A BOUNDARY VALUE PROBLEM IN THE HYPERBOLIC SPACE

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We consider a nonlinear problem for the mean curvature equation in the hyperbolic space with a Dirichlet boundary data g. We find solutions in a Sobolev space under appropriate conditions on g.

1. Introduction

Let M be the open unit ball in $\mathbb{R}^3$ of center 0 and let

$$g_{ij}(x) = \frac{4\delta_{ij}}{\left(1-|x|^2\right)^2} \tag{1.1}$$

be the hyperbolic metric on M. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega \in C^{1,1}$, and let (u, v) be the variables in $\mathbb{R}^2$. We consider in this paper the Dirichlet problem for a function $X : \overline{\Omega} \to M$ which satisfies the equation of prescribed mean curvature

$$\begin{gathered} \nabla_{X_u} X_u + \nabla_{X_v} X_v = -2H(X) X_u \wedge X_v \quad \text{in } \Omega, \\ X = g \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $H : M \to \mathbb{R}$ is a given continuous function, and $g \in W^{2,p}(\Omega, \mathbb{R}^3)$ for $1 < p < \infty$, with $\|g\|_\infty < 1$.

In the above equation X_u, X_v, and $X_u \wedge X_v : \Omega \to TM$ are the vector fields given by

$$\begin{gathered} X_u(u, v) = \sum_{k=1}^{3} \frac{\partial X_k}{\partial u}\bigg|_{(u,v)} \frac{\partial}{\partial x_k}\bigg|_{X(u,v)}, \qquad X_v(u, v) = \sum_{k=1}^{3} \frac{\partial X_k}{\partial v}\bigg|_{(u,v)} \frac{\partial}{\partial x_k}\bigg|_{X(u,v)}, \\ X_u \wedge X_v(u, v) = \sum_{k=1}^{3} \left(X_u \wedge X_v\right)^k (u, v) \frac{\partial}{\partial x_k}\bigg|_{X(u,v)}, \end{gathered} \tag{1.3}$$

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where

$$\begin{aligned}(X_u\wedge X_v)^1(u,v)&=\varphi^{1/2}\big(X(u,v)\big)\left(\frac{\partial X_2}{\partial u}\Big|_{(u,v)}\frac{\partial X_3}{\partial v}\Big|_{(u,v)}-\frac{\partial X_3}{\partial u}\Big|_{(u,v)}\frac{\partial X_2}{\partial v}\Big|_{(u,v)}\right),\\(X_u\wedge X_v)^2(u,v)&=\varphi^{1/2}\big(X(u,v)\big)\left(\frac{\partial X_3}{\partial u}\Big|_{(u,v)}\frac{\partial X_1}{\partial v}\Big|_{(u,v)}-\frac{\partial X_1}{\partial u}\Big|_{(u,v)}\frac{\partial X_3}{\partial v}\Big|_{(u,v)}\right),\\(X_u\wedge X_v)^3(u,v)&=\varphi^{1/2}\big(X(u,v)\big)\left(\frac{\partial X_1}{\partial u}\Big|_{(u,v)}\frac{\partial X_2}{\partial v}\Big|_{(u,v)}-\frac{\partial X_2}{\partial u}\Big|_{(u,v)}\frac{\partial X_1}{\partial v}\Big|_{(u,v)}\right),\end{aligned}\tag{1.4}$$

for $\varphi(x)=4/(1-|x|^2)^2$.

We remark that if X_u and X_v are linearly independent, then $X(\Omega)\subset M$ is an imbedded submanifold and $X_u\wedge X_v(u,v)$ is the only vector orthogonal to $X(\Omega)$ at $X(u,v)$ that satisfies, for any $z=\sum_{k=1}^3 z^k(\partial/\partial x_k)|_{X(u,v)}$

$$\big\langle z, X_u\wedge X_v(u,v)\big\rangle=\omega\big(X(u,v)\big)\big(z,X_u(u,v),X_v(u,v)\big),\tag{1.5}$$

where ω is the volume element of $(M,\langle\,,\rangle)$, namely

$$\omega=\sqrt{\det\big(g_{ij}\big)}\,dx_1\wedge dx_2\wedge dx_3=\varphi^{3/2}dx_1\wedge dx_2\wedge dx_3.\tag{1.6}$$

If ∇ is the Levi-Civita connection associated to $\langle\,,\rangle$ and $\Gamma^k_{ij}:M\to\mathbb{R}$ are the Christoffel symbols

$$\Gamma^k_{ij}=\sum_{r=1}^3\frac{g^{rk}}{2}\left(\frac{\partial g_{rj}}{\partial x_i}+\frac{\partial g_{ri}}{\partial x_j}-\frac{\partial g_{ij}}{\partial x_r}\right)\tag{1.7}$$

with $(g^{ij})=(g_{ij})^{-1}$, then a simple computation shows that

$$\Gamma^i_{ij}(x)=\Gamma^i_{ji}(x)=\frac{2x_j}{1-|x|^2},\qquad \Gamma^k_{ii}(x)=\begin{cases}-\dfrac{2x_k}{1-|x|^2} & \text{if } k\neq i,\\ 0 & \text{otherwise.}\end{cases}\tag{1.8}$$

Let $E,F,G:\Omega\to\mathbb{R}$ be the coefficients of the first fundamental form, and the unit normal $N:\Omega\to TM$ be given by

$$N=\frac{1}{\sqrt{EG-F^2}}X_u\wedge X_v\tag{1.9}$$

which is orthogonal to the tangent space $\{X(\Omega)\}_x$ for any $x=X(u,v)$. Then, if $H:\Omega\to\mathbb{R}$ is the mean curvature of $X(\Omega)$ we obtain

$$\left\langle N,\frac{G}{EG-F^2}\nabla_{X_u}X_u+\frac{E}{EG-F^2}\nabla_{X_v}X_v-2\frac{F}{EG-F^2}\nabla_{X_u}X_v\right\rangle=-2H.\tag{1.10}$$

In particular, if X is isothermal, that is, $E=G$, $F=0$, then $\langle\nabla_{X_u}X_u+\nabla_{X_v}X_v,X_u\rangle=0=\langle\nabla_{X_u}X_u+\nabla_{X_v}X_v,X_v\rangle$ and consequently

$$\nabla_{X_u}X_u+\nabla_{X_v}X_v=-2HX_u\wedge X_v.\tag{1.11}$$

Thus, (1.11) is the equation of prescribed mean curvature for an imbedded submanifold of M.

2. A Dirichlet problem for (1.11)

With the notations of the previous section, we consider the Dirichlet problem (1.2). The equation of prescribed mean curvature for a surface in $\mathbb{R}^3$ has been studied for constant H in [3, 5], and for H nonconstant in [1, 2].

Without loss of generality, we may assume that g is harmonic in Ω. Our existence result reads as follows.

THEOREM 2.1. *Let c_0 and c_1 be some positive constants to be specified. Then (1.2) is solvable for any $g \in W^{2,p}(\Omega,\mathbb{R}^3)$ harmonic such that*

$$\|g\|_\infty + 2\Big(c_1 + \sqrt{c_1\big(c_1+c_0\big)}\Big)\|\operatorname{grad}(g)\|_{2p} \le 1. \tag{2.1}$$

In the proof of Theorem 2.1, we ignore the canonical isomorphism $\partial/\partial x_k|_{X(u,v)} \to e_k$ (with $\{e_k\}$ the usual basis of $\mathbb{R}^3$), and considering $X_u, X_v \in \mathbb{R}^3$ we may write (1.2) as a system

$$\begin{aligned} -\Delta X_k &= \psi_k\big(X, X_u, X_v\big) \quad \text{in } \Omega, \\ X_k &= g_k \quad \text{on } \partial\Omega \end{aligned} \tag{2.2}$$

with $\psi_k(X,X_u,X_v) = 2H(X)(X_u \wedge X_v)^k + \sum_{i,j}\Gamma_{ij}^k(X)\operatorname{grad}(X_i)\operatorname{grad}(X_j)$, $1 \le k \le 3$. For fixed $\overline{X} \in W_0^{1,2p}(\Omega,\mathbb{R}^3)$ such that $\|g+\overline{X}\|_\infty < 1$, we define $X = T\overline{X}$ as the unique solution in $W^{2,p}(\Omega,\mathbb{R}^3) \hookrightarrow W^{1,2p}(\Omega,\mathbb{R}^3)$ of the linear problem

$$\begin{aligned} -\Delta X_k &= \psi_k\Big(\overline{X}+g, \big(\overline{X}+g\big)_u, \big(\overline{X}+g\big)_v\Big) \quad \text{in } \Omega, \\ X_k &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.3}$$

Then, for $B = \{X \in W_0^{1,2p}(\Omega,\mathbb{R}^3) \mid \|g+X\|_\infty < 1\}$ the operator $T : B \to W_0^{1,2p}(\Omega,\mathbb{R}^3)$ is well defined and a strong solution of (1.2) in $W^{2,p}$ can be regarded as $Y = g + X$, where X is a fixed point of T. By the usual a priori bounds for the Laplacian and the compactness of the imbedding $W^{2,p}(\Omega,\mathbb{R}^3) \hookrightarrow W_0^{1,2p}(\Omega,\mathbb{R}^3)$ we get the following lemma.

LEMMA 2.2. *$T : B \to W_0^{1,2p}(\Omega,\mathbb{R}^3)$ is continuous. Furthermore, if*

$$C_{R_1,R_2} = \Big\{X \in W_0^{1,2p}\big(\Omega,\mathbb{R}^3\big) \mid \|g+X\|_\infty \le R_1, \|\operatorname{grad}(X)\|_{2p} \le R_2\Big\} \tag{2.4}$$

with $R_1 < 1$, then $T(C_{R_1,R_2})$ is precompact.

Proof. For $X = T(\overline{X})$, $Y = T(\overline{Y})$, as $X = Y$ on $\partial\Omega$ we obtain that

$$\begin{aligned} \big\|\operatorname{grad}\big(X_k - Y_k\big)\big\|_{2p} &\le c\big\|\Delta\big(X_k-Y_k\big)\big\|_p \\ &= c\Big\|\psi_k\Big(\overline{X}+g,\big(\overline{X}+g\big)_u,\big(\overline{X}+g\big)_v\Big) - \psi_k\Big(\overline{Y}+g,\big(\overline{Y}+g\big)_u,\big(\overline{Y}+g\big)_v\Big)\Big\|_p \end{aligned} \tag{2.5}$$

and the continuity of T follows. On the other hand, if $\overline{X} \in C_{R_1,R_2}$, then

$$\begin{aligned} \|\operatorname{grad}(X_k)\|_{2p} \leq c\|\Delta X_k\|_p = c\left\|\psi_k\left(\overline{X}+g, (\overline{X}+g)_u, (\overline{X}+g)_v\right)\right\|_p \\ \leq \bar{c}\left(R_2+\|\operatorname{grad}(g)\|_{2p}\right)^2 \end{aligned} \tag{2.6}$$

for some constant $\bar{c}$ and the result follows. □

Remark 2.3. By definition of ψ_k, it is clear that $\bar{c} \leq c_1/(1-R_1)$ for some constant c_1.

Proof of Theorem 2.1. With the notation of the previous lemma, by Schauder fixed point theorem, it suffices to see that C_{R_1,R_2} is T-invariant for some R_1, R_2. From the previous computations, we have

$$\|\operatorname{grad}(X)\|_{2p} \leq \frac{c_1}{1-R_1}\left(R_2+\|\operatorname{grad}(g)\|_{2p}\right)^2. \tag{2.7}$$

Moreover, by Poincaré's inequality

$$\|g+X\|_\infty \leq \|g\|_\infty + c_0\|\operatorname{grad}(X)\|_{2p}. \tag{2.8}$$

Thus, a sufficient condition for obtaining $T(C_{R_1,R_2}) \subset C_{R_1,R_2}$ is that

$$\frac{c_1}{1-R_1}\left(R_2+\|\operatorname{grad}(g)\|_{2p}\right)^2 \leq R_2, \qquad \|g\|_\infty + c_0 R_2 \leq R_1. \tag{2.9}$$

For R small enough we may fix $R_1 = \|g\|_\infty + c_0 R < 1$, and then the theorem is proved if

$$c_1\left(R+\|\operatorname{grad}(g)\|_{2p}\right)^2 \leq R\left(1-\|g\|_\infty - c_0 R\right) \tag{2.10}$$

for some $R>0$. As last condition is equivalent to our hypothesis, the result holds. □

3. Regularity of the solutions of problem (1.2)

In this section, we state the following regularity result.

THEOREM 3.1. *Let $X \in W^{1,2p}(\Omega,\mathbb{R}^3)$ be a solution of (1.2). Then*

(a) *if $g \in W^{2,q}(\Omega,\mathbb{R}^3)$ for some $q>1$, then $X \in W^{2,q}(\Omega,\mathbb{R}^3)$,*
(b) *if $\partial\Omega \in C^{k+2,\alpha}$, $H \in C^{k,\alpha}(\mathbb{R}^3,\mathbb{R})$, $g \in C^{k+2,\alpha}(\overline{\Omega},\mathbb{R}^3)$ for some $0<\alpha<1$, $k \geq 0$, then $X \in C^{k+2,\alpha}(\overline{\Omega},\mathbb{R}^3)$.*

Proof. (a) Let $\Delta X = f \in L^p$. If $p \geq q$, let Z be the unique solution in $W^{2,q}$ of the problem $\Delta Z = f$, $Z|_{\partial\Omega} = g$. As $\Delta(X-Z)=0$ and $X=Z$ on $\partial\Omega$ the result follows. On the other hand, if $p<q$, we obtain in the same way that $X \in W^{2,p}$. For $2 \leq p < q$ this implies that $X \in W^{1,2q}$ and the result follows.

Now we consider the case $p < 2, q$. Let $p_0 = p$ and define

$$p_{n+1} = \begin{cases} \dfrac{p_n^*}{2} & \text{if } p_n < 2, q, \\ q & \text{otherwise,} \end{cases} \tag{3.1}$$

where p_n^* is the critical Sobolev exponent $2p_n/(2-p_n)$. Then $\{p_n\}$ is bounded, and $X \in W^{1,2p_n}$ for every n. If $p_n < 2, q$ for every n, then p_n is increasing and taking $r = \lim_{n\to\infty} p_n$, we obtain that $r/(2-r) = r$, a contradiction. Hence, $p_n \geq q$ or $q > p_n \geq 2$ for some n, and the proof is complete.

(b) Case $k = 0$: by part (a), choosing $q > 2/(1-\alpha)$ we obtain that $X \in W^{2,q} \hookrightarrow C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$. Then $\Delta X = f \in C^{\alpha}(\overline{\Omega}, \mathbb{R}^3)$. By [4, Theorem 6.14] the equation $\Delta Z = f$ in Ω, $Z = g$ in $\partial\Omega$ is uniquely solvable in $C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$, and the result follows from the uniqueness in [4, Theorem 9.15].

The general case is now immediate, from [4, Theorem 6.19]. □

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