NONLINEAR ERGODIC THEOREMS FOR A SEMITOPOLOGICAL SEMIGROUP OF NON-LIPSCHITZIAN MAPPINGS WITHOUT CONVEXITY

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Let G be a semitopological semigroup, C a nonempty subset of a real Hilbert space H, and $\Im = \{T_t : t \in G\}$ a representation of G as asymptotically nonexpansive type mappings of C into itself. Let $L(x) = \{z \in H : \inf_{s\in G} \sup_{t\in G} \|T_{ts}x - z\| = \inf_{t\in G} \|T_t x - z\|\}$ for each $x \in C$ and $L(\Im) = \bigcap_{x\in C} L(x)$. In this paper, we prove that $\bigcap_{s\in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \bigcap L(\Im)$ is nonempty for each $x \in C$ if and only if there exists a unique nonexpansive retraction P of C into $L(\Im)$ such that $PT_s = P$ for all $s \in G$ and $P(x) \in \overline{\text{conv}}\{T_s x : s \in G\}$ for every $x \in C$. Moreover, we prove the ergodic convergence theorem for a semitopological semigroup of non-Lipschitzian mappings without convexity.

1. Introduction and preliminaries

Let H be a Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Let G be a semitopological semigroup, that is, a semigroup with a Hausdorff topology such that for each $s \in G$ the mappings $s \mapsto s\cdot t$ and $s \mapsto t \cdot s$ of G into itself are continuous. Let C be a nonempty subset of H and let $\Im = \{T_t : t \in G\}$ be a semigroup on C, that is, $T_{st}(x) = T_s T_t(x)$ for all $s, t \in G$ and $x \in C$. Recall that a semigroup $\Im$ is said to be

(a) nonexpansive if $\|T_t x - T_t y\| \le \|x - y\|$ for $x, y \in C$ and $t \in G$.

(b) asymptotically nonexpansive [6] if there exists a function $k : G \mapsto [0,\infty)$ with $\inf_{s\in G} \sup_{t\in G} k_{ts} \le 1$ such that $\|T_t x - T_t y\| \le k_t \|x - y\|$ for $x, y \in C$ and $t \in G$.

(c) of asymptotically nonexpansive type [6] if for each x in C, there is a function $r(\cdot, x) : G \mapsto [0,\infty)$ with $\inf_{s\in G} \sup_{t\in G} r(ts, x) = 0$ such that $\|T_t x - T_t y\| \le \|x - y\| + r(t, x)$ for all $y \in C$ and $t \in G$.

It is easily seen that (a)⇒(b)⇒(c) and that both the inclusions are proper (cf. [6, page 112]).

Baillon [1] proved the first nonlinear mean ergodic theorem for nonexpansive mappings in a Hilbert space: let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T

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is nonempty, then for each $x \in C$, the Cesáro means

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x \tag{1.1}$$

converge weakly as $n \to \infty$ to a point of $F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and $Px \in \overline{\text{conv}}\{T^n x : n = 0, 1, 2, \ldots\}$ for each $x \in C$, where $\overline{\text{conv}} A$ is the closure of the convex hull of A. The analogous results are given for nonexpansive semigroups on C by Baillon [2] and Breźis-Browder [3]. In [10], Mizoguchi-Takahashi proved a nonlinear ergodic retraction theorem for Lipschitzian semigroups by using the notion of submean. Recently, Li and Ma [8, 9] proved the nonlinear ergodic retraction theorems for non-Lipschitzian semigroups in a Banach space without using the notion of submean. Also, in 1992, Takahashi [13] proved the ergodic theorem for nonexpansive semigroups on condition that $\bigcap_{s\in G} \overline{\text{conv}}\{T_{st}x : t \in G\} \subset C$ for some $x \in C$.

In this paper, without using the concept of submean, we prove nonlinear ergodic theorem for semitopological semigroup of non-Lipschitzian mappings without convexity in a Hilbert space. We first prove that if C is a nonempty subset of a Hilbert space H, G a semitopological semigroup, and $\Im = \{T_t : t \in G\}$ a representation of G as asymptotically nonexpansive type mappings of C into itself, then $\bigcap_{s\in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \bigcap L(\Im)$ is nonempty for each $x \in C$ if and only if there exists a unique nonexpansive retraction P of C into $L(\Im)$ such that $PT_s = P$ for all $s \in G$ and Px is in the closed convex hull of $\{T_s x : s \in G\}$, where $L(x) = \{z : \inf_{s\in G}\sup_{t\in G}\|T_{ts}x - z\| = \inf_{t\in G}\|T_t x - z\|\}$ and $L(\Im) = \bigcap_{x\in C} L(x)$. By using this result, we also prove the ergodic convergence theorem for semitopological semigroup of non-Lipschitzian mapping without convexity. Our results are generalizations and improvements of the previously known results of Brézis-Browder [3], Hirano-Takahashi [4], Mizoguchi-Takahashi [10], Takahashi-Zhang [14], and Takahashi [11, 12, 13] in many directions. Further, it is safe to say that in the results [1, 2, 3, 4, 5, 7, 10, 11, 12, 13, 14], many key conditions are not necessary.

2. Ergodic convergence theorems

Throughout this paper, we assume that C is a nonempty subset of a real Hilbert space H, G a semitopological semigroup, and $\Im = \{T_t : t \in G\}$ an asymptotically nonexpansive type semigroup on C. For each $x \in C$, define $L(x)$ and $L(\Im)$ by

$$L(x) = \left\{z : \inf_{s\in G}\sup_{t\in G}\left\|T_{ts}x - z\right\| = \inf_{t\in G}\left\|T_t x - z\right\|\right\}, \qquad L(\Im) = \bigcap_{x\in C} L(x), \tag{2.1}$$

respectively. We denote $F(\Im)$ by the set $\{x \in C : T_s(x) = x \text{ for all } s \in G\}$ of common fixed point of $\Im$. We begin with the following lemma.

Lemma 2.1. *Let C be a nonempty subset of a Hilbert space H and $\Im = \{T_t : t \in G\}$ an asymptotically nonexpansive type semigroup on C. Then $F(\Im) \subset L(\Im)$.*

Proof. Let $x \in C$ and $f \in F(\Im)$. Since $\Im$ is asymptotically nonexpansive type, for an arbitrary $\varepsilon > 0$, there exists $s_0 \in G$ such that for all $t \in G$

$$r\big(ts_0, f\big) < \varepsilon. \tag{2.2}$$

Hence, for each $a \in G$,

$$\begin{aligned}\inf_{s\in G}\sup_{t\in G}\big\|T_{ts}x - f\big\| \le \sup_{t\in G}\big\|T_{ts_0a}x - f\big\| &\le \sup_{t\in G}\big(\big\|T_a x - f\big\| + r\big(ts_0, f\big)\big) \\ &\le \big\|T_a x - f\big\| + \varepsilon.\end{aligned} \tag{2.3}$$

Since $\varepsilon > 0$ is arbitrary, we have $\inf_{s\in G}\sup_{t\in G}\|T_{ts}x - f\| \le \inf_{t\in G}\|T_t x - f\|$. Therefore, $f \in L(x)$. This completes the proof. □

Remark 2.2. It is not easy to prove that $F(\Im)$ is nonempty when C is not a convex subset. However, we can show that $L(\Im)$ is nonempty under some conditions and it is important for the ergodic convergence theorem.

The following proposition plays a crucial role in the proof of our main theorems in this paper.

PROPOSITION 2.3. *Let G be a semitopological semigroup, C a nonempty subset of a Hilbert space H, and $\Im = \{T_t : t \in G\}$ an asymptotically nonexpansive type semigroup on C. Then, for every $x \in C$, the set*

$$\bigcap_{s\in G}\overline{\text{conv}}\big\{T_{ts}x : t \in G\big\}\bigcap L(x), \tag{2.4}$$

consists of at most one point.

Proof. Let $u, v \in \bigcap_{s\in G}\overline{\text{conv}}\{T_{ts}x : t \in G\}\bigcap L(x)$, without loss of generality, we assume that

$$\inf_{t\in G}\big\|T_t x - u\big\|^2 \le \inf_{t\in G}\big\|T_t x - v\big\|^2. \tag{2.5}$$

Now, for each $t, s \in G$, since

$$\|u - v\|^2 + 2\big(T_{ts}x - u, u - v\big) = \big\|T_{ts}x - v\big\|^2 - \big\|T_{ts}x - u\big\|^2, \tag{2.6}$$

we have

$$\begin{aligned}\|u - v\|^2 + 2\inf_{t\in G}\big(T_{ts}x - u, u - v\big) &\ge \inf_{t\in G}\big\|T_{ts}x - v\big\|^2 - \sup_{t\in G}\big\|T_{ts}x - u\big\|^2 \\ &\ge \inf_{t\in G}\big\|T_t x - v\big\|^2 - \sup_{t\in G}\big\|T_{ts}x - u\big\|^2.\end{aligned} \tag{2.7}$$

From $u \in L(x)$, we have

$$\begin{aligned}\|u - v\|^2 + 2\sup_{s\in G}\inf_{t\in G}\big(T_{ts}x - u, u - v\big) &\ge \inf_{t\in G}\big\|T_t x - v\big\|^2 - \inf_{s\in G}\sup_{t\in G}\big\|T_{ts}x - u\big\|^2 \\ &= \inf_{t\in G}\big\|T_t x - v\|^2 - \inf_{t\in G}\big\|T_t x - u\big\|^2 \ge 0.\end{aligned} \tag{2.8}$$

Therefore, for $\varepsilon > 0$ there is an $s_1 \in G$ such that

$$\|u-v\|^2 + 2\big(T_{ts_1}x - u, u - v\big) > -\varepsilon \quad \forall t \in G. \tag{2.9}$$

From $v \in \overline{\text{conv}}\{T_{ts_1}x : t \in G\}$, we have

$$\|u-v\|^2 + 2(v-u, u-v) \geq -\varepsilon. \tag{2.10}$$

This inequality implies that $\|u-v\|^2 \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $u = v$. This completes the proof. □

Remark 2.4. In the Takahashi-Zhang's result [14], it is assumed that C is a closed convex subset, G a reversible semigroup, and $\Im$ an asymptotically nonexpansive semigroup. Proposition 2.3 shows those key conditions are not necessary.

Let $m(G)$ be the Banach space of all bounded real-valued functions on a semi-topological semigroup G with the supremum norm and let X be a subspace of $m(G)$ containing constants. Then, an element μ of X^* (the dual space of X) is called a mean on X if $\|\mu\| = \mu(1) = 1$. Let μ be a mean on X and $f \in X$. Then, according to time and circumstances, we use $\mu_t(f(t))$ instead of $\mu(f)$. For each $s \in G$ and $f \in m(G)$, we define elements $l_s f$ and $r_s f$ in $m(G)$ given by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for all $t \in G$, respectively.

Throughout the rest of this section, let X be a subspace of $m(G)$ containing constants invariant under l_s and r_s for each $s \in G$. Furthermore, suppose that for each $x \in C$ and $y \in H$, a function $f(t) = \|T_t x - y\|^2$ is in X. For $\mu \in X^*$, we define the value $\mu_t(T_t x, y)$ of μ at this function. By Riesz theorem, there exists a unique element $\Im_\mu x$ in X such that

$$\mu_t\big(T_t x, y\big) = \big(\Im_\mu x, y\big) \quad \forall y \in H. \tag{2.11}$$

Lemma 2.5. *Suppose that X has an invariant mean μ. Then we have*

$$\bigcap_{s\in G} \overline{\text{conv}}\big\{T_{ts}x : t \in G\big\} \bigcap L(x) = \big\{\Im_\mu x\big\} \quad \textit{for every } x \in C. \tag{2.12}$$

Further, if T_t is continuous for each $t \in G$ and $\bigcap_{s\in G} \overline{\text{conv}}\{T_{st}x : t \in G\} \subset C$ for some $x \in C$, then $\Im_\mu x \in F(\Im)$.

Proof. Since μ is an invariant mean, it is easy to show that $\Im_\mu x \in \bigcap_{s\in G} \overline{\text{conv}}\{T_{ts}x : t \in G\}$ for each $x \in C$. By Proposition 2.3, it is enough to prove that $\Im_\mu x \in L(x)$ for each $x \in C$. To this end, let $\varepsilon > 0$, since $\Im$ is an asymptotically nonexpansive type semigroup, for each $t \in G$ there is an $h_t \in G$ such that for each $h \in G$,

$$r\big(hh_t, T_t x\big) < \varepsilon. \tag{2.13}$$

Put $M = \sup_{t,s\in G} \|T_t x - T_s x\|$, then we have

$$\begin{aligned} \big\|T_{hh_t t}x - \Im_\mu x\big\|^2 - \big\|T_t x - \Im_\mu x\big\|^2 &= \mu_s\Big(\big\|T_{hh_t t}x - T_s x\big\|^2 - \big\|T_t x - T_s x\big\|^2\Big) \\ &= \mu_s\Big(\big\|T_{hh_t t}x - T_{hh_t s}x\big\|^2 - \big\|T_t x - T_s x\big\|^2\Big) \\ &\leq 2M\varepsilon \quad \text{for each } h \in G. \end{aligned} \tag{2.14}$$

Hence, we have

$$\inf_{s\in G}\sup_{h\in G}\left\|T_{hs}x-\Im_{\mu}x\right\|^{2}\leq\left\|T_{t}x-\Im_{\mu}x\right\|^{2}+2M\varepsilon\quad\forall t\in G. \tag{2.15}$$

Since $\varepsilon>0$ is arbitrary, we have $\Im_{\mu}x\in L(x)$. Finally, suppose that $\bigcap_{s\in G}\overline{\text{conv}}\{T_{st}x:t\in G\}\subset C$ and each T_t is continuous from C into itself. Then, we can easily prove that $\Im_{\mu}x\in\bigcap_{s\in G}\overline{\text{conv}}\{T_{st}x:t\in G\}$ and hence we have $\Im_{\mu}x\in C$. For each $h\in G$ and $\varepsilon\in(0,1)$, there exists $0<\delta<\varepsilon$ such that $\|T_hy-T_h\Im_{\mu}x\|<\varepsilon$ whenever $y\in C$ and $\|y-\Im_{\mu}x\|\leq\delta$. Since $\Im$ is an asymptotically nonexpansive type semigroup, there is $s_0\in G$ such that

$$r\big(ts_0,\Im_{\mu}x\big)<\frac{1}{2\big(M_1+1\big)}\delta^{2}\quad\forall t\in G, \tag{2.16}$$

where $M_1=\sup_{t\in G}\|T_tx-\Im_{\mu}x\|$. Then for each $t,s\in G$, we have

$$\begin{aligned}&\left\|T_{ss_0}\Im_{\mu}x-\Im_{\mu}x\right\|^{2}+2\big(T_tx-\Im_{\mu}x,\Im_{\mu}x-T_{ss_0}\Im_{\mu}x\big)\\&\quad=\left\|T_tx-T_{ss_0}\Im_{\mu}x\right\|^{2}-\left\|T_tx-\Im_{\mu}x\right\|^{2}\\&\quad=\left\|T_{ss_0t}x-T_{ss_0}\Im_{\mu}x\right\|^{2}-\left\|T_tx-\Im_{\mu}x\right\|^{2}-\left\|T_{ss_0t}x-T_{ss_0}\Im_{\mu}x\right\|^{2}+\left\|T_tx-T_{ss_0}\Im_{\mu}x\right\|^{2}\\&\quad\leq\delta^{2}-\left\|T_{ss_0t}x-T_{ss_0}\Im_{\mu}x\right\|^{2}+\left\|T_tx-T_{ss_0}\Im_{\mu}x\right\|^{2}.\end{aligned} \tag{2.17}$$

It follows that

$$\left\|T_{ss_0}\Im_{\mu}x-\Im_{\mu}x\right\|\leq\delta\quad\forall s\in G. \tag{2.18}$$

This implies that

$$\left\|T_h\Im_{\mu}x-\Im_{\mu}x\right\|\leq\left\|T_h\Im_{\mu}x-T_hT_{ss_0}\Im_{\mu}x\right\|+\left\|T_{hss_0}\Im_{\mu}x-\Im_{\mu}x\right\|<2\varepsilon. \tag{2.19}$$

Since $\varepsilon>0$ is arbitrary, we have $T_h\Im_{\mu}x=\Im_{\mu}x$. This completes the proof. □

Now, we prove a nonlinear ergodic theorem for asymptotically nonexpansive type semigroups without convexity. Before doing this, we give a definition concerning means. Let $\{\mu_\alpha:\alpha\in A\}$ be a net of means on X, where A is a directed set. Then $\{\mu_\alpha:\alpha\in A\}$ is said to be asymptotically invariant if for each $f\in X$ and $s\in G$,

$$\mu_\alpha(f)-\mu_\alpha\big(l_sf\big)\longrightarrow 0,\qquad \mu_\alpha(f)-\mu_\alpha\big(r_sf\big)\longrightarrow 0. \tag{2.20}$$

Theorem 2.6. *Let C be a nonempty subset of a Hilbert space H, X an invariant subspace of $m(G)$ containing constants, and $\Im=\{T_t:t\in G\}$ an asymptotically nonexpansive type semigroup on C. If for each $x\in C$ and $y\in H$, the function f on G defined by $f(t)=\|T_tx-y\|^2$ belong to X, then for an asymptotically invariant net $\{\mu_\alpha:\alpha\in A\}$ on X, the net $\{\Im_{\mu_\alpha}x\}_{\alpha\in A}$ converges weakly to an element $x_0\in L(x)$.*

Further, if T_t is continuous for each $t \in G$ and $\bigcap_{s\in G} \overline{\text{conv}}\{T_{st}x : t \in G\} \subset C$, then $x_0 \in F(\Im)$.

Proof. Let W be the set of all weak limit points of subnet of the net $\{\Im_{\mu_\alpha} x : \alpha \in A\}$. By Proposition 2.3, it is enough to prove that

$$W \subset \bigcap_{s\in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \bigcap L(x). \tag{2.21}$$

To show this, let $z \in W$ and let $\{\Im_{u_{\alpha_\beta}} x\}$ be a subnet of $\{\Im_{\mu_\alpha} x\}$ such that $\{\Im_{\mu_{\alpha_\beta}} x\}$ converges weakly to z. Now, without loss of generality, we can suppose that $\{\Im_{\mu_{\alpha_\beta}} x\}$ converges weakly* to $\mu \in X^*$. It is easily seen that μ is an invariant mean on X and then Lemma 2.5 implies that $z = \Im_\mu x \in \bigcap_{s\in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \bigcap L(x)$. This completes the proof. □

Let $C(G)$ be the Banach space of all bounded continuous real-valued functions on G and let $RUC(G)$ be the space of all bounded right uniformly continuous functions on G, that is, all $f \in C(G)$ such that the mapping $s \mapsto r_s f$ is continuous. Then $RUC(G)$ is a closed subalgebra of $C(G)$ containing constants and invariant under l_s and r_s.

As a direct consequence of Theorem 2.6, we obtain the following corollary.

COROLLAFRY 2.7 (see [13]). *Let C be a nonempty subset of a Hilbert space H and let G be a semitopological semigroup such that $RUC(G)$ has an invariant mean. Let $\Im = \{T_t : t \in G\}$ be a nonexpansive semigroup on C such that $\{T_t x : t \in G\}$ is bounded and $\bigcap_{s\in G} \overline{\text{conv}}\{T_{st}x : t \in G\} \subset C$ for some $x \in C$. Then, $F(\Im) \neq \emptyset$. Further, for an asymptotically invariant net $\{\mu_\alpha\}_{\alpha\in A}$ of means on $RUC(G)$, the net $\{\Im_{\mu_\alpha}\}_{\alpha\in A}$, converges weakly to an element $x_0 \in F(\Im)$.*

Remark 2.8. For the proof of Corollary 2.7, Takahashi [13] used the condition $\bigcap_{s\in G} \overline{\text{conv}}\{T_{st}x : t \in G\} \subset C$. But, from Theorem 2.6, we can prove the result without this condition except proving the fact that the weak limit of $\{\Im_{\mu_\alpha} x\}$ is in $F(\Im)$.

3. Nonexpansive retractions

In this section, we prove an ergodic retraction theorem for a semitopological semigroup of asymptotically nonexpansive type mappings without convexity.

THEOREM 3.1. *Let C be a nonempty subset of a Hilbert space H and let $\Im = \{T_t : t \in G\}$ be a semitopological semigroup of asymptotically nonexpansive type mappings on C such that $L(\Im) \neq \emptyset$. Then the following statements are equivalent:*

(a) *$\bigcap_{s\in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \bigcap L(\Im) \neq \emptyset$ for each $x \in C$.*

(b) *There is a unique nonexpansive retraction P of C into $L(\Im)$ such that $PT_t = P$ for every $t \in G$ and $Px \in \overline{\text{conv}}\{T_t x : t \in G\}$ for every $x \in C$.*

Proof. (b)⇒(a). Let $x \in C$, then $Px \in L(\Im)$. Also $Px \in \bigcap_{s\in G} \overline{\text{conv}}\{T_{ts}x : t \in G\}$. In fact, for each $s \in G$, $Px = PT_s x \in \overline{\text{conv}}\{T_t T_s x : t \in G\} = \overline{\text{conv}}\{T_{ts}x : t \in G\}$.

(a)⇒(b). Let $x \in C$. Then by Proposition 2.3, $\bigcap_{s\in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \bigcap L(\Im)$ contains exactly one point Px. For each $a \in G$, we have

$$\begin{aligned} \{PT_a x\} &= \bigcap_{s\in G} \overline{\text{conv}}\{T_{tsa}x : t \in G\} \bigcap L(\Im) \\ &\supseteq \bigcap_{s\in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \bigcap L(\Im) = \{Px\} \end{aligned} \tag{3.1}$$

and hence we have $PT_a = P$ for every $a \in G$.

Finally, we have to show that P is nonexpansive. Let $x, y \in C$ and $0 < \lambda < 1$. Then for any $\varepsilon > 0$, there exists $s_1 \in G$ such that

$$\sup_{t\in G} \|T_{ts_1}x - Py\| \leq \inf_{t\in G} \|T_t x - Py\| + \varepsilon, \tag{3.2}$$

from $Py \in L(\Im)$. Hence, we have

$$\begin{aligned} &\|\lambda T_{tss_1}x + (1-\lambda)Px - Py\|^2 \\ &\quad = \|\lambda(T_{tss_1}x - Py) + (1-\lambda)(Px - Py)\|^2 \\ &\quad = \lambda\|T_{tss_1}x - Py\|^2 + (1-\lambda)\|Px - Py\|^2 - \lambda(1-\lambda)\|T_{tss_1}x - Px\|^2 \\ &\quad \leq \lambda(\|T_{ab}x - Py\| + \varepsilon)^2 + (1-\lambda)\|Px - Py\|^2 - \lambda(1-\lambda)\inf_{t\in G}\|T_t x - Px\|^2, \end{aligned} \tag{3.3}$$

for each $t, s, a, b \in G$. Since $\varepsilon > 0$ is arbitrary, this implies

$$\begin{aligned} &\inf_{s\in G}\sup_{t\in G}\|\lambda T_{ts}x + (1-\lambda)Px - Py\|^2 \\ &\quad \leq \lambda\|T_{ab}x - Py\|^2 + (1-\lambda)\|Px - Py\|^2 - \lambda(1-\lambda)\inf_{t\in G}\|T_t x - Px\|^2 \\ &\quad = \|\lambda T_{ab}x + (1-\lambda)Px - Py\|^2 + \lambda(1-\lambda)\|T_{ab}x - Px\|^2 - \lambda(1-\lambda)\inf_{t\in G}\|T_t x - Px\|^2. \end{aligned} \tag{3.4}$$

Then it is easily seen that

$$\begin{aligned} &\inf_{s\in G}\sup_{t\in G}\|\lambda T_{ts}x + (1-\lambda)Px - Py\|^2 - \lambda(1-\lambda)\inf_{b\in G}\sup_{a\in G}\|T_{ab}x - Px\|^2 \\ &\quad \leq \sup_{b\in G}\inf_{a\in G}\|\lambda T_{ab}x + (1-\lambda)Px - Py\|^2 - \lambda(1-\lambda)\inf_{t\in G}\|T_t x - Px\|^2. \end{aligned} \tag{3.5}$$

Since $Px \in L(\Im)$, we have

$$\inf_{s\in G}\sup_{t\in G}\|\lambda T_{ts}x + (1-\lambda)Px - Py\|^2 \leq \sup_{s\in G}\inf_{t\in G}\|\lambda T_{ts}x + (1-\lambda)Px - Py\|^2. \tag{3.6}$$

Let

$$h(\lambda) = \inf_{s\in G}\sup_{t\in G}\|\lambda T_{ts}x + (1-\lambda)Px - Py\|^2. \tag{3.7}$$

Then for any $\varepsilon > 0$, there exists $s_2 \in G$ such that for all $t \in G$,

$$\left\|\lambda T_{ts_2}x + (1-\lambda)Px - Py\right\|^2 \le h(\lambda) + \varepsilon \tag{3.8}$$

and hence

$$\big(\lambda T_{ts_2}x + (1-\lambda)Px - Py, Px - Py\big) \le \big(h(\lambda)+\varepsilon\big)^{1/2}\|Px - Py\| \quad \forall t \in G. \tag{3.9}$$

From $Px \in \overline{\text{conv}}\{T_{ts_2}x : t \in G\}$, we have

$$\big(\lambda Px + (1-\lambda)Px - Py, Px - Py\big) \le \big(h(\lambda)+\varepsilon\big)^{1/2}\|Px - Py\|. \tag{3.10}$$

Since $\varepsilon > 0$ is arbitrary, this yields that

$$\|Px - Py\|^2 \le h(\lambda). \tag{3.11}$$

That is,

$$\|Px - Py\|^2 \le \inf_{s\in G}\sup_{t\in G}\left\|\lambda T_{ts}x + (1-\lambda)Px - Py\right\|^2. \tag{3.12}$$

Now, one can choose an $s_3 \in G$ such that $\|T_{ts_3}x - Px\| \le M$ for all $t \in G$, where $M = 1 + \inf_{t\in G}\|T_t x - Px\|$. Then, we have

$$\begin{aligned}
&\left\|\lambda T_{tss_3}x + (1-\lambda)Px - Py\right\|^2\\
&\quad = \left\|\lambda\big(T_{tss_3}x - Px\big) + (Px - Py)\right\|^2\\
&\quad = \lambda^2\left\|T_{tss_3}x - Px\right\|^2 + \|Px - Py\|^2 + 2\lambda\big(T_{tss_3}x - Px, Px - Py\big)\\
&\quad \le M^2\lambda^2 + \|Px - Py\|^2 + 2\lambda\big(T_{tss_3}x - Px, Px - Py\big).
\end{aligned} \tag{3.13}$$

It then follows from (3.6) and (3.12) that

$$\begin{aligned}
&2\lambda \sup_{s\in G}\inf_{t\in G}\big(T_{ts}x - Px, Px - Py\big)\\
&\quad \ge 2\lambda \sup_{s\in G}\inf_{t\in G}\big(T_{tss_3}x - Px, Px - Py\big)\\
&\quad \ge \sup_{s\in G}\inf_{t\in G}\left\|\lambda T_{tss_3}x + (1-\lambda)Px - Py\right\|^2 - \|Px - Py\|^2 - M^2\lambda^2\\
&\quad = \sup_{s\in G}\inf_{t\in G}\left\|\lambda T_{ts}T_{s_3}x + (1-\lambda)PT_{s_3}x - Py\right\|^2 - \|Px - Py\|^2 - M^2\lambda^2\\
&\quad \ge \left\|PT_{s_3}x - Py\right\|^2 - \|Px - Py\|^2 - M^2\lambda^2\\
&\quad = -M^2\lambda^2.
\end{aligned} \tag{3.14}$$

Hence, we have

$$\sup_{s\in G}\inf_{t\in G}\big(T_{ts}x - Px, Px - Py\big) \ge -\frac{1}{2}M^2\lambda. \tag{3.15}$$

Letting $\lambda \to 0$, then we have

$$\sup_{s\in G}\inf_{t\in G}\big(T_{ts}x - Px, Px - Py\big) \ge 0. \tag{3.16}$$

Let $\varepsilon > 0$, then there is $s_4 \in G$ such that

$$r\big(ts_4, x\big) < \varepsilon \quad \forall t \in G. \tag{3.17}$$

For such an $s_4 \in G$, from (3.16), we have

$$\sup_{s\in G}\inf_{t\in G}\big(T_{ts}T_{s_4}x - PT_{s_4}x,\, PT_{s_4}x - Py\big) \geq 0 \tag{3.18}$$

and hence there is $s_5 \in G$ such that

$$\inf_{t\in G}\big(T_{ts_5}T_{s_4}x - PT_{s_4}x,\, PT_{s_4}x - Py\big) > -\varepsilon. \tag{3.19}$$

Then, from $PT_{s_4}x = Px$, we have

$$\inf_{t\in G}\big(T_{ts_5s_4}x - Px,\, Px - Py\big) > -\varepsilon. \tag{3.20}$$

Similarly, from (3.16), we also have

$$\sup_{s\in G}\inf_{t\in G}\big(T_{ts}T_{s_5s_4}y - PT_{s_5s_4}y,\, PT_{s_5s_4}y - Px\big) \geq 0, \tag{3.21}$$

and there exists $s_6 \in G$ such that

$$\inf_{t\in G}\big(T_{ts_6s_5s_4}y - PT_{s_5s_4}y,\, PT_{s_5s_4}y - Px\big) \geq -\varepsilon, \tag{3.22}$$

that is,

$$\inf_{t\in G}\big(Py - T_{ts_6s_5s_4}y,\, Px - Py\big) \geq -\varepsilon. \tag{3.23}$$

On the other hand, from (3.20)

$$\inf_{t\in G}\big(T_{ts_6s_5s_4}x - Px,\, Px - Py\big) > -\varepsilon. \tag{3.24}$$

Combining (3.23) and (3.24), we have

$$\begin{aligned}
-2\varepsilon &< \big(T_{ts_6s_5s_4}x - T_{ts_6s_5s_4}y,\, Px - Py\big) - \|Px - Py\|^2 \\
&\leq \big\|T_{ts_6s_5s_4}x - T_{ts_6s_5s_4}y\big\| \cdot \|Px - Py\| - \|Px - Py\|^2 \\
&\leq \big(r\big(ts_6s_5s_4, x\big) + \|x - y\|\big) \cdot \|Px - Py\| - \|Px - Py\|^2 \\
&\leq \big(\varepsilon + \|x - y\|\big) \cdot \|Px - Py\| - \|Px - Py\|^2.
\end{aligned} \tag{3.25}$$

Since $\varepsilon > 0$ is arbitrary, this implies $\|Px - Py\| \leq \|x - y\|$. The proof is completed. □

Using Lemma 2.1, we have the following ergodic retraction theorem for asymptotically nonexpansive type semigroups.

THEOREM 3.2. *Let C be a nonempty subset of a real Hilbert space H and let $\Im = \{T_t : t \in G\}$ be a semitopological semigroup of asymptotically nonexpansive type mappings on C such that $F(\Im) \neq \emptyset$. Then the following statements are equivalent:*

(a) $\bigcap_{s\in G}\overline{\text{conv}}\{T_{ts}x : t \in G\} \bigcap F(\Im) \neq \emptyset$ *for each* $x \in C$.

(b) *There is a unique nonexpansive retraction P of C onto $F(\Im)$ such that $PT_t = T_tP = P$ for every $t \in G$ and $Px \in \overline{\text{conv}}\{T_tx : t \in G\}$ for every $x \in C$.*

We denote by $B(G)$ the Banach space of all bounded real-valued functions on G with supremum norm. Let X be a subspace of $B(G)$ containing constants. Then, according to Mizoguchi-Takahashi [10], a real-valued function μ on X is called a submean on X if the following conditions are satisfied:

(1) $\mu(f+g) \leq \mu(f)+\mu(g)$ for every $f, g \in X$;
(2) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in X$ and $\alpha \geq 0$;
(3) for $f, g \in X$, $f \leq g$ implies $\mu(f) \leq \mu(g)$;
(4) $\mu(c) = c$ for every constant c.

The following corollaries are immediately deduced from Theorem 3.2.

COROLLAFRY 3.3 (see [10]). *Let C be a closed convex subset of a Hilbert space H and let X be an r_s-invariant subspace of $B(G)$ containing constants which has a right invariant submean. Let $\Im = \{T_t : t \in G\}$ be a Lipschitzian semigroup on C with $\inf_s \sup_t k_{ts}^2 \leq 1$ and $F(\Im) \neq \emptyset$, where k_t is the Lipschitzian constants. If for each $x, y \in C$, the function f on G defined by*

$$f(t) = \left\|T_t x - y\right\|^2 \quad \forall t \in G \tag{3.26}$$

and the function g on G defined by

$$g(t) = k_t^2 \quad \forall t \in G \tag{3.27}$$

belong to X, then the following statements are equivalent:

(a) $\bigcap_{s\in G} \overline{\operatorname{conv}}\{T_{ts}x : t \in G\} \bigcap F(\Im) \neq \emptyset$ *for each $x \in C$.*

(b) *There is a nonexpansive retraction P of C onto $F(\Im)$ such that $PT_t = T_t P = P$ for every $t \in G$ and $Px \in \overline{\operatorname{conv}}\{T_t x : t \in G\}$ for every $x \in C$.*

COROLLAFRY 3.4 (see [7]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $\Im = \{T_t : t \in G\}$ be a continuous representation of a semitopological semigroup as nonexpansive mappings from C into itself. If for each $x \in C$, the set $\bigcap_{s\in G} \overline{\operatorname{conv}}\{T_{ts}x : t \in G\} \bigcap F(\Im) \neq \emptyset$, then there exists a nonexpansive retraction P of C onto $F(\Im)$ such that $PT_t = T_t P = P$ for every $t \in G$ and $Px \in \overline{\operatorname{conv}}\{T_t x : t \in G\}$ for every $x \in C$.*

Remark 3.5. By Theorem 3.2, many key conditions, in Corollaries 3.3 and 3.4, such as C is convex closed subset and $\Im$ is continuous Lipschitzian semigroup, are not necessary.

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