j(t)\right) dx^j \\ &= \left[\frac{dp_j}{dt} - \Gamma_{ij}^k p_k \dot{x}^i(0)\right] dx^j, \end{aligned} \quad (27)$$

where Γ_{ij}^k are the connection coefficients of ∇. We evaluate this covector on $d\pi(X)$ to get the number

$$\begin{aligned} Q(X,X) &= \left(\nabla_{d\pi(X)}\omega(t)\right)(d\pi(X)) \\ &= -\Gamma_{ij}^k p_k \dot{x}^i \dot{x}^j + \dot{p}_j \dot{x}^j. \end{aligned} \quad (28)$$

If the equation which is obtained above taken into account in (26), we get for $G(X,X)$the following:

$$\begin{aligned} G(X,X) &= Q(2X,2X) - 2Q(X,X) \\ &= 2Q(X,X) \\ &= -2\Gamma_{ij}^k p_k \dot{x}^i \dot{x}^j + 2\dot{p}_j \dot{x}^j \\ &= \left(-2\Gamma_{ij}^k p_k dx^i dx^j + 2dp_j dx^j\right)(X,X). \end{aligned} \quad (29)$$

Therefore G has local expression

$$G = -2\Gamma_{ij}^k p_k dx^i dx^j + 2dp_j dx^j \quad (30)$$

with respect to the induced coordinates (x^i, p_i) in $\pi^{-1}(U)$. Since the adapted dual frame on T^*M is $(dx^i, \delta p_i)$, where

$$\delta p_i = dp_i - p_k \Gamma_{ij}^k dx^j, \quad (31)$$

we get $G = 2\delta p_i dx^j$. □

Secondly, the pseudo-Riemann metric on the cotangent bundle T^*M is obtained as follows.

Theorem 3. *Let M be a manifold with a (pseudo)Riemann metric g. Then the pseudo-Riemannian metric g^C on TM corresponds to the pseudo-Riemannian metric $G = 2\delta p_i dx^k$ on T^*M.*

Proof. Let g be a (pseudo)Riemannian metric on M then g^C given by $g^C = 2g_{jk}\delta y^j dx^k$ is a pseudo-Riemann metric on TM. By using the equalities in (4), we get

$$\begin{aligned} &\left(\varphi^{-1}\right)^*\left(g^C\right) \\ &\quad = 2\left(\varphi^{-1}\right)^*\left(g_{jk}\right)\left(\varphi^{-1}\right)^*\left(\delta y^j\right)\left(\varphi^{-1}\right)^*\left(dx^k\right) \\ &\quad = 2\underbrace{g_{jk}g^{ji}}_{\delta_k^i}\delta p_i dx^k, \end{aligned} \quad (32)$$

which gives a pseudo-Riemann metric $G = 2\delta p_i dx^k$on T^*M. □

In order to understand the relation between the pseudo-Riemann manifold (T^*M, G) and the symplectic manifold (T^*M, θ), we need to define an almost symplectic structure θ on the cotangent bundle T^*M of the Hamilton space and an almost product structure P on T^*M. The definition of P and θ was obtained as we were inspired by the studies of Miron [10] for the Hamilton space.

Definition 4. Let $w = p_i dx^i$ be globally defined as 1-form on T^*M. The exterior differential dw of the 1-form w is called an almost symplectic structure θ on the cotangent bundle T^*M of the Hamilton space given by

$$\theta = dw = \delta p_i \wedge dx^i. \quad (33)$$

Definition 5. Let T^*M be a $2n$-dimensional manifold. A mixed tensor field defines an endomorphism on each tangent space of T^*M. If there exists a mixed tensor field P which satisfies

$$P \circ P = I, \quad (34)$$

we say that the field gives an almost product structure to T^*M.

We can consider the tensor field with type (1, 1) on T^*M:

$$P = \frac{\delta}{\delta x^i} \otimes dx^j - \frac{\partial}{\partial p_i} \otimes \delta p_j. \quad (35)$$

Theorem 6. *P is an almost product structure on T^*M.*

Proof. We have

$$P\left(\frac{\delta}{\delta x^i}\right) = \frac{\delta}{\delta x^i}, \qquad P\left(\frac{\partial}{\partial p_i}\right) = -\frac{\partial}{\partial p_i} \quad (36)$$

from which $P \circ P = I$. □

Theorem 7. *Let G be a pseudo-Riemann metric defined as Riemann extension of ∇ in M and let P be an almost product structure on T^*M. θ is an almost symplectic structure associated with (G, P). Nondegenerate skew-symmetric 2-form θ on T^*M is given by following equation:*

$$\theta(X,Y) = G(PX, Y). \quad (37)$$

Proof. By using (33), the value of the vector fields X, Y on T^*M under θ is

$$\begin{aligned}\theta\left(X^V+X^H, Y^V+Y^H\right) &= \delta p_i \wedge dx^i\left(X^i\delta_i + X^{n+i}\partial_i, Y^j\delta_j + Y^{n+j}\partial_j\right)\\ &= \delta p_i\left(X^{n+i}\partial_i\right)\cdot dx^i\left(Y^j\delta_j\right) - \delta p_i\left(Y^{n+j}\partial_j\right)\cdot dx^i\left(X^i\delta_i\right)\\ &= X^{n+i}Y^i - Y^{n+i}X^i.\end{aligned} \tag{38}$$

On the other hand, the value of $G(PX, Y)$ is

$$\begin{aligned}G\left(P\left(X^V+X^H\right), Y^V+Y^H\right) &= G\left(X^V-X^H, Y^V+Y^H\right)\\ &= G\left(X^V, Y^H\right) - G\left(X^H, Y^V\right)\\ &= X^{n+i}Y^i - Y^{n+i}X^i.\end{aligned} \tag{39}$$

Therefore, it is seen forward the accuracy of the claim of the theorem. □

Theorem 8. *Let M be Riemann manifold and let H be Hamiltonian. For any vector field X on T^*M,*

$$dH(X) = \theta(V, X). \tag{40}$$

Proof. dH is a 1-form on T^*M with local expression

$$dH = \frac{\delta H}{\delta x^i}dx^i + \frac{\partial H}{\partial p_i}\delta p_i, \tag{41}$$

with respect to adapted local dual frame, and X has local expression

$$X = X^j\frac{\delta}{\delta x^j} + X^{n+j}\frac{\partial}{\partial p_j}. \tag{42}$$

From (40) and (41), we get

$$dH(X) = \frac{\delta H}{\delta x^i}X^i + \frac{\partial H}{\partial p_i}X^{n+i}. \tag{43}$$

Since V is a Hamilton vector field, V has local expression with respect to adapted frame on T^*M:

$$V = \frac{dx^j}{dt}\frac{\delta}{\delta x^j} + \frac{\delta p_j}{dt}\frac{\partial}{\partial p_j}, \tag{44}$$

where $\delta p_i/dt = (dp_i/dt) - p_k\Gamma^k_{ij}(dx^j/dt)$. Thus we have

$$\theta(V, X) = G(PV, X) = \frac{dx^i}{dt}X^{n+i} - \left(\frac{dp_i}{dt} - p_k\Gamma^k_{ij}\frac{dx^j}{dt}\right)X^i. \tag{45}$$

If we substitute the above equation into (3), we get

$$\theta(V, X) = \frac{\partial H}{\partial p_i}X^{n+i} + \underbrace{\left(\frac{\partial H}{\partial x^i} + p_k\Gamma^k_{ij}\frac{\partial H}{\partial p_j}\right)}_{\delta H/\delta x^i}X^i. \tag{46}$$

From (43) and (46) it is easily seen that $dH(X) = \theta(V, X)$.

By using the differential geometric objects H, G, θ, and P on the cotangent bundle of the Hamilton space considered in this section, we obtain

$$dH(X) = \theta(V, X) = G(PV, X) = \nabla_{d\pi(PV)}\omega(d\pi(X)), \tag{47}$$

where G is the pseudo-Riemannian metric defining the Riemann extension of the Levi-Civita connection on M. If we put V instead of X in the above equation, we obtain

$$\begin{aligned}\theta(V, V) &= G(PV, V) = \nabla_{d\pi(PV)}\omega(d\pi(V))\\ &= dH(V) = V(H) = 0.\end{aligned} \tag{48}$$

Since $V(H) = 0$, the Hamilton function which gives the total energy of each stage of the system is constant. □

4. The Differential Geometry of (T^*M, G)

In this section, we obtained that the coefficients of the Levi-Civita connection $\overset{\circ}{\nabla}$ and Riemannian curvature tensor K of (T^*M, G) and we found the condition under which T^*M is locally flat.

Theorem 9. *The Lie brackets of the horizontal base vector fields $\delta/\delta x^i = \partial/\partial x^i - N_{ij}\partial/\partial p_j$, $i, j = 1, \ldots, n$ and vertical base vector fields $\partial/\partial p_i$ on T^*M are given by*

(i) $[\delta/\delta x^i, \delta/\delta x^j] = -R_{kij}\partial/\partial p_k$,

(ii) $[\delta/\delta x^i, \partial/\partial p_j] = -\Gamma^j_{ik}\partial/\partial p_k$,

(iii) $[\partial/\partial p_i, \partial/\partial p_j] = 0$, *where* $R_{kij} = \delta N_{kj}/\delta x^i - \delta N_{ki}/\delta x^j$ *[2].* $R_{kij} = -p_h R^h_{kij}$ *for* $N_{ki} = -\Gamma^h_{ki}p_h$.

Theorem 10. *Let (M, H) be Hamilton space, T^*M the cotangent bundle of M, G a pseudo-Riemann metric defined as Riemann extension of Levi-Civita connection ∇ in M, and $\overset{\circ}{\nabla}$ the Levi-Civita connection on T^*M. Then the connection coefficients of the Levi-Civita connection of the pseudo-Riemannian metric G on T^*M are given by*

$$\begin{aligned}\overset{\circ}{\nabla}_{\delta_i}\delta_j &= -R_{kij}\partial_k, &\quad \overset{\circ}{\nabla}_{\delta_i}\partial_j &= -\Gamma^j_{ik}\partial_k,\\ \overset{\circ}{\nabla}_{\partial_i}\delta_j &= \Gamma^i_{jk}\partial_k, &\quad \overset{\circ}{\nabla}_{\partial_i}\partial_j &= 0,\end{aligned} \tag{49}$$

where

$$\delta_i = \frac{\delta}{\delta x^i}, \qquad \partial_i = \frac{\partial}{\partial p_i}. \tag{50}$$

Proof. Let X, Y, and Z be vector fields on T^*M. According to the Koszul formula, we get

$$2G\left(\overset{\circ}{\nabla}_X Y, Z\right) = XG(Y,Z) + YG(Z,X) - ZG(X,Y) - G(X,[Y,Z]) - G(Y,[Z,X]) + G(Z,[X,Z]). \quad (51)$$

We put δ_i, δ_j, and δ_k instead of X, Y, and Z in (51); then we get

$$2G\left(\overset{\circ}{\nabla}_{\delta_i}\delta_j, \delta_k\right) = R_{ijk} + R_{jki} - R_{kij}. \quad (52)$$

By the equality $\sum_{(i,jk)} R_{ijk} = 0$, we find

$$G\left(\overset{\circ}{\nabla}_{\delta_i}\delta_j, \delta_k\right) = -R_{kij}, \quad (53)$$

and we put δ_i, δ_j, and ∂_k instead of X, Y, and Z in (51). So we get

$$2G\left(\overset{\circ}{\nabla}_{\delta_i}\delta_j, \partial_k\right) = \Gamma^k_{ji} - \Gamma^k_{ij}. \quad (54)$$

Since the Levi-Civita connection ∇ which is defined on M is torsion-free, we have $\Gamma^k_{ji} = \Gamma^k_{ij}$. Subsequently we find

$$G\left(\overset{\circ}{\nabla}_{\delta_i}\delta_j, \partial_k\right) = 0. \quad (55)$$

Thus we get

$$\overset{\circ}{\nabla}_{\delta_i}\delta_j = -R_{kij}\partial_k, \quad (56)$$

and the rest of the equalities can be obtained similarly. □

Theorem 11. *Let M be a Hamilton space, T^*M the cotangent bundle of M, G a pseudo-Riemann metric defined as Riemann extension of Levi-Civita connection ∇ in M, $\overset{\circ}{\nabla}$ the Levi-Civita connection on T^*M, and K the Riemann curvature tensor on T^*M. Then the components of the Riemann curvature tensor on T^*M are given by*

$$\begin{aligned}
K\left(\delta_i,\delta_j\right)\delta_k &= \left(-\delta_i R_{hjk} + \delta_j R_{hik} + R_{ljk}\Gamma^l_{ih} - R_{lik}\Gamma^l_{jh} + R_{lij}\Gamma^l_{kh}\right)\partial_h,\\
K\left(\delta_i,\delta_j\right)\partial_k &= \left(\partial_k R_{hij}\right)\partial_h,\\
K\left(\delta_i,\partial_j\right)\delta_k &= \delta_k\left(\Gamma^j_{ih}\circ\pi\right)\partial_h,\\
K\left(\partial_i,\delta_j\right)\delta_k &= -\delta_k\left(\Gamma^i_{jh}\circ\pi\right)\partial_h,\\
K\left(\delta_i,\partial_j\right)\partial_k &= K\left(\partial_i,\delta_j\right)\partial_k = K\left(\partial_i,\partial_j\right)\delta_k = K\left(\partial_i,\partial_j\right)\partial_k = 0.
\end{aligned} \quad (57)$$

Proof. Let X, Y, and Z be vector fields on T^*M. Then

$$K(X,Y,)Z = \overset{\circ}{\nabla}_X\overset{\circ}{\nabla}_Y Z - \overset{\circ}{\nabla}_Y\overset{\circ}{\nabla}_X Z - \overset{\circ}{\nabla}_{[X,Y]}Z. \quad (58)$$

If we put δ_i, δ_j, and ∂_k instead of X, Y, and Zin (58), we get

$$\begin{aligned}
K\left(\delta_i,\delta_j\right)\partial_k &= \overset{\circ}{\nabla}_{\delta_i}\overset{\circ}{\nabla}_{\delta_j}\partial_k - \overset{\circ}{\nabla}_{\delta_j}\overset{\circ}{\nabla}_{\delta_i}\partial_k - \overset{\circ}{\nabla}_{\underbrace{[\delta_i,\delta_j]}_{\in VT^*M}}\partial_k\\
&= -\delta_i\left(\Gamma^k_{jl}\circ\pi\right)\partial_l + \delta_j\left(\Gamma^k_{il}\circ\pi\right)\partial_l - \Gamma^k_{jl}\Gamma^l_{ih}\partial_h + \Gamma^k_{il}\Gamma^l_{jh}\partial_h\\
&= \left(-\frac{\partial\Gamma^k_{jh}}{\partial x^i} + \frac{\partial\Gamma^k_{ih}}{\partial x^j} - \Gamma^k_{jl}\Gamma^l_{ih} + \Gamma^k_{il}\Gamma^l_{jh}\right)\partial_h\\
&= -R^k_{hij}\partial_h.
\end{aligned} \quad (59)$$

By the equality $R_{hij} = -p_k R^k_{hij}$, we obtain

$$K\left(\delta_i,\delta_j\right)\partial_k = \frac{\partial R_{hij}}{\partial p_k}\partial_h. \quad (60)$$

The other coefficients of the curvature tensor can be obtained similarly. □

Theorem 12. *The pseudo-Riemann manifold (T^*M, G) is flat if and only if the Riemann manifold (M, g) is Euclidean.*

Proof. If the Riemann manifold (M, g) is Euclidean, the Christoffel symbols must be zero. Thus, Riemann curvature tensor R on M and K on T^*M must be zero. □

5. Concluding Remarks

The projected curves in the Hamilton space of the integral curves of the Hamilton vector field are geodesics. Furthermore, the total energy of each stage of the system is constant. The cotangent bundle of the Hamilton space is flat if and only if the Hamilton space is Euclidean.

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