BROUWER DEGREE, EQUIVARIANT MAPS AND TENSOR POWERS

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ABSTRACT. A construction of equivariant maps based on factorization through symmetric powers of a faithful representation is presented together with several examples of related equivariant maps. Applications to differential equations are also discussed.

1. INTRODUCTION

Let G be a compact Lie group and V, W two orthogonal G-representations of the same dimensions. We denote by $S(V)$ the unit sphere in V.

The following question was raised by J.F. Adams in 1963 (cf. [1,9]) whether *there exists a G-equivariant map $f : S(V) \to S(W)$ of a given degree* $\deg(f) = k$. From the point of view of applications in analysis, the case $k \neq 0$ has particular importance. For a finite p-group G, a complete answer to this question was given by M. Atiyah and D. Tall (cf. [2]). To be more specific, a map f with degree $\deg(f) \not\equiv 0 \pmod{p}$ exists if and only if the irreducible components of V and W are conjugate in pairs by (possibly different) elements of the Galois group "associated" to G.

In the general case of a compact Lie group G, the congruence relations between possible values of degrees of equivariant maps are described in the following result:

Theorem 1.1. *Let $f_1, f_2 : S(V) \to S(W)$ be two G-equivariant maps. Then*

(1.1) $\deg(f_1) \equiv \deg(f_2) \pmod{\alpha}$,

where $\alpha = \gcd\left(\chi(G/H_1), \ldots, \chi(G/H_n)\right)$, $\chi(\cdot)$ stands for the Euler characteristic, and $\{(H_1), \ldots, (H_n)\}$ are all the orbit types in $S(V)$ satisfying

  (i) H_i *is of maximal rank;*
  (ii) $\dim V^{H_i} = \dim W^{H_i}$;
  (iii) *there exists an H_i-equivariant map between "complementing" spheres to V^{H_i} and W^{H_i}, respectively[1].*

The proof as well as the more general formulation of Theorem 1.1 can be found in [8]. We also refer to [5,6] for the discussion and related results in the case where G is an abelian group.

Evidently, in order to use this result, one has to answer the question if there exists an equivariant map f_1 with degree "simple" to calculate for which the congruence relation (1.1) would imply the nontriviality of $\deg(f_2)$.

On the other hand, computations of important invariants of variational problems with symmetries (such as genus, G-category, etc.) lead to the problem of (non)existence of equivariant map $f : S(V) \to S(W)$ with $\dim V > \dim W$ (we refer to [3] for a general discussion of this question and several elegant examples). We would like to point out that this problem is also closely related to the verification of condition (iii) of Theorem 1.1 (cf. [8] and [5,6]).

This paper is intended to reveal common features behind certain examples of equivariant maps related to the above issues. Actually, we would like to describe an approach to *explicit* construction of equivariant maps by exploiting classic methods of the representation/invariant theory (for all relevant information, we refer to [4,13]).

In what follows we will always assume that all the considered complex (resp. real) representation are unitary (resp. orthogonal). A map $f : V \to W$ is said to be *nonsingular* if $f(x) \neq 0$ for $x \neq 0$ (cf. [7]).

2. EXAMPLES

We denote by $\mathbb{H} = \mathbb{C}^2 = \{z_1 + jz_2; \ z_1, z_2 \in \mathbb{C}\}$ the algebra of quaternions, where the multiplication satisfies the standard properties $j^2 = -1$, $ji = -ij$. We also identify the cyclic group $\mathbb{Z}_n$ with $\{\gamma \in \mathbb{C}; \ \gamma^n = 1\}$. The group $\mathbf{Q}_8$ of quaternionic units is defined as $\{\pm 1, \pm i, \pm j, \pm ji\} \subset \mathbb{H}$, that is $\mathbf{Q}_8 = \mathbb{Z}_4 \cup j\mathbb{Z}_4$. For the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ we use the notation:

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[1]We refer to [8] for the precise formulation of this rather technical condition which is related to the "secondary obstructions" in the equivariant context (cf. [5,6])

$1 = (1,1)$, $-1 = (-1,-1)$, $\kappa = (1,-1)$, $-\kappa = (-1,1)$. Notice, $\mathbb{Z}_2$ and $\mathbb{Z}_4$ are normal in $\mathbf{Q}_8$. The homomorphisms $\alpha : \mathbf{Q}_8 \to \mathbf{Q}_8/\mathbb{Z}_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, $\alpha(z) = z^2$ and $\alpha(jz) = \kappa z^2$, where $z \in \{\pm 1, \pm i\}$, and $\beta : \mathbf{Q}_8 \to \mathbf{Q}_8/\mathbb{Z}_4 \simeq \mathbb{Z}_2$, $\beta(z_1 + jz_2) = z_1^2 - z_2^2$, are well defined quotient projections.

Clearly, $\mathbf{Q}_8$ acts on $\mathbb{H}$ by the quaternionic multiplication and we denote by $(\mathbb{H}, \mathbf{Q}_8)$ the corresponding $\mathbf{Q}_8$-representation. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on $\mathbb{C}^2$ given by

$$\big(\pm 1, (z_1, z_2)\big) \mapsto (\pm z_1, \pm z_2), \qquad \big(\pm \kappa, (z_1, z_2)\big) \mapsto (\mp z_1, \pm z_2),$$

induces via α a (complex) $\mathbf{Q}_8$-representation which we will denote by $(\mathbb{C}^2, \mathbf{Q}_8/\mathbb{Z}_2)$.

A $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on $\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}$ is defined by

$$\big(\pm 1, (z, t)\big) \mapsto (\pm z, t), \qquad \big(\pm \kappa, (z, t)\big) \mapsto (\mp \bar{z}, \mp t),$$

and the corresponding (real) $\mathbf{Q}_8$-representation will be denoted by $(\mathbb{R}^3, \mathbf{Q}_8/\mathbb{Z}_2)$. Finally, the antipodal $\mathbb{Z}_2$-action on $\mathbb{R}^3$ induces via β a (real) $\mathbf{Q}_8$-representation $(\mathbb{R}^3, \mathbf{Q}_8/\mathbb{Z}_4)$.

The following examples of equivariant maps are the starting point for our discussion.

Example 2.1. We define a map $\tilde{f} : \mathbb{H} \to \mathbb{C}^2$ by

$$(2.1) \qquad \tilde{f}(z_1 + jz_2) := (z_1^2 + z_2^2, z_1^2 - z_2^2), \quad z_1 + jz_2 \in \mathbb{H}.$$

By direct verification $\tilde{f}$ is a $\mathbf{Q}_8$-equivariant nonsingular map from $(\mathbb{H}, \mathbf{Q}_8)$ to $(\mathbb{C}^2, \mathbf{Q}_8/\mathbb{Z}_2)$. Consequently, we have a $\mathbf{Q}_8$-equivariant map $f : S(\mathbb{H}) \to S(\mathbb{C}^2)$, given by $f(x) := \tilde{f}(x)/\|\tilde{f}(x)\|$, $x \in S(\mathbb{H})$. Clearly, $\deg f = 4$, hence Theorem 1.1 yields

Proposition 2.2. *For any $\mathbf{Q}_8$-equivariant map $\psi : S(\mathbb{H}) \to S(\mathbb{C}^2)$ one has* $\deg \psi \equiv 4 \pmod 8$. *In particular, $\deg \psi$ is different from zero.*

Remark 2.3. Notice that the statement of Proposition 2.2 does not follow from the well known results for abelian groups (see, for example, [5,6]). Indeed, there are only two abelian subgroups in $\mathbf{Q}_8$, namely $\mathbb{Z}_2$ and $\mathbb{Z}_4$ for which the "restriction" of the actions leads to the congruencies $\deg \psi \equiv 4 \pmod 2$ and $\deg \psi \equiv 4 \pmod 4$, respectively. This information is insufficient to conclude that $\deg \psi \neq 0$, by the same token, the situation described in Example 2.1 is of non-abelian "nature".

Remark 2.4. Observe also that Example 2.1 is compatible with the above mentioned Atiyah-Tall result (cf. [2]). For, since $(\mathbb{H}, \mathbf{Q}_8)$ and $(\mathbb{C}^2, \mathbf{Q}_8/\mathbb{Z}_2)$ are two non-conjugate representations (by an element of the Galois group) of the 2-group $\mathbf{Q}_8$, every equivariant map $\psi : S(\mathbb{H}) \to S(\mathbb{C}^2)$ satisfies $\deg \psi \equiv 0 \pmod 2$. Notice that in Example 2.1 $\deg f = 4 = 2^2 \equiv 0 \pmod 2$.

Example 2.5. We consider the map $g : \mathbb{H} \to \mathbb{C} \oplus \mathbb{R} \simeq \mathbb{R}^3$ given by

(2.2) $g(z_1 + jz_2) = (z_1^2 - \bar{z}_2^2, \Re e(z_1 z_2)), \quad z_1 + jz_2 \in \mathbb{H}.$

Clearly, g is a $\mathbf{Q}_8$-equivariant nonsingular map from $(\mathbb{H}, \mathbf{Q}_8)$ to $(\mathbb{R}^3, \mathbf{Q}_8/\mathbb{Z}_2)$.

Remark 2.6. The map g was originally constructed by T. Bartsch (cf. [3], Example 3.21) for the representations $(\mathbb{H}, \mathbb{Z}_4)$ and $(\mathbb{R}^3, \mathbb{Z}_4/\mathbb{Z}_2)$, where $\mathbb{Z}_4 \simeq \{1, -1, j, -j\} \subset \mathbf{Q}_8$ acts on $\mathbb{H}$ by the quaternionic multiplication (which is the standard action of $\mathbb{Z}_4$ on $\mathbb{C}^2 \simeq \mathbb{H}$), and $\mathbb{Z}_4/\mathbb{Z}_2$ acts antipodally on $\mathbb{R}^3$. For more information on the connection between decreasing dimension equivariant maps and estimates of certain topological invariants (such as genus, G-category, etc.) related to variational problems with symmetries, we refer the interested reader to the book [3] (see also the references therein). By direct computation one can verify that the Hopf invariant of g is equal to -1.

Remark 2.7. We should also mention Example 4.1 from [6] (which was partially inspired by Bartsch's example) describing possible values of Brouwer degrees of equivariant maps "generic" in a certain sense. The problem of existence of decreasing dimension equivariant maps is also related to the "secondary obstructions" in the equivariant extension context (see [5,6,8] for more details).

Example 2.8. *It is easy to verify that the map* $h : \mathbb{H} \to \mathbb{C} \oplus \mathbb{R} = \mathbb{R}^3$, *given by*

(2.3) $h(z_1 + jz_2) = (z_1^2 - z_2^2, \Re e(\bar{z}_1 z_2)),$

is a $\mathbf{Q}_8$*-equivariant nonsingular map from* $(\mathbb{H}, \mathbf{Q}_8)$ *to* $(\mathbb{R}^3, \mathbf{Q}_8/\mathbb{Z}_4)$.

Example 2.8. *Let* $\mathbb{H}$ *be a* $\mathbf{Q}_{8n}$*-representation ($n \geq 2$), where* $\mathbf{Q}_{8n} = \mathbb{Z}_{4n} \cup j\mathbb{Z}_{2n} \subset \mathbb{H}$ *acts on* $\mathbb{H}$ *by quaternionic multiplication. If* $\mathbb{Z}_2$ *acts antipodally on* $\mathbb{R}^3$, *the natural homomorphism* $\mathbf{Q}_{8n} \to \mathbf{Q}_{8n}/\mathbb{Z}_{4n} \simeq \mathbb{Z}_2$, $(z_1 + jz_2) \mapsto z_1^{2n} - z_2^{2n}$, $z_i \in \mathbb{Z}_{4n}$, $i = 1, 2$, *gives us a* $\mathbf{Q}_{8n}$*-representation on* $\mathbb{R}^3$. *Notice that the formula*

(2.4) $h(z_1, z_2) = (z_1^{2n} - z_2^{2n}, \Re e(\bar{z}_1^n z_2^n)), \quad z_1, z_2 \in \mathbb{C},$

determines evidently a $\mathbf{Q}_{8n}$*-equivariant nonsingular map from* $(\mathbb{H}, \mathbf{Q}_{8n})$ *to* $(\mathbb{R}^3, \mathbf{Q}_{8n}/\mathbb{Z}_{4n})$. *The Hopf invariant of* h *is* n.

3. CONSTRUCTION

Let V be a faithful (complex or real) finite-dimensional representation of a compact Lie group G. We are interested in describing G-representations W which admit G-equivariant maps from $S(V)$ into $S(W)$. It is well known that in the most interesting cases, for any irreducible G-representation U,

there exists a tensor power $V^{\otimes k} := V \underbrace{\otimes \cdots \otimes}_{k \text{ times}} V$, with the standard G-action, which contains U.

Fix $k \geq 2$ and consider the "diagonal" map $\triangle : V \to V^{\otimes k}$, $\triangle(v) = v \underbrace{\otimes \cdots \otimes}_{k \text{ times}} v$, $v \in V$. Clearly, in general $\triangle(V)$ is not a linear subspace of $V^{\otimes k}$, however, it is G-invariant and the set $\triangle(V)$ spans the symmetric k-power $\mathrm{Sym}^k(V)$ of the space V (see [4]). Thus $\triangle : V \to \mathrm{Sym}^k(V)$ is a G-equivariant map. Let $W \subset \mathrm{Sym}^k(V)$ be a subrepresentation and $A : \mathrm{Sym}^k(V) \to W$ a G-equivariant linear operator.

We make the following assumption:

(A) $\ker A \cap \triangle(V) = \{0\}$.

As an immediate consequence we obtain

Proposition 3.1. *Under the assumption (A) the map $\tilde{f} : V \to W$, given by $\tilde{f}(v) = A(\triangle(v))$, $v \in V$, is a G-equivariant nonsingular map.*

Remark 3.2. Suppose that V is a complex representation and assume that the center $Z(G)$ of the group G is a discrete subgroup of order k. Then, clearly, $Z(G)$ acts trivially on $\mathrm{Sym}^k(V)$ (this fact in certain cases is also true for real representations). Consequently, every subrepresentation $W \subset \mathrm{Sym}^k(V)$ is not equivalent to V.

Let us reproduce the equivariant maps from Examples 2.1, 2.5 and 2.8 using the above construction.

Example 3.3. We consider the (complex) representation $(\mathbb{H}, \mathbf{Q}_8)$ and its second complex symmetric power $\mathrm{Sym}^2_{\mathbb{C}}(\mathbb{H})$. Notice, the center $\mathbb{Z}_2$ of $\mathbf{Q}_8$ acts trivially on $\mathrm{Sym}^2_{\mathbb{C}}(\mathbb{H})$, i.e. the second symmetric power "kills" the center, hence $\mathbf{Q}_8$ acts on $\mathrm{Sym}^2_{\mathbb{C}}(\mathbb{H})$ as $\mathbf{Q}_8/\mathbb{Z}_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. One can easily check that

$$e_1 = \tfrac{1 \otimes 1 + j \otimes j}{2}, \quad e_2 = \tfrac{1 \otimes 1 - j \otimes j}{2}, \quad e_3 = \tfrac{j \otimes 1 + 1 \otimes j}{2},$$

is an invariant basis for $\mathrm{Sym}^2_{\mathbb{C}}(\mathbb{H})$ and

$$\triangle(z_1 + jz_2) = e_1(z_1^2 + z_2^2) + e_2(z_1^2 - z_2^2) + e_3(2z_1 z_2).$$

Let P_1, P_2 and P_3 denote the natural $\mathbf{Q}_8$-projections onto the subspaces of $\mathrm{Sym}^2_{\mathbb{C}}(\mathbb{H})$ spanned by $\{e_1, e_2\}$, $\{e_2, e_3\}$ and $\{e_1, e_3\}$, respectively. A direct calculation shows that $\ker P_i \cap \triangle(\mathbb{H}) = \{0\}$ for $i = 1, 2, 3$. Consequently $f_i = P_i \circ \triangle$, $i = 1, 2, 3$, are $\mathbf{Q}_8$-equivariant nonsingular maps. Notice that the map f_1 is exactly the map $\tilde{f}$ from Example 2.1.

Remark 3.4. Observe that in Example 3.3 the coordinates of the "diagonal" map are *complex* homogeneous polynomials. It follows immediately from the Hilbert Nullstellen Theorem that it is impossible to construct, using *complex* symmetric powers, an equivariant nonsingular map $f : V \to W$

decreasing dimension (i.e. $\dim V > \dim W$). However, as it is shown in Examples 2.5 and 2.8 and Remark 2.9, there are equivariant nonsingular maps decreasing dimension with coordinates given by *real* homogeneous polynomials. This indicates the use of real symmetric powers for the construction of equivariant maps decreasing dimension.

Example 3.5. The quaternionic algebra $\mathbb{H}$ is a 4-dimensional *real* vector space where a vector $h = z_1 + jz_2$, $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, can be represented as the linear combination $h = 1 \cdot x_1 + i \cdot y_1 + j \cdot x_2 + ji \cdot y_2$. Therefore, the second real symmetric power $\mathrm{Sym}^2_{\mathbb{R}}(\mathbb{H})$ admits the following invariant basis:

$$
\begin{aligned}
e_1 &= \tfrac{1\otimes 1 - j\otimes j - i\otimes i + ji\otimes ji}{4}, & e_2 &= \tfrac{1\otimes i + i\otimes 1 + j\otimes ji + ji\otimes j}{4}, \\
e_3 &= \tfrac{1\otimes j + j\otimes 1 - i\otimes ji - ji\otimes i}{4}, & e_4 &= \tfrac{i\otimes j + j\otimes i + 1\otimes ji + ji\otimes 1}{4}, \\
e_5 &= \tfrac{1\otimes 1 + j\otimes j - i\otimes i - ji\otimes ji}{4}, & e_6 &= \tfrac{1\otimes i + i\otimes 1 - j\otimes ji - ji\otimes j}{4}, \\
e_7 &= \tfrac{1\otimes j + j\otimes 1 + i\otimes ji + ji\otimes i}{4}, & e_8 &= \tfrac{i\otimes j + j\otimes i - 1\otimes ji - ji\otimes 1}{4}, \\
e_9 &= \tfrac{1\otimes 1 - j\otimes j + i\otimes i - ji\otimes ji}{4}, & e_{10} &= \tfrac{1\otimes 1 + j\otimes j + i\otimes i + ji\otimes ji}{4},
\end{aligned}
$$

and the "diagonal" map $\triangle : \mathbb{H} \to \mathrm{Sym}^2_{\mathbb{R}}(\mathbb{H})$ in the above basis $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$ is given by

$$
\begin{aligned}
\triangle(z_1 + jz_2) =&(x_1^2 - x_2^2 - y_1^2 + y_2^2, 2(x_1y_1 + x_2y_2), 2(x_1x_2 - y_1y_2), \\
&2(x_2y_1 + x_1y_2), \\
&x_1^2 + x_2^2 - y_1^2 - y_2^2, 2(x_1y_1 - x_2y_2), 2(x_1x_2 + y_1y_2), \\
&2(x_2y_1 - x_1y_2), \\
&x_1^2 - x_2^2 + y_1^2 - y_2^2, x_1^2 + x_2^2 + y_1^2 + y_2^2) \\
=&(z_1^2 - \bar{z}_2^2, 2z_1z_2, z_1^2 + \bar{z}_2^2, 2z_1\bar{z}_2, |z_1|^2 - |z_2|^2, |z_1|^2 + |z_2|^2).
\end{aligned}
$$

Notice that if we change the order of the basis to $\{e_5, e_2, e_1, e_6, e_7, e_8, e_9, e_3, e_4, e_{10}\}$ then the map $\triangle$ can be written as follows:

$$
\triangle(z_1 + jz_2) = (z_1^2 + z_2^2, z_1^2 - z_2^2, 2z_1\bar{z}_2, |z_1|^2 - |z_2|^2, 2z_1z_2, |z_1|^2 + |z_2|^2).
$$

Let P_1 and P_2 be the natural projections on the subspaces spanned by $\{e_1, e_2, e_3\}$ and $\{e_1, e_6, e_7\}$, respectively. By direct verification one can easily check that $\ker P_i \cap \triangle(\mathbb{H}) = \{0\}$, $i = 1, 2$, and therefore, $f_i = P_i \circ \triangle$, $i = 1, 2$, are nonsingular $\mathbf{Q}_8$-equivariant maps which were presented in (2.2) and (2.3), respectively.

Remark 3.6. Notice that, if $f : V \to W$ is a nonsingular G-equivariant map with coordinates being homogeneous polynomials of degree k, then there exists a G-equivariant linear operator $T : W \to \mathrm{Sym}^k(V)$ such that

$T \circ f = \triangle$. In this sense the above construction of "homogeneous" nonsingular equivariant maps is generic. Clearly, it is possible to combine different symmetric powers in order to construct more complicated examples of equivariant maps (see [6], Example 4.1).

Practical applications of the construction described above are related to several classical problems of the representation theory:
(i) decomposition of symmetric power $\mathrm{Sym}^k(V)$ into irreducible components (*spectral decomposition*);
(ii) finding invariant bases in $\mathrm{Sym}^k(V)$ (*Clebsch-Gardon problem*);
(iii) verification of the condition (A) (a sort of an *equivariant transversality*).
   There is a vast literature devoted to the problems (i) and (ii) for the classical groups (see [4] and [13] and the references therein). It is our intention to elaborate the above construction for certain important representations of classical (infinite) groups (for example, $SU(n)$ and $\mathrm{Sp}(n)$). Observe also that Remarks 2.6 and 2.9 give rise to the question if there is any congruence relation, similar to (1.1), for Hopf invariants of equivariant maps from S^{4n-1} to S^{2n} (see also [3,5,6]).

4. Applications

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : [0, \omega] \times \mathbb{R}^n \to \mathbb{R}^n$ be two maps. We are making the following assumptions:
(H1) f is a positively homogeneous map with constant $\alpha \geq 1$, i.e. $f(tv) = t^\alpha f(v)$ for every $t > 0$ and $v \in \mathbb{R}^n$;
(H2) g satisfies the Carathéodory conditions;
(H3) $v \equiv 0$ is the only bounded solution of $\dot{v} = f(v)$;
(H4) $\lim_{\|v\| \to \infty} \|g(t, v)\| / \|v\|^\alpha = 0$ uniformly a.e. in $t \in [0, \omega]$;
(H5) $\deg(f, B(0, \varepsilon)) \neq 0$ for some $\varepsilon > 0$, where $B(0, \varepsilon) = \{v \in \mathbb{R}^n; \ \|v\| < \varepsilon\}$.
   We consider the following system of ODEs:

$$(4.1) \qquad \dot{v} = f(v) + g(t, v), \quad v \in \mathbb{R}^n.$$

The systems of type (4.1), i.e. perturbation of autonomous systems with positively homogeneous nonlinearity, were widely studied (see [7,11] and references therein).

Proposition 4.1. (cf. [11], [12], Cor. 4.7). *Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : [0, \omega] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the assumptions (H1) – (H5). Then (4.1) has at least one ω-periodic solution.*

As an immediate consequence of Propositions 4.1 and 2.2 we obtain

Theorem 4.2. *Let $f : \mathbb{H} \to \mathbb{C}^2$ and $g : [0, \omega] \times \mathbb{R}^4 \to \mathbb{R}^4$ satisfy assumptions (H1)–(H4). If f is a nonsingular $\mathbf{Q}_8$-equivariant map, then (4.1) has at least one ω-periodic solution.*

Example 4.3. Take $f : \mathbb{H} \to \mathbb{C}^2$, $f(z_1 + jz_2) = (\bar{z}_1^2 + \bar{z}_2^2, \bar{z}_1^2 - \bar{z}_2^2)$, $(z_1, z_2) \in \mathbb{C}^2 = \mathbb{R}^4$. Clearly, f is a nonsingular $\mathbf{Q}_8$-equivariant map. Assume that $g : [0, \omega] \times \mathbb{R}^4 \to \mathbb{R}^4$ satisfies conditions (H2) and (H4) with $\alpha = 2$. Then system (4.1) has at least one ω-periodic solution. Indeed, we only need to show that the system $\dot{v} = f(v)$, $v \in \mathbb{R}^4$, does not have any bounded solution except $v \equiv 0$. Indeed, assume that $\nu(t) = (z_1(t), z_2(t))$ is a non-zero bounded solution of $\dot{v} = f(v)$ and put $\varphi(t) := \Phi(\nu(t))$, where $\Phi(z_1, z_2) = \operatorname{Re}(z_1^3 - z_2^3)$. Notice that $\varphi'(t) = <\nabla\Phi(z_1(t), z_2(t)), \dot{\nu}(t)> = 3(|z_1(t)|^4 + |z_2(t)|^4)$, thus the function $\varphi(t)$ is strictly increasing and bounded. In particular, $\lim_{t \to \pm\infty} \varphi(t) = a_\pm$ ($|a_\pm| < \infty$) and hence $\lim_{t \to \pm\infty} \varphi'(t) = 0$. Since $\varphi'(t) = 3(|z_1(t)|^4 + |z_2(t)|^4)$, it follows that $\lim_{t \to \pm\infty} \nu(t) = 0$, but this contradicts $\varphi(t)$ is a strictly increasing function.

Remark 4.4. Consider the map $\psi(z_1, z_2) = (\bar{z}_1^{2n} + \bar{z}_2^{2n}, \bar{z}_1^{2n} - \bar{z}_2^{2n})$, where $n \geq 1$ is an odd integer. It is clear that ψ is $\mathbf{Q}_8$-equivariant, positively homogeneous and nonsingular. It may be verified that it also satisfies the assumption (H3).

As another application we consider the following boundary value problem

$$(4.2) \qquad \begin{cases} x^{(n)} = F(x, x', \dots, x^{(n-1)}, t), \\ x^{(i)}(0) = x^{(i)}(\omega), \quad i = 0, 1, \dots, n-1, \end{cases}$$

where $x \in \mathbb{R}^4$, $F : \mathbb{R}^{4n} \times [0, 1] \to \mathbb{R}^4$ is continuously differentiable and $\omega \in (0, 1)$ but is not specified. Similar problems, in more general setting (cf. [10] and the references therein) were considered by many authors. By the well known result of K. Schmitt (cf. [10], Theorem 9.4.1) and Proposition 2.2 we immediately obtain

Theorem 4.5. *Let $D \subset \mathbb{H}$ be an open bounded set containing zero and $h : \mathbb{H} \to \mathbb{H}$ be defined by $h(x) = F(x, 0, \dots, 0)$. If h is homotopic to a $\mathbf{Q}_8$-equivariant nonsingular map $f : (\mathbb{H}, \mathbf{Q}_8) \to (\mathbb{C}^2, \mathbf{Q}_8/\mathbb{Z}_2)$, by a homotopy without zeros on ∂D, then there exists $\omega_0 > 0$ such that (4.2) has a solution for each $\omega \in (0, \omega_0)$.*

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