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Research Article

Some Inequalities for Multiple Integrals on the n-Dimensional Ellipsoid, Spherical Shell, and Ball

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The authors establish some new inequalities of Pólya type for multiple integrals on the n-dimensional ellipsoid, spherical shell, and ball, in terms of bounds of the higher order derivatives of the integrands. These results generalize the main result in the paper by Feng Qi, Inequalities for a multiple integral, *Acta Mathematica Hungarica* (1999).

1. Introduction

In [1], it was obtained that if f is differentiable and if $f(a) = f(b) = 0$, then

$$f'(\tau) > \frac{4}{(b-a)^2}\int_a^b f(t)\,\mathrm{d}t, \tag{1}$$

for a certain τ between a and b. This inequality can be found in [2–4] and many other textbooks. It can be reformulated as follows. If $f(x)$ is differentiable and not identically constant, such that $f(a) = f(b) = 0$ and $|f'(x)| \le M$ on $[a,b]$, then

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{4}M. \tag{2}$$

In the literature, the inequalities (1) or (2) is called the Pólya integral inequality.

In [5], the inequality (1), or say (2), was generalized as

$$\begin{aligned}\left|\int_a^b f(x)\,\mathrm{d}x - \frac{1}{2}(b-a)\left[f(a)+f(b)\right]\right| \\ \le \frac{M(b-a)^2}{4} - \frac{\left[f(b)-f(a)\right]^2}{4M},\end{aligned} \tag{3}$$

where $f : [a,b] \to \mathbb{R}$ is a differentiable function and $|f'(x)| \le M$.

In [6–9], the above inequalities were refined and generalized as follows.

Theorem 1 (see [9, Proposition 1]). *Let $f(x)$ be continuous on $[a,b]$ and differentiable in (a,b). Suppose that $f(a) = f(b) = 0$, and that $m \le f'(x) \le M$ in (a,b). If $f(x)$ is not identically zero, then $m < 0 < M$ and*

$$\left|\int_a^b f(x)\,dx\right| \le -\frac{(b-a)^2}{2}\frac{mM}{M-m}. \tag{4}$$

Theorem 2 (see [6, 7, 9]). *Let $f(x)$ be continuous on $[a,b]$ and differentiable in (a,b). Suppose that $f(x)$ is not identically a constant, and that $m \le f'(x) \le M$ in (a,b). Then,*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\right| &\le \left[f(b)-f(a)-m(b-a)\right] \\ &\quad\times\left[M(b-a)-f(b)+f(a)\right] \\ &\quad\times\left(2(M-m)(b-a)\right)^{-1} \\ &= -\frac{\left[M-S_0(a,b)\right]\left[m-S_0(a,b)\right]}{2(M-m)}(b-a),\end{aligned} \tag{5}$$

where

$$S_0(a,b)=\frac{f(b)-f(a)}{b-a}. \tag{6}$$

Theorem 3 (see [8]). *For $a=(a_1,\dots,a_m)\in\mathbb{R}^m$ and $b=(b_1,\dots,b_m)\in\mathbb{R}^m$ with $a_i<b_i$ for $i=1,2,\dots,m$, denote the m-rectangles by*

$$Q_m=\prod_{i=1}^{m}[a_i,b_i],\qquad Q_m(t)=\prod_{i=1}^{m}[a_i,c_i(t)],$$
$$\mathring{Q}_m=\prod_{i=1}^{m}(a_i,b_i), \tag{7}$$

where $c_i(t)=(1-t)a_i+tb_i$ for $i=1,2,\dots,m$ and $t\in(0,1)$. Let $\nu=(\nu_1,\dots,\nu_m)$ be a multi-index; that is, ν_i is a nonnegative integer, with $|\nu|=\sum_{i=1}^{m}\nu_i$. Let $f\in C^{(n+1)}(Q_m)$ be a function of m variables on Q_m, and let its partial derivatives of $(n+1)$th order remain between $M_{n+1}(\nu)$ and $N_{n+1}(\nu)$ in $\mathring{Q}_m$; that is,

$$N_{n+1}(\nu)\le D^{\nu}f(x)\le M_{n+1}(\nu),\quad x\in\mathring{Q}_m, \tag{8}$$

where $|\nu|=n+1$ and

$$D^{\nu}f(x)=\frac{\partial^{|\nu|}f(x)}{\prod_{i=1}^{m}\partial x_i^{\nu_i}}. \tag{9}$$

Let

$$A(\nu)=\prod_{i=1}^{m}\frac{(b_i-a_i)^{\nu_i+1}}{(\nu_i+1)!}M_{n+1}(\nu),$$
$$B(\nu,f(x))=\prod_{i=1}^{m}\left[\frac{(b_i-a_i)^{\nu_i+1}}{(\nu_i+1)!}\left(\frac{\partial}{\partial x_i}\right)^{\nu_i}\right]f(x),$$
$$C(\nu)=\prod_{i=1}^{m}\frac{(b_i-a_i)^{\nu_i+1}}{(\nu_i+1)!}N_{n+1}(\nu), \tag{10}$$
$$T(\nu,t)=\prod_{i=1}^{m}\left\{1-(1-t)^{\nu_i+1}\right\}-1,$$

for $t\in(0,1)$. Then, for any $t\in(0,1)$,

(1) *when n is even, one has*

$$\begin{aligned}\sum_{|\nu|=n+1}C(\nu)t^{m+n+1}&+\sum_{|\nu|=n+1}A(\nu)T(\nu,t)\\ &\le\int_{Q_m}f(x)\,dx-\sum_{k=0}^{n}\sum_{|\nu|=k}B(\nu,f(a))t^{m+k}\\ &\quad+\sum_{k=0}^{n}(-1)^k\sum_{|\nu|=k}B(\nu,f(b))T(\nu,t)\\ &\le\sum_{|\nu|=n+1}A(\nu)t^{m+n+1}+\sum_{|\nu|=n+1}C(\nu)T(\nu,t).\end{aligned} \tag{11}$$

(2) *When n is odd, one has*

$$\begin{aligned}\sum_{|\nu|=n+1}&C(\nu)\left[t^{m+n+1}+T(\nu,t)\right]\\ &\le\int_{Q_m}f(x)\,dx-\sum_{k=0}^{n}\sum_{|\nu|=k}B(\nu,f(a))t^{m+k}\\ &\quad+\sum_{k=0}^{n}(-1)^k\sum_{|\nu|=k}B(\nu,f(b))T(\nu,t)\\ &\le\sum_{|\nu|=n+1}A(\nu)\left[t^{m+n+1}+T(\nu,t)\right].\end{aligned} \tag{12}$$

We remark that Theorem 2 has been applied in [10] to give bounds for the complete elliptic integrals of the first and second kinds.

For more information on this topic, please refer to [11–18] and [19, pp. 558–561], especially to the preprint [20].

In what follows, we will continue to use some notations from Theorem 3. Assume that $b_i,r_i>0$ for $i=1,2,\dots,n$ and $\rho,\rho_1,\rho_2>0$ with $\rho_1<\rho_2$, and adopt the following notations:

$$\begin{gathered}x=(x_1,x_2,\dots,x_n),\qquad r=(r_1,r_2,\dots,r_n)\\ \nu=(\nu_1,\nu_2,\dots,\nu_n),\end{gathered}$$
$$\begin{aligned}&\Omega(a,b,2r)\\ &\quad=\left\{x:\sum_{i=1}^{n}\frac{(x_i-a_i)^{2r_i}}{b_i^{2r_i}}\le1,x_1\ge a_1,\dots,x_n\ge a_n\right\}\\ &\quad=\Omega_{2r},\\ &\Omega_1(a,b)=\left\{x:\sum_{i=1}^{n}\frac{(x_i-a_i)^2}{b_i^2}\le1\right\}=\Omega_1,\\ &\Omega_2(\rho_1,\rho_2)=\left\{x:\rho_1^2\le\sum_{i=1}^{n}x_i^2\le\rho_2^2\right\}=\Omega_2,\\ &\Omega_3(a,\rho)=\left\{x:\sum_{i=1}^{n}(x_i-a_i)^2\le\rho^2\right\}=\Omega_3,\\ &\Omega_4(t)=\left\{x:\sum_{i=1}^{n}(x_i-a_i)^2\le\rho^2(t),\rho(t)=t\rho,t\in(0,1]\right\}\\ &\quad=\Omega_4.\end{aligned} \tag{13}$$

Moreover, let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be an $(m+1)$-times differentiable function, and let

$$\begin{aligned}g_1(x)&=\sqrt{\sum_{i=1}^{n}\frac{(x_i-a_i)^2}{b_i^2}},\quad x\in\Omega_1,\\ g_2(x)&=\sqrt{\sum_{i=1}^{n}x_i^2},\quad x\in\Omega_2,\\ g_3(x)&=\sqrt{\sum_{i=1}^{n}(x_i-a_i)^2},\quad x\in\Omega_3.\end{aligned} \tag{14}$$

In this paper, we will establish some new inequalities of Pólya type for multiple integrals of the composition function $f \circ g_1$ on the n-dimensional ellipsoid Ω_1, of the composition function $f \circ g_2$ on the spherical shell Ω_2, and of the composition function $f \circ g_3$ on the n-dimensional ball Ω_3. We also obtain a general inequality for the multiple integral $\int_{\Omega_{2r}} f(x)\mathrm{d}x$.

2. A Lemma

In order to establish some new inequalities of Pólya type for multiple integrals, we need the following lemma.

Lemma 4. *For $b_i, r_i > 0$, and $v_i > -1$, one has*

$$\int_{\Omega_{2r}} \prod_{i=1}^{n} (x_i - a_i)^{v_i} dx = \frac{\prod_{i=1}^{n} \left(b_i^{v_i+1}/r_i\right)}{2^{n-1} \sum_{i=1}^{n} \left((v_i+1)/r_i\right)} \frac{\prod_{i=1}^{n} \Gamma\left((v_i+1)/2r_i\right)}{\Gamma\left(\sum_{i=1}^{n} (v_i+1)/2r_i\right)}, \tag{15}$$

where

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \qquad \Re(z) > 0 \tag{16}$$

is the classical Euler gamma function.

Proof. Using the spherical coordinates on the region Ω_{2r} yields

$$\begin{aligned} x_1 &= b_1 s^{1/r_1} \cos^{1/r_1} \varphi_1 + a_1, \\ x_i &= b_i \left[s \cos \varphi_i \prod_{k=1}^{i-1} \sin \varphi_k \right]^{1/r_i} + a_i, \quad 2 \le i \le n-1, \\ x_n &= b_n \left[s \prod_{k=1}^{n-1} \sin \varphi_k \right]^{1/r_n} + a_n, \end{aligned} \tag{17}$$

where $0 \le s \le 1$ and $0 \le \varphi_1, \varphi_2, \ldots, \varphi_{n-1} \le \pi/2$, and

$$F_i \equiv s^2 \prod_{k=1}^{i-1} \sin^2 \varphi_k - \sum_{k=i}^{n} \left(\frac{x_k - a_k}{b_k} \right)^{2r_k} = 0, \quad 1 \le i \le n. \tag{18}$$

We note that when $i = 1$, the empty product in (18) is understood to be 1. It is clear that the expressions in (17) are solutions of (18), and that

$$\begin{aligned} J &= \frac{Dx}{D(s, \varphi_1, \varphi_2, \ldots, \varphi_{n-1})} \\ &= (-1)^n \frac{D(F_1, F_2, \ldots, F_n) / D(s, \varphi_1, \varphi_2, \ldots, \varphi_{n-1})}{D(F_1, F_2, \ldots, F_n)/Dx}. \end{aligned} \tag{19}$$

A straightforward computation gives

$$J = \prod_{k=1}^{n} \frac{b_k}{r_k} s^{\sum_{i=1}^{n}(1/r_i)-1} \prod_{k=1}^{n-1} \sin^{\sum_{i=k+1}^{n}(1/r_i)-1} \varphi_k \cos^{(1/r_k)-1} \varphi_k. \tag{20}$$

Since

$$\int_0^{\pi/2} \cos^m \varphi \sin^n \varphi \, d\varphi = \frac{\Gamma((m+1)/2)\Gamma((n+1)/2)}{2\Gamma((m+n+2)/2)}, \tag{21}$$

we obtain

$$\begin{aligned} &\int_{\Omega_{2r}} \prod_{i=1}^{n} (x_i - a_i)^{v_i} dx \\ &= \frac{\prod_{i=1}^{n} b_i^{v_i+1}}{\prod_{i=1}^{n} r_i} \int_0^1 s^{\sum_{k=1}^{n}((v_k+1)/r_k)-1} ds \\ &\quad \times \prod_{k=1}^{n-1} \int_0^{\pi/2} \sin^{\sum_{i=k+1}^{n}((v_i+1)/r_i)-1} \varphi_k \cos^{((v_k+1)/r_k)-1} \varphi_k d\varphi_k \\ &= \frac{\prod_{i=1}^{n} \left(b_i^{v_i+1}/r_i\right)}{2^{n-1} \sum_{i=1}^{n} \left((v_i+1)/r_i\right)} \frac{\prod_{i=1}^{n} \Gamma\left((v_i+1)/2r_i\right)}{\Gamma\left(\sum_{i=1}^{n} (v_i+1)/2r_i\right)}. \end{aligned} \tag{22}$$

The proof of Lemma 4 is complete. □

3. Main Results

Now, we start out to state and prove our main results.

Theorem 5. *Let $f : [0,1] \to \mathbb{R}$ be an $(m+1)$-times differentiable function satisfying*

$$N(m) \le f^{(m+1)}(u) \le M(m). \tag{23}$$

Then, one has

$$\begin{aligned} &\frac{2\pi^{n/2}(n-1)!\prod_{i=1}^{n} b_i}{\Gamma(n/2)(n+m+1)!} \min\left\{(-1)^{m+1} M(m), (-1)^{m+1} N(m)\right\} \\ &\quad \le \int_{\Omega_1} f(g_1(x)) \, dx \\ &\quad - \sum_{k=0}^{m} \frac{(-1)^k 2\pi^{n/2}(n-1)!\prod_{i=1}^{n} b_i}{(n+k)!\Gamma(n/2)} f^{(k)}(1) \\ &\quad \le \frac{2\pi^{n/2}(n-1)!\prod_{i=1}^{n} b_i}{\Gamma(n/2)(n+m+1)!} \\ &\quad \times \max\left\{(-1)^{m+1} M(m), (-1)^{m+1} N(m)\right\}. \end{aligned} \tag{24}$$

Proof. Using the transformation in (17) on Ω_1 and letting $r_i = 1$ for $i = 1, 2, \ldots, n$ yield the Jacobian determinant

$$J = s^{n-1} \prod_{k=1}^{n} b_k \prod_{k=1}^{n-2} \sin^{n-k-1} \varphi_k, \tag{25}$$

$$\begin{gathered} 0 \le s \le 1, \qquad 0 \le \varphi_1, \varphi_2, \ldots, \varphi_{n-2} \le \pi, \\ 0 \le \varphi_{n-1} \le 2\pi. \end{gathered} \tag{26}$$

Because

$$\int_0^{\pi} \sin^n t dt = 2 \int_0^{\pi/2} \cos^n t dt = \frac{\sqrt{\pi}\Gamma((n+1)/2)}{\Gamma((n+2)/2)}, \tag{27}$$

we have

$$\prod_{k=1}^{n-2}\int_0^{\pi}\sin^{n-k-1}\varphi_k d\varphi_k\int_0^{2\pi}d\varphi_{n-1}=\frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{28}$$

By integration by parts, one has

$$\begin{aligned}\int_\alpha^\beta & s^{n-1}f(s)\,ds\\ &=\sum_{k=0}^{m}\frac{(-1)^k(n-1)!\left[\beta^{n+k}f^{(k)}(\beta)-\alpha^{n+k}f^{(k)}(\alpha)\right]}{(n+k)!}\\ &\quad+(-1)^{m+1}\frac{(n-1)!}{(n+m)!}\int_\alpha^\beta f^{(m+1)}(s)\,s^{n+m}ds.\end{aligned} \tag{29}$$

Choosing $\alpha=0$ and $\beta=1$ in the above equality shows that

$$\begin{aligned}\int_{\Omega_1} & f(g_1(x))\,dx\\ &=\prod_{k=1}^{n}b_k\int_0^1 s^{n-1}f(s)\,ds\prod_{k=1}^{n-2}\int_0^{\pi}\sin^{n-k-1}\varphi_k d\varphi_k\int_0^{2\pi}d\varphi_{n-1}\\ &=\frac{2\pi^{n/2}\prod_{i=1}^{n}b_i}{\Gamma(n/2)}\sum_{k=0}^{m}\frac{(-1)^k(n-1)!f^{(k)}(1)}{(n+k)!}\\ &\quad+(-1)^{m+1}\frac{2\pi^{n/2}\prod_{i=1}^{n}b_i}{\Gamma(n/2)}\frac{(n-1)!}{(n+m)!}\int_0^1 f^{(m+1)}(s)\,s^{n+m}ds.\end{aligned} \tag{30}$$

Further utilizing the condition (23) leads to the inequality (24). The proof of Theorem 5 is completed. □

Theorem 6. *Let $f:[\rho_1,\rho_2]\to\mathbb{R}$ be an $(m+1)$-times differentiable function satisfying the inequality* (23). *Then, one has*

$$\begin{aligned}&\frac{2\pi^{n/2}\left(\rho_2^{n+m+1}-\rho_1^{n+m+1}\right)(n-1)!}{\Gamma(n/2)(n+m+1)!}\\ &\qquad\times\min\left\{(-1)^{m+1}M(m),(-1)^{m+1}N(m)\right\}\\ &\quad\le\int_{\Omega_2}f(g_2(x))\,dx-\frac{2\pi^{n/2}}{\Gamma(n/2)}\\ &\qquad\times\sum_{k=0}^{m}\frac{(-1)^k(n-1)!\left[\rho_2^{n+k}f^{(k)}(\rho_2)-\rho_1^{n+k}f^{(k)}(\rho_1)\right]}{(n+k)!}\\ &\quad\le\frac{2\pi^{n/2}\left(\rho_2^{n+m+1}-\rho_1^{n+m+1}\right)(n-1)!}{\Gamma(n/2)(n+m+1)!}\\ &\qquad\times\max\left\{(-1)^{m+1}M(m),(-1)^{m+1}N(m)\right\}.\end{aligned} \tag{31}$$

Proof. Using the transformation in (17) on Ω_2 and choosing $r_i=1$, $a_i=0$, and $b_i=1$ for $i=1,2,\ldots,n$ yield

$$\begin{gathered}J=s^{n-1}\prod_{k=1}^{n-2}\sin^{n-k-1}\varphi_k,\\ \rho_1\le s\le\rho_2,\qquad 0\le\varphi_1,\varphi_2,\ldots,\varphi_{n-2}\le\pi,\\ 0\le\varphi_{n-1}\le 2\pi.\end{gathered} \tag{32}$$

Further letting $\alpha=\rho_1$ and $\beta=\rho_2$ in (29) gives

$$\begin{aligned}\int_{\Omega_2} & f(g_2(x))\,dx\\ &=\int_{\rho_1}^{\rho_2}s^{n-1}f(s)\,ds\prod_{k=1}^{n-2}\int_0^{\pi}\sin^{n-k-1}\varphi_k d\varphi_k\int_0^{2\pi}d\varphi_{n-1}\\ &=\frac{2\pi^{n/2}}{\Gamma(n/2)}\sum_{k=0}^{m}(-1)^k(n-1)!\\ &\qquad\times\left[\rho_2^{n+k}f^{(k)}(\rho_2)-\rho_1^{n+k}f^{(k)}(\rho_1)\right]\\ &\qquad\times((n+k)!)^{-1}\\ &\quad+\frac{(-1)^{m+1}2\pi^{n/2}(n-1)!}{\Gamma(n/2)(n+m)!}\int_{\rho_1}^{\rho_2}f^{(m+1)}(s)\,s^{n+m}ds.\end{aligned} \tag{33}$$

Hence, by virtue of the condition (23), the inequality (31) follows immediately. The proof of Theorem 6 is completed. □

Theorem 7. *Let $f:[0,\rho]\to\mathbb{R}$ be an $(m+1)$-times differentiable function satisfying* (23). *Then, one has*

$$\begin{aligned}&\frac{2\pi^{n/2}(n-1)!\rho^{n+m+1}}{\Gamma(n/2)(n+m+1)!}\min\left\{(-1)^{m+1}M(m),(-1)^{m+1}N(m)\right\}\\ &\quad\le\int_{\Omega_3}f(g_3(x))\,dx\\ &\qquad-\sum_{k=0}^{m}\frac{(-1)^k2\pi^{n/2}(n-1)!\rho^{n+k}}{(n+k)!\Gamma(n/2)}f^{(k)}(\rho)\\ &\quad\le\frac{2\pi^{n/2}(n-1)!\rho^{n+m+1}}{\Gamma(n/2)(n+m+1)!}\\ &\qquad\times\max\left\{(-1)^{m+1}M(m),(-1)^{m+1}N(m)\right\}.\end{aligned} \tag{34}$$

Proof. Similar to the proof of Theorem 5, by choosing $b_1=b_2=\cdots=b_n=\rho$ and $0\le s\le\rho$, we obtain the inequality (34). The proof is complete. □

Corollary 8. *Under the conditions of Theorem 7, if $f^{(k)}(\rho) = 0$ for $k = 0, 1, 2, \ldots, m$, then*

$$\frac{2\pi^{n/2}(n-1)!\rho^{n+m+1}}{\Gamma(n/2)(n+m+1)!}N(m) \le (-1)^{m-1}\int_{\Omega_3} f(g_3(x))\,dx \le \frac{2\pi^{n/2}(n-1)!\rho^{n+m+1}}{\Gamma(n/2)(n+m+1)!}M(m). \tag{35}$$

4. A More General Inequality

Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ be an n-tuple index; that is, the numbers $\nu_1, \nu_2, \ldots, \nu_n$ are nonnegative and denote $|\nu| = \sum_{i=1}^n \nu_i$. Let $f : \Omega_{2r} \to \mathbb{R}$ be a function which has an $m+1$ times continuous derivative on Ω_{2r}, and let

$$D^\nu f(x) = \frac{\partial^{|\nu|} f(x)}{\prod_{i=1}^n \partial x_i^{\nu_i}},$$

$$H(\nu, b, r) = \frac{\prod_{i=1}^n \left(b_i^{\nu_i+1}/r_i\right)}{2^{n-1}\sum_{i=1}^n ((\nu_i+1)/r_i)}\frac{\prod_{i=1}^n \Gamma((\nu_i+1)/2r_i)}{\Gamma(\sum_{i=1}^n (\nu_i+1)/2r_i)}, \tag{36}$$

and $|\nu| = m+1$.

Theorem 9. *Let $f \in C^{m+1}(\Omega_{2r})$ satisfy*

$$N_{m+1}(\nu) \le D^\nu f(x) \le M_{m+1}(\nu), \quad x \in \Omega_{2r}. \tag{37}$$

Then

$$H(\nu, b, r)\sum_{|\nu|=m+1}\frac{N_{m+1}(\nu)}{\prod_{i=1}^n \nu_i!} \le \int_{\Omega_{2r}} f(x)\,dx - H(\nu, b, r)\sum_{j=0}^m\sum_{|\nu|=j}\frac{D^\nu f(a)}{\prod_{i=1}^n \nu_i!} \le H(\nu, b, r)\sum_{|\nu|=m+1}\frac{M_{m+1}(\nu)}{\prod_{i=1}^n \nu_i!}. \tag{38}$$

Proof. By Taylor's formula, we obtain

$$f(x) = \sum_{j=0}^m \frac{1}{j!}\left[\sum_{i=1}^n (x_i - a_i)\frac{\partial}{\partial x_i}\right]^j f(a) + R_m(x), \tag{39}$$

where

$$R_m(x) = \frac{1}{(m+1)!}\left[\sum_{i=1}^n (x_i - a_i)\frac{\partial}{\partial x_i}\right]^{|\nu|} f(a + \theta(x-a)), \quad \theta \in (0,1). \tag{40}$$

Using

$$\left(\sum_{i=1}^n q_i\right)^j = j!\sum_{|\nu|=j}\prod_{i=1}^n \frac{q_i^{\nu_i}}{\nu_i!}, \tag{41}$$

we have

$$f(x) = \sum_{j=0}^m\sum_{|\nu|=j}\frac{1}{\prod_{i=1}^n \nu_i!}\prod_{i=1}^n\left[(x_i - a_i)\frac{\partial}{\partial x_i}\right]^{\nu_i} f(a) + \sum_{|\nu|=m+1}\frac{1}{\prod_{i=1}^n \nu_i!} \times \prod_{i=1}^n\left[(x_i - a_i)\frac{\partial}{\partial x_i}\right]^{\nu_i} f(a + \theta(x-a)). \tag{42}$$

Integrating on both sides of the above equality leads to

$$\begin{aligned}\int_{\Omega_{2r}} f(x)\,\mathrm{d}x &= \sum_{j=0}^m\sum_{|\nu|=j}\frac{1}{\prod_{i=1}^n \nu_i!} \times \int_{\Omega_{2r}}\prod_{i=1}^n\left[(x_i - a_i)\frac{\partial}{\partial x_i}\right]^{\nu_i} f(a)\,\mathrm{d}x \\ &\quad + \sum_{|\nu|=m+1}\frac{1}{\prod_{i=1}^n \nu_i!} \times \int_{\Omega_{2r}}\prod_{i=1}^n\left[(x_i - a_i)\frac{\partial}{\partial x_i}\right]^{\nu_i} \times f(a + \theta(x-a))\,\mathrm{d}x \\ &= \sum_{j=0}^m\sum_{|\nu|=j}\frac{1}{\prod_{i=1}^n \nu_i!}\frac{\partial^{|\nu|} f(a)}{\prod_{i=1}^n \partial x_i^{\nu_i}} \times \int_{\Omega_{2r}}\prod_{i=1}^n (x_i - a_i)^{\nu_i}\,\mathrm{d}x \\ &\quad + \sum_{|\nu|=m+1}\frac{1}{\prod_{i=1}^n \nu_i!} \times \int_{\Omega_{2r}}\prod_{i=1}^n (x_i - a_i)^{\nu_i}\frac{\partial^{|\nu|} f(a + \theta(x-a))}{\prod_{i=1}^n \partial x_i^{\nu_i}}\,\mathrm{d}x \\ &= I_1 + I_2,\end{aligned} \tag{43}$$

where

$$I_1 = \sum_{j=0}^m\sum_{|\nu|=j}\frac{1}{\prod_{i=1}^n \nu_i!}\frac{\partial^{|\nu|} f(a)}{\prod_{i=1}^n \partial x_i^{\nu_i}}\int_{\Omega_{2r}}\prod_{i=1}^n (x_i - a_i)^{\nu_i}\,\mathrm{d}x, \tag{44}$$

$$I_2 = \sum_{|\nu|=m+1}\frac{1}{\prod_{i=1}^n \nu_i!}\int_{\Omega_{2r}}\prod_{i=1}^n (x_i - a_i)^{\nu_i}\frac{\partial^{|\nu|} f(a + \theta(x-a))}{\prod_{i=1}^n \partial x_i^{\nu_i}}\,\mathrm{d}x. \tag{45}$$

By Lemma 4 and (44), one has

$$\begin{aligned} I_1 &= \sum_{j=0}^{m} \sum_{|\nu|=j} \frac{1}{\prod_{i=1}^{n} \nu_i !} \frac{\partial^{|\nu|} f(a)}{\prod_{i=1}^{n} \partial x_i^{\nu_i}} \\ &\quad \times \frac{\prod_{i=1}^{n} \left(b_i^{\nu_i+1}/r_i\right)}{2^{n-1} \sum_{i=1}^{n} \left((\nu_i+1)/r_i\right)} \frac{\prod_{i=1}^{n} \Gamma\left((\nu_i+1)/2r_i\right)}{\Gamma\left(\sum_{i=1}^{n} (\nu_i+1)/2r_i\right)} \\ &= \sum_{j=0}^{m} \sum_{|\nu|=j} \frac{D^{\nu} f(a)}{\prod_{i=1}^{n} \nu_i !} H(\nu, b, r). \end{aligned} \tag{46}$$

From (37) and

$$I_2 = \sum_{|\nu|=m+1} \frac{1}{\prod_{i=1}^{n} \nu_i !} \int_{\Omega_{2r}} \prod_{i=1}^{n} (x_i - a_i)^{\nu_i} D^{\nu} f(a + \theta(x-a))\, dx, \tag{47}$$

we have

$$\sum_{|\nu|=m+1} \frac{N_{m+1}(\nu)}{\prod_{i=1}^{n} \nu_i !} H(\nu, b, r) \le I_2 \le \sum_{|\nu|=m+1} \frac{M_{m+1}(\nu)}{\prod_{i=1}^{n} \nu_i !} H(\nu, b, r). \tag{48}$$

Consequently, the proof of Theorem 9 is complete. □

Corollary 10. *Let $|\nu| = m+1$, and let $f \in C^{m+1}(\Omega_4)$ with (37). Then, for $t \in (0,1]$ one has*

$$\begin{aligned} &H(\nu, t) \sum_{|\nu|=m+1} \frac{N_{m+1}(\nu)}{\prod_{i=1}^{n} \nu_i !} \\ &\quad \le \int_{\Omega_4(t)} f(x)\, dx - H(\nu, t) \sum_{j=0}^{m} \sum_{|\nu|=j} \frac{D^{\nu} f(a)}{\prod_{i=1}^{n} \nu_i !} \\ &\quad \le H(\nu, t) \sum_{|\nu|=m+1} \frac{M_{m+1}(\nu)}{\prod_{i=1}^{n} \nu_i !}, \end{aligned} \tag{49}$$

where

$$\begin{aligned} H(\nu, t) = &\frac{\rho^{n+m+1}}{n+m+1} \frac{\prod_{i=1}^{n} \left[1 + (-1)^{\nu_i}\right]}{2^{n-1}} \\ &\times \frac{\prod_{i=1}^{n} \Gamma\left((\nu_i+1)/2\right)}{\Gamma\left(\sum_{i=1}^{n} (\nu_i+1)/2\right)} t^{n+m+1}. \end{aligned} \tag{50}$$

5. An Application

Now, we list some special cases of Ω_{2r} as follows.

(1) If we take $r_1 = r_2 = \cdots = r_n = 1/2$, the body Ω_{2r} becomes a closed region between the n-dimensional pyramid and the rectangle $x_i = a_i$ for $i = 1, 2, \ldots, n$.

(2) If we take $r_1 = r_2 = \cdots = r_n = 1$, the body Ω_{2r} is a closed region between the n-dimensional ellipsoid $\Omega_1(a, b)$ and the rectangle $x_i = a_i$ for $i = 1, 2, \ldots, n$.

(3) If we take $r_1 = r_2 = \cdots = r_n = 1$ and $b_1 = b_2 = \cdots = b_n = \rho$, the body Ω_{2r} is a closed region between the n-dimensional ball $\Omega_3(a, \rho)$ and the rectangle $x_i = a_i$ for $i = 1, 2, \ldots, n$.

In the calculation of the uniform n-dimensional volume, static moment, the moment of inertia, the centrifugal moment, and so on, have important applications. See [21, 22].

To show the applicability of the above main results, we now estimate the value of a triple integral

$$I = \iiint_V \sin\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^{5/2} dx\, dy\, dz, \tag{51}$$

where V is the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1. \tag{52}$$

Choosing $n = 3$, $b_1 = a$, $b_2 = b$, and $b_3 = c$ in (25), the Jacobian determinant is

$$J = abcs^2 \sin\varphi_1, \tag{53}$$

$$\begin{aligned} I &= \int_0^{2\pi} d\varphi_2 \int_0^{\pi} d\varphi_1 \int_0^1 abc\, s^2 \sin\varphi_1 \sin s^5 ds \\ &= 4\pi abc \int_0^1 s^2 \sin s^5 ds. \end{aligned} \tag{54}$$

Using Taylor's formula, it follows that

$$\sin x = \sum_{k=1}^{m} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} + (-1)^m \frac{x^{2m+1}}{(2m+1)!} \cos(\theta x), \quad 0 < \theta < 1. \tag{55}$$

Specially, we have

$$\begin{aligned} \sin x &= \sum_{k=1}^{3} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} - \frac{x^7}{7!} \cos\theta_1 x, \\ \sin x &= \sum_{k=1}^{6} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} + \frac{x^{13}}{13!} \cos\theta_2 x, \end{aligned} \tag{56}$$

where $0 < \theta_1, \theta_2 < 1$ and $0 < x < 1$. Therefore,

$$\sum_{k=1}^{6} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} \le \sin x \le \sum_{k=1}^{3} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}. \tag{57}$$

By (54) and the above inequality, we have

$$\begin{aligned}
&\sum_{k=1}^{6}\frac{(-1)^{k-1}}{(2k-1)!}\int_0^1 s^{10k-3}\mathrm{d}s \\
&\quad\le \int_0^1 s^2\sin s^5\mathrm{d}s \le \sum_{k=1}^{3}\frac{(-1)^{k-1}}{(2k-1)!}\int_0^1 s^{10k-3}\mathrm{d}s, \\
&\sum_{k=1}^{6}\frac{(-1)^{k-1}}{(2k-1)!\,(10k-2)} \\
&\quad\le \int_0^1 s^2\sin s^5\mathrm{d}s \le \sum_{k=1}^{3}\frac{(-1)^{k-1}}{(2k-1)!\,(10k-2)}, \\
&\quad\frac{61249255037}{131964940800}\pi abc \le I \le \frac{3509}{7560}\pi abc.
\end{aligned} \tag{58}$$

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