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Research Article

Global Strong Solution to the Density-Dependent 2-D Liquid Crystal Flows

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The initial-boundary value problem for the density-dependent flow of nematic crystals is studied in a 2-D bounded smooth domain. For the initial density away from vacuum, the existence and uniqueness is proved for the global strong solution with the large initial velocity u_0 and small ∇d_0. We also give a regularity criterion $\nabla d \in L^p(0,T;L^q(\Omega))$ $((2/q)+(2/p)=1,\ 2<q\le\infty)$ of the problem with the Dirichlet boundary condition $u=0$, $d=d_0$ on $\partial\Omega$.

1. Introduction and Main Results

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$, and ν is the unit outward normal vector on $\partial\Omega$. We consider the global strong solution to the density-dependent incompressible liquid crystal flow [1–4] as follows:

$$\operatorname{div} u = 0, \tag{1}$$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{2}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla\pi - \Delta u = -\nabla\cdot(\nabla d \odot \nabla d), \tag{3}$$

$$\partial_t d + u\cdot\nabla d - \Delta d = |\nabla d|^2 d, \tag{4}$$

in $(0,\infty)\times\Omega$ with initial and boundary conditions

$$(\rho, u, d)(\cdot, 0) = (\rho_0, u_0, d_0) \quad \text{in } \Omega, \tag{5}$$

$$u = 0, \quad \partial_\nu d = 0 \quad \text{on } \partial\Omega, \tag{6}$$

where ρ denotes the density, u the velocity, d the unit vector field that represents the macroscopic molecular orientations, and π the pressure. The symbol $\nabla d \odot \nabla d$ denotes a matrix whose (i,j)th entry is $\partial_i d \partial_j d$, and it is easy to find that $\nabla d \odot \nabla d = \nabla d^T \nabla d$.

When d is a given constant unit vector, then (1), (2), and (3) represent the well-known density-dependent Navier-Stokes system, which has received many studies; see [5–7] and references therein.

When $\rho \equiv 1$ and $\Omega := \mathbb{R}^2$, Xu and Zhang [8] proved global existence of weak solutions to the problem if $u_0 \in L^2, \nabla d_0 \in L^2, |d_0| = 1$, and

$$\exp\left(216\left(\|u_0\|_{L^2}^2 + \frac{1}{16}\right)^2\right)\|\nabla d_0\|_{L^2}^2 < \frac{1}{16}. \tag{7}$$

When $\rho \equiv 1$ and (6) is replaced by

$$u = 0, \qquad d = d_0 \quad \text{on } \partial\Omega. \tag{8}$$

Lin et al. [9] proved the global existence of weak solutions to the system (1)–(5) and (8), which are smooth away from at most finitely many singular times, and they also prove a regularity criterion

$$d \in L^2\left(0,T;H^2(\Omega)\right). \tag{9}$$

When $\rho = 1$ and the term $|\nabla d|^2$ in (4) is replaced by $(1-|d|^2)d$, then the problem has been studied in [10–15].

Very recently, Wen and Ding [16] proved the global existence and uniqueness of strong solutions to the problem (1)–(6) with small u_0 and ∇d_0 and the local strong solutions with large initial data when $\Omega \subseteq \mathbb{R}^2$ is a smooth bounded domain.

Fan et al. [17] studied the regularity criterion of the Cauchy problem (1)–(5) when $\Omega := \mathbb{R}^2$.

We will prove the following.

Theorem 1. *Let $0 < m \le \rho_0 \le M < \infty$, $\rho_0 \in W^{1,r}$ for some $r \in (2, \infty)$, $u_0 \in H_0^1 \cap H^2$, and $d_0 \in H^3$ with $\operatorname{div} u_0 = 0$, and $|d_0| = 1$ in Ω. If*

$$\|\nabla d_0\|_{L^2}^2 \exp\left[216 \frac{C_0^2}{m}\left(\|\sqrt{\rho_0} u_0\|_{L^2}^2 + \frac{1}{8C_0^2}\right)^2\right] \le \frac{1}{8C_0^2}, \quad (10)$$

with an absolute constant C_0 in (22), then the problem (1)–(6) has a unique global-in-time strong solution (ρ, u, d) satisfying

$$\|\rho\|_{L^\infty(0,T;W^{1,r})} \le C, \quad \|\rho_t\|_{L^\infty(0,T;L^r)} \le C,$$
$$\|u\|_{L^\infty(0,T;H^2) \cap L^2(0,T;W^{2,s})} \le C, \quad \text{for some } s > 2, \quad (11)$$
$$\|d\|_{L^\infty(0,T;H^3)} \le C.$$

Remark 2. When $\Omega := \mathbb{R}^2$, Theorem 1 is also correct, thus improving the result in [18], where u_0 and ∇d_0 are assumed to be small.

Next, we consider (1)–(4) with $\rho \equiv 1$ as follows:

$$\operatorname{div} u = 0, \quad (12)$$

$$\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u = -\nabla \cdot (\nabla d \odot \nabla d), \quad (13)$$

$$\partial_t d + u \cdot \nabla d - \Delta d = |\nabla d|^2 d, \quad (14)$$

$$u = 0, \quad d = d_0 \quad \text{on } \partial\Omega, \quad (15)$$

$$(u, d)(\cdot, 0) = (u_0, d_0) \quad \text{in } \Omega. \quad (16)$$

We will prove the following.

Theorem 3. *Let $u_0 \in L^2$ and $d_0 \in H^1$ with $\operatorname{div} u_0 = 0$ and $|d_0| = 1$ in Ω and $d_0 \in C^{2,\beta}(\partial\Omega)$ for some $\beta \in (0,1)$. If d satisfies*

$$\nabla d \in L^q(0, T; L^p), \quad \frac{2}{q} + \frac{2}{p} = 1, \quad 2 < p \le \infty, \quad (17)$$

then the strong solution (u, d) can be extended beyond $T > 0$.

Remark 4. In [9], the authors prove the regularity criterion (9) for the problem (12)–(16), and our condition (17) is weaker than (9). Moreover, (17) is scaling invariant for (12)–(14).

2. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. Since the local-in-time well-posedness has been proved in [16], we only need to establish a priori estimates. Also, by the local well-posedness result in [16], we note that ∇d is absolutely continuous on $[0, T]$ for any given $T > 0$.

By the maximum principle, it follows from (1) and (2) that

$$0 < m \le \rho \le M < \infty. \quad (18)$$

Testing (3) by u and using (1) and (2), we see that

$$\frac{1}{2}\frac{d}{dt}\int \rho u^2 dx + \int |\nabla u|^2 dx = -\int (u \cdot \nabla) d \cdot \Delta d \, dx. \quad (19)$$

Testing (4) by $-\Delta d - |\nabla d|^2 d$, using $|d| = 1$, we find that

$$\frac{1}{2}\frac{d}{dt}\int |\nabla d|^2 dx + \int \left|\Delta d + |\nabla d|^2 d\right|^2 dx = \int (u \cdot \nabla) d \cdot \Delta d \, dx. \quad (20)$$

Summing up (19) and (20) and integrating over $(0, T)$, we get

$$\int \left(\rho u^2 + |\nabla d|^2\right) dx + 2\int_0^T \int \left(|\nabla u|^2 + \left|\Delta d + |\nabla d|^2 d\right|\right) dx \, dt \le \int \left(\rho_0 u_0^2 + |\nabla d_0|^2\right) dx. \quad (21)$$

Since $\partial_\nu d = 0$ on $(0, \infty) \times \partial\Omega$, we have the following Gagliardo-Nirenberg inequality:

$$\|\nabla d\|_{L^4}^2 \le C_0 \|\nabla d\|_{L^2} \|\Delta d\|_{L^2}. \quad (22)$$

By (20) and the Ladyzhenskaya inequality in 2D, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla d|^2 dx + \int \left|\Delta d + |\nabla d|^2 d\right|^2 dx \\
&\le \|u\|_{L^4} \|\nabla d\|_{L^4} \|\Delta d\|_{L^2} \\
&\le \sqrt{2}\|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2} \cdot \sqrt{C_0} \|\nabla d\|_{L^2}^{1/2} \|\Delta d\|_{L^2}^{3/2} \\
&\le \frac{\|\Delta d\|_{L^2}^2}{8} + 216 C_0^2 \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla d\|_{L^2}^2 \\
&\le \frac{\|\Delta d\|_{L^2}^2}{8} + 216 \frac{C_0^2}{m}\left(\|\sqrt{\rho_0} u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2\right) \|\nabla u\|_{L^2}^2 \|\nabla d\|_{L^2}^2.
\end{aligned} \quad (23)$$

On the other hand, since $(a + b)^2 \ge (a^2/2) - b^2$, we have

$$\begin{aligned}
\int \left|\Delta d + |\nabla d|^2 d\right|^2 dx &\ge \frac{\|\Delta d\|_{L^2}^2}{2} - \|\nabla d\|_{L^4}^4 \\
&\ge \frac{\|\Delta d\|_{L^2}^2}{2} - C_0^2 \|\nabla d\|_{L^2}^2 \|\Delta d\|_{L^2}^2.
\end{aligned} \quad (24)$$

If the initial data $\|\nabla d_0\|_{L^2}^2 < (1/C_0^2)(1/8)$, then there exists $T_1 > 0$ such that for any $t \in [0, T_1]$,

$$\|\nabla d(t)\|_{L^2}^2 \le \frac{1}{C_0^2} \cdot \frac{1}{4}. \quad (25)$$

We denote by T_1^* the maximal time such that (25) holds on $[0, T_1^*]$. Therefore, by (23), (24), and (25), it follows that for any $t \in [0, T_1^*]$,

$$\begin{aligned}
&\frac{d}{dt}\int |\nabla d|^2 dx + \frac{1}{4}\|\Delta d\|_{L^2}^2 \\
&\quad \le 432\frac{C_0^2}{m}\left(\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2\right)\|\nabla u\|_{L^2}^2\|\nabla d\|_{L^2}^2 \\
&\quad \le 432\frac{C_0^2}{m}\left(\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \frac{1}{8C_0^2}\right)\|\nabla u\|_{L^2}^2\|\nabla d\|_{L^2}^2,
\end{aligned} \tag{26}$$

which gives

$$\begin{aligned}
&\|\nabla d(t)\|_{L^2}^2 + \frac{1}{4}\int_0^t \|\Delta d(\tau)\|_{L^2}^2 d\tau \\
&\quad \le \|\nabla d_0\|_{L^2}^2 \exp\left[432\frac{C_0^2}{m}\left(\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \frac{1}{8C_0^2}\right)\right. \\
&\qquad\qquad \left.\times \int_0^{T_1^*} \|\nabla u\|_{L^2}^2 d\tau\right] \\
&\quad \le \|\nabla d_0\|_{L^2}^2 \exp\left[216\frac{C_0^2}{m}\left(\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \frac{1}{8C_0^2}\right)^2\right] \\
&\quad \le \frac{1}{8C_0^2},
\end{aligned} \tag{27}$$

which implies that $T_1^* = T$ if the initial data satisfies

$$\|\nabla d_0\|_{L^2}^2 \exp\left[216\frac{C_0^2}{m}\left(\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \frac{1}{8C_0^2}\right)^2\right] \le \frac{1}{8C_0^2}. \tag{28}$$

Let T^* be a maximal existence time for the solution (ρ, u, d). Then, (18), (21), and (27) ensure that $T^* = \infty$ by continuity argument.

Testing (3) by u_t, using (1), (18), (21), (22), $|d| = 1$, and the Gagliardo-Nirenberg inequalities, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2 dx + \int \rho u_t^2 dx \\
&\quad = -\int \rho u\cdot\nabla u\cdot u_t dx - \int u_t\cdot\nabla d\cdot\Delta d\, dx \\
&\quad \le C\|\sqrt{\rho}u_t\|_{L^2}\left(\|u\|_{L^4}\|\nabla u\|_{L^4} + \|\nabla d\|_{L^4}\|\Delta d\|_{L^4}\right) \\
&\quad \le C\|\sqrt{\rho}u_t\|_{L^2}\left[\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}\left(\|\Delta u\|_{L^2}^{1/2} + \|u\|_{L^2}^{1/2}\right)\right. \\
&\qquad\qquad \left. + \|\nabla d\|_{L^2}^{1/2}\|\Delta d\|_{L^2}\left(\|\nabla\Delta d\|_{L^2}^{1/2} + \|d\|_{L^2}^{1/2}\right)\right] \\
&\quad \le C\|\sqrt{\rho}u_t\|_{L^2}\left(\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{1/2} + \|\nabla u\|_{L^2} + \|\Delta d\|_{L^2}\right. \\
&\qquad\qquad \left.\times\|\nabla\Delta d\|_{L^2}^{1/2} + \|\Delta d\|_{L^2}\right).
\end{aligned} \tag{29}$$

On the other hand, (3) can be rewritten as

$$-\Delta u + \nabla\pi = f := -\rho u_t - \rho u\cdot\nabla u - \nabla\cdot(\nabla d\odot\nabla d). \tag{30}$$

By the H^2-theory of Stokes system, we have

$$\begin{aligned}
\|\Delta u\|_{L^2} &\le C\|f\|_{L^2} \\
&\le C\|\sqrt{\rho}u_t\|_{L^2} + C\|u\|_{L^4}\|\nabla u\|_{L^4} + C\|\nabla d\|_{L^4}\|\Delta d\|_{L^4} \\
&\le C\|\sqrt{\rho}u_t\|_{L^2} + C\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{1/2} + C\|\nabla u\|_{L^2} \\
&\quad + C\|\Delta d\|_{L^2}\|\nabla\Delta d\|_{L^2}^{1/2} + C\|\Delta d\|_{L^2},
\end{aligned} \tag{31}$$

which yields

$$\begin{aligned}
\|\Delta u\|_{L^2} &\le C\|\sqrt{\rho}u_t\|_{L^2} + C\|\nabla u\|_{L^2}^2 + C \\
&\quad + C\|\Delta d\|_{L^2}\|\nabla\Delta d\|_{L^2}^{1/2} + C\|\Delta d\|_{L^2}.
\end{aligned} \tag{32}$$

Inserting (32) into (29), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2 dx + \int \rho u_t^2 dx \\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^{3/2}\|\nabla u\|_{L^2} + C\|\sqrt{\rho}u_t\|_{L^2}\left(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}\right) \\
&\quad + C\|\sqrt{\rho}u_t\|_{L^2}\|\Delta d\|_{L^2}\|\nabla\Delta d\|_{L^2}^{1/2} + C\|\sqrt{\rho}u_t\|_{L^2}\|\Delta d\|_{L^2} \\
&\le \frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 + C + \frac{1}{8}\|\nabla\Delta d\|_{L^2}^2 + C\|\Delta d\|_{L^2}^4.
\end{aligned} \tag{33}$$

Applying Δ to (4), testing by Δd, using $|d| = 1$, (21) and (22), and the Gagliardo-Nirenberg inequalities, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Delta d|^2 dx + \int |\nabla\Delta d|^2 dx \\
&\quad \le \int \left|\nabla\left(|\nabla d|^2 d\right)\right| |\nabla\Delta d|\, dx + \int |\nabla(u\cdot\nabla d)|\,|\nabla\Delta d|\, dx \\
&\quad \le C\left(\|\nabla d\|_{L^6}^3 + \|\nabla d\|_{L^4}\|\Delta d\|_{L^4} + \|u\|_{L^4}\|\Delta d\|_{L^4}\right. \\
&\qquad\quad \left. + \|\nabla u\|_{L^2}\|\nabla d\|_{L^\infty}\right)\|\nabla\Delta d\|_{L^2} \\
&\quad \le C\left(\|\nabla d\|_{L^2}\|\Delta d\|_{L^2}^2 + \|\Delta d\|_{L^2}\|\nabla\Delta d\|_{L^2}^{1/2} + \|\Delta d\|_{L^2}\right. \\
&\qquad\quad + \|\nabla u\|_{L^2}^{1/2}\|\Delta d\|_{L^2}^{1/2}\|\nabla\Delta d\|_{L^2}^{1/2} \\
&\qquad\quad + \|\nabla u\|_{L^2}^{1/2}\|\Delta d\|_{L^2}^{1/2} + \|\nabla u\|_{L^2} \\
&\qquad\quad \left.\times\|\nabla d\|_{L^2}^{1/2}\|\nabla\Delta d\|_{L^2}^{1/2}\right)\|\nabla\Delta d\|_{L^2} \\
&\quad \le \frac{1}{8}\|\nabla\Delta d\|_{L^2}^2 + C\|\Delta d\|_{L^2}^4 + C + C\|\nabla u\|_{L^2}^4.
\end{aligned} \tag{34}$$

Here, we have used the Gagliardo-Nirenberg inequalities

$$\begin{aligned}
\|\nabla d\|_{L^6}^3 &\le C\|\nabla d\|_{L^2}\|\Delta d\|_{L^2}^2, \\
\|\nabla d\|_{L^\infty}^2 &\le \|\nabla d\|_{L^2}\|\nabla\Delta d\|_{L^2}, \\
\|\Delta d\|_{L^4}^2 &\le C\|\Delta d\|_{L^2}\|\nabla\Delta d\|_{L^2} + C\|\Delta d\|_{L^2}.
\end{aligned} \tag{35}$$

Combining (33) and (34) and using the Gronwall inequality, we have

$$\|u\|_{L^\infty(0,T;H^1)} + \|u\|_{L^2(0,T;H^2)} \le C, \tag{36}$$

$$\left\|\sqrt{\rho}u_t\right\|_{L^2(0,T;L^2)} \le C, \tag{37}$$

$$\|d\|_{L^\infty(0,T;H^2)} + \|d\|_{L^2(0,T;H^3)} \le C. \tag{38}$$

Now, by the similar calculations as those in [17], we arrive at

$$\begin{gathered}\left\|(u_t, \nabla d_t)\right\|_{L^\infty(0,T;L^2)\cap L^2(0,T;H^1)} \le C,\\ \|(u, \nabla d)\|_{L^\infty(0,T;H^2)} \le C,\\ \|u\|_{L^2(0,T;W^{2,s})} \le C \quad \text{for some } s > 2,\\ \|\rho\|_{L^\infty(0,T;W^{1,r})} \le C, \qquad \|\rho_t\|_{L^\infty(0,T;L^r)} \le C.\end{gathered} \tag{39}$$

This completes the proof.

3. Proof of Theorem 3

This section is devoted to the proof of Theorem 3. By the results in [9], we only need to prove (9).

Similar to (21), we still have

$$\begin{aligned}&\int \left(u^2 + |\nabla d|^2\right) dx + 2\int_0^T \int \left(|\nabla u|^2 + \left|\Delta d + |\nabla d|^2 d\right|\right) dx\, dt\\ &\quad \le \int \left(u_0^2 + \left|\nabla d_0\right|^2\right) dx.\end{aligned} \tag{40}$$

We will use the following Gagliardo-Nirenberg inequalities:

$$\|u\|_{L^{2p/(p-2)}} \le C\|u\|_{L^2}^{1-(2/p)} \|\nabla u\|_{L^2}^{2/p}, \tag{41}$$

$$\|\nabla d\|_{L^{2p/(p-2)}} \le C\|\nabla d\|_{L^2}^{1-(2/p)} \|\Delta d\|_{L^2}^{2/p} + C\|\nabla d\|_{L^2}. \tag{42}$$

Testing (14) by $-\Delta d$, using $|d| = 1$, (40), (41), and (42), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int |\nabla d|^2 dx + \int |\Delta d|^2 dx\\ &\quad = \int \left(u\cdot\nabla d - |\nabla d|^2 d\right)\Delta d\, dx\\ &\quad \le \left(\|u\|_{L^{2p/(p-2)}}\|\nabla d\|_{L^p} + \|\nabla d\|_{L^p}\|\nabla d\|_{L^{2p/(p-2)}}\right)\|\Delta d\|_{L^2}\\ &\quad \le C\|\nabla d\|_{L^p}\left(\|u\|_{L^2}^{1-(2/p)}\|\nabla u\|_{L^2}^{2/p} + \|\nabla d\|_{L^2}\right.\\ &\qquad\qquad \left. + \|\nabla d\|_{L^2}^{1-(2/p)}\|\Delta d\|_{L^2}^{2/p}\right)\|\Delta d\|_{L^2}\\ &\quad \le C\|\nabla d\|_{L^p}\left(\|\nabla u\|_{L^2}^{2/p} + 1 + \|\Delta d\|_{L^2}^{2/p}\right)\|\Delta d\|_{L^2}\\ &\quad \le \frac{1}{4}\|\Delta d\|_{L^2}^2 + C\|\nabla d\|_{L^p}^2\left(\|\nabla u\|_{L^2}^{4/p} + 1 + \|\Delta d\|_{L^2}^{4/p}\right)\\ &\quad \le \frac{1}{2}\|\Delta d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + C\|\nabla d\|_{L^p}^{2p/(p-2)} + C,\end{aligned} \tag{43}$$

which gives (9).

This completes the proof.

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