ATTRACTORS OF MULTIVALUED SEMIFLOWS GENERATED BY DIFFERENTIAL INCLUSIONS AND THEIR APPROXIMATIONS

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We prove the existence of global compact attractors for differential inclusions and obtain some results concerning the continuity and upper semicontinuity of the attractors for approximating and perturbed inclusions. Applications are given to a model of regional economic growth.

1. Introduction

The theory of multivalued dynamical systems is motivated by differential equations for which it is not known whether the solution corresponding to each initial data is unique or not. In such a case it is not possible to define a semigroup of operators. However, by taking the union of all solutions belonging to a certain class we can define a multivalued semiflow and study in this way the asymptotic behavior of the trajectories. We will recall some results of the abstract theory of attractors for multivalued semiflows developed in [11, 13, 14] (see also [3, 5]).

Denote by X a complete metric space with the metric ρ and by 2^X ($\beta(X)$; $C_v(X)$; $\operatorname{comp}(X)$) the family of all (nonempty bounded; nonempty, bounded, closed, convex; nonempty compact) subsets of X. As usual, $\operatorname{dist}(A,B) = \sup_{y\in A}\inf_{x\in B}\rho(y,x)$ and $\operatorname{dist}_H(A,B) = \max\{\operatorname{dist}(A,B),\operatorname{dist}(B,A)\}$, $A, B \in \beta(X)$, is the Hausdorff metric. Let $B_\epsilon(A) = \{y \in X \mid \operatorname{dist}(y,A) \le \epsilon\}$ be an ϵ-neighborhood of the set $A \subset X$.

A multivalued map $F : X \to 2^X$ is said to be w-upper semicontinuous if $\forall x_0 \in D(F)$, $\forall\epsilon > 0$, $\exists\delta > 0$ such that $F(x) \subset B_\epsilon(F(x_0))$, $\forall x \in B_\delta(x_0)$, where $D(F) = \{x \mid F(x) \in P(X)\}$. It is said to be upper semicontinuous if $\forall x_0 \in D(F)$ and any neighborhood $O(F(x_0))$ there exists $\delta > 0$ such that $F(x) \subset O(F(x_0))$, $\forall x \in B_\delta(x_0)$. Obviously, any upper semicontinuous map is w-upper semicontinuous, the converse being valid if F has compact values [1, page 45].

A multivalued map $G : \mathbb{R}_+ \times X \to P(X)$ is said to be a multivalued semiflow (m-semiflow for short) if $G(0,\cdot) = Id$ and $G(t_1+t_2,x) \subset G(t_1,G(t_2,x))$, $\forall t_1,t_2 \in \mathbb{R}_+$, $\forall x \in X$. The set Ξ is called a global attractor of G if $\Xi \subset G(t,\Xi)$, $\forall t \in \mathbb{R}_+$, and

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$\operatorname{dist}(G(t,B),\Xi) \xrightarrow{t\to\infty}$, $\forall B \in \beta(X)$. It is said to be invariant if $\Xi = G(t,\Xi), \forall t \in \mathbb{R}_+$. If Ξ is compact then it is the minimal closed set attracting all bounded sets.

The m-semiflow G is called point dissipative if there exists $B_0 \in \beta(X)$ such that $\operatorname{dist}(G(t,x),B_0) \xrightarrow{t\to\infty} 0$, $\forall x \in X$.

THEOREM 1.1 (see [14, Theorem 3 and Proposition 1]). *Let for any $t \in \mathbb{R}_+$, $G(t,\cdot) : X \to C(X)$ be upper semicontinuous. Suppose that G is point dissipative and that for some $t_0 > 0$ the operator $G(t_0,\cdot)$ is compact. Then G has the global compact attractor Ξ.*

Concerning the dependence of attractors on a parameter from the proof of [11, Theorem 4] it follows the following theorem.

THEOREM 1.2. *Let Λ be a metric space, λ_0 be a non-isolated point and $G_\lambda : \mathbb{R}_+ \times X \to P(X)$, $\lambda \in \Lambda$, be a family of m-semiflows satisfying:*

(1) *for each $\lambda \in \Lambda$, G_λ has a global attractor Ξ_λ and $\cup_{\lambda\in\Lambda}\Xi_\lambda \in \beta(X)$;*
(2) *the map $\lambda \mapsto G_\lambda(t,\Xi)$, $\Xi = \overline{\cup_{\lambda\in\Lambda}\Xi_\lambda}$, is w-upper semicontinuous at λ_0 for large t.*

Then $\operatorname{dist}(\Xi_\lambda, \Xi_{\lambda_0}) \to 0$, *as $\lambda \to \lambda_0$.*

Other approaches to the problem of non-uniqueness is the construction of the so-called trajectory attractors (see [8, 15, 18]) or multivalued semiflows via the non-standard framework [7].

Whereas in [4, 21] are considered differential inclusions generating a semigroup of operators in this paper we study, as in [14], inclusions generating a multivalued semiflow. This paper is organized as follows. In Section 2, we extend the results of [14] on existence of a global compact attractor Ξ for the differential inclusion

$$\begin{gathered}\frac{dy}{dt} \in -\partial\phi(y) + F(y), \qquad t \in [0;T],\\ y(0) = y_0,\end{gathered} \tag{1.1}$$

where $F : H \to 2^H$ is a multivalued map in a Hilbert space H. In Sections 3 and 4, we prove that for a certain class of approximating maps F_n of the multivalued right-hand side F the corresponding attractors Ξ_n converge in the Hausdorff metric to Ξ. Finally, in Section 5 we prove the upper semicontinuity of the global attractor under a small perturbation of the map F, $F_\epsilon = F + \epsilon S$, $\epsilon > 0$. All these results are applied to boundary value problems and in particular to a model of regional economic growth.

2. Existence of the global attractor

Let H be a real separable Hilbert space, $(\cdot,\cdot)$, $\|\cdot\|$ be the scalar product and norm in H, respectively, $\phi : H \mapsto (-\infty,+\infty]$ be a proper, convex, lower semicontinuous function and let $\partial\phi : D(\partial\phi) \subset H \mapsto 2^H$ be its subdifferential.

Consider the problem

$$\frac{dy}{dt} \in -\partial\phi(y) + F(y), \quad t \in [0;T], \\ y(0) = y_0 \in H, \tag{2.1}$$

where $F : H \to 2^H$ and satisfy the properties:

(G1) $F : H \to C_v(H)$;
(G2) $\exists D_1, D_2 \geq 0$ such that $\sup_{u \in F(v)} \|u\| \leq D_1 + D_2\|v\|$, $\forall v \in H$;
(G3) F is w-upper semicontinuous;
(G4) $\exists \delta > 0$, $M > 0$ such that $\forall u \in D(\partial\phi)$, $\|u\| > M$, $\forall y \in -\partial\phi(u) + F(u)$,

$$(y, u) \leq -\delta; \tag{2.2}$$

(G5) $\forall R > 0$ the set $M_R = \{u \in H \mid \|u\| \leq R, \phi(u) \leq R\}$ is compact in H.

Further we denote $X = \overline{D(\phi)}$.

Definition 2.1. The continuous function $y : [0, T] \to X$ is called an integral solution of problem (2.1) if $y(0) = y_0$ and there exists $f \in L_1([0, T], X)$, $f(\tau) \in F(y(\tau))$, a.e. $\tau \in (0, T)$, such that $\forall u \in D(\partial\varphi)$, $\forall v \in -\partial\varphi(u)$,

$$\|y(t) - u\|^2 \leq \|y(s) - u\|^2 + 2\int_s^t \big(f(\tau) + v, y(\tau) - u\big)\, d\tau, \quad t \geq s. \tag{2.3}$$

Further we shall denote each integral solution by $y(\cdot) = I(y_0)f(\cdot)$. The integral solution $y(\cdot)$ is called a strong one if it is absolutely continuous on $(0, T)$ and $dy/dt \in -\partial\phi(y(\tau)) + f(\tau)$, a.e. on $(0, T)$.

According to [20, Theorem 2.1] $\forall x_0 \in X$, $\forall T > 0$, there exists an integral solution of (2.1), $x(\cdot) = I(x_0)f(\cdot)$, $x(0) = x_0$. Moreover, the set of all integral solutions on $[0, T]$ starting from the point x_0 (denoted by $\Theta_F^T(x_0)$) is a connected compact set in the space $C(0, T; X)$ and the map $x \mapsto \Theta_F^T(x)$ is w-upper semicontinuous [20, Theorems 2.1 and 4.3].

Lemma 2.2. *Under condition (G2) each integral solution of (2.1) is a strong solution.*

Proof. According to [6, page 189] it is sufficient to prove that any selection $f(\cdot) \in F(y(\cdot))$, where $y(\cdot) = I(u_0)f(\cdot)$, belongs to $L_2(0, T; X)$. It follows from (G2) that $\|f(t)\| \leq D_1 + D_2\|y(t)\|$, but $y(\cdot) \in C(0, T; X)$, so that $f(\cdot) \in L_2(0, T; X)$. $\square$

Now in the same way as in [14] we define the m-semiflow $G : \mathbb{R}_+ \times X \to P(X)$, $G(t, y_0) = \{y(t) \mid y(\cdot)$ is a strong solution of (2.1), $y(0) = y_0\}$. Following [14, Lemma 6] we can prove that $G(t_1 + t_2, x) = G(t_1, G(t_2, x))$, $\forall x \in X$, $\forall t_1, t_2 \in \mathbb{R}_+$.

Theorem 2.3. *Let (G1)–(G5) hold. Then G has the global compact invariant attractor Ξ, which is the minimal closed set attracting all bounded sets.*

Proof. We obtain some properties of G. First we prove that $\forall t \geq 0$, $\forall x \in X$, $G(t,x)$ is compact in X. Indeed, from the fact that $\Theta_F^T(x)$ is compact in $C(0,T;X)$ we have that $\forall \{y_n(\cdot)\} \subset \Theta_F^T(x)$ there exist a subsequence and $y(\cdot) \in \Theta_F^T(x)$ such as $y_n \to y$ in $C(0,T;X)$. Hence, $y_n(t) \to y(t)$, $\forall t \in [0,T]$, in X. It follows that $G(t,x)$ is compact $\forall t \in [0,T]$. On the other hand, we obtain that $G(t,\cdot) : X \to P(X)$ is upper semicontinuous. Indeed, from the fact that $x \mapsto \Theta_F^T(x)$ is w-upper semicontinuous, we have that $\forall \epsilon > 0, \forall x \in X, \exists \delta > 0$ such that $\|x - x_0\| < \delta$ implies $\Theta_F^T(x) \subset B_\epsilon(\Theta_F^T(x_0))$, that is, for an arbitrary $y(\cdot) \in \Theta_F^T(x)$, $\exists y_0(\cdot) \in \Theta_F^T(x_0)$ such that $\max_{t\in[0,T]} \|y(t) - y_0(t)\| \leq \epsilon$ and then $\forall t \in [0,T]$, $\|y(t) - y_0(t)\| \leq \epsilon$. Thus $G(t,x) \subset B_\epsilon(G(t,x_0))$ and by virtue of the compactness of $G(t,x)$ the upper semicontinuity is proved.

Let $B_0 = \{u \in X \mid \|u\| \leq M + \epsilon\}$, $\epsilon > 0$. We show that $G(t,B_0) \subset B_0$, $\forall t \geq 0$. Let $x_0 \in B_0, x(\cdot) \in \Theta_F^T(x_0)$ be such that $\exists t > 0$ for which $x(t) \notin B_0$, that is, $\|x(t)\| > M + \epsilon$. As $x(\cdot)$ is continuous, then there exists t_0 such that $\|x(t_0)\| = M + \epsilon$, $\|x(\tau)\| \geq M + \epsilon$, $\forall \tau \in [t_0, t]$. Therefore, using (G4) and the fact that $x(\cdot)$ is a strong solution of (2.1), in a standard way we obtain that $(1/2)(d/d\tau)\|x(\tau)\|^2 \leq -\delta$, $\forall \tau \in [t_0,t]$, so that $\|x(t)\|^2 \leq \|x(t_0)\|^2 - 2\delta(t - t_0)$, which is a contradiction. Hence, $G(t,B_0) \subset B_0$, $\forall t \geq 0$. Thus, repeating the proof of [14, Theorem 7], we obtain that $\forall x \in X$, $\exists t_x > 0$ such that $G(t,x) \subset B_0$, $\forall t \geq t_x$. In the same way we also prove that $G(t,B_N) \subset B_N$, $\forall N > M$, $\forall t \geq 0$, where $B_N = \{u \in X \mid \|u\| \leq N\}$. Therefore, G is pointwise dissipative and $\bigcup_{\tau\geq 0} G(\tau,B) \in \beta(X)$, $\forall B \in \beta(X)$.

Now we prove that $G(t,B)$ is precompact in X for any $t > 0$ and $B \in \beta(X)$. According to (G5) it is sufficient to prove that $\exists R = R(t,B)$ such that $G(t,B) \subset M_R$. First we shall show that the set $M(B,T) = \{f(\cdot) \mid y(\cdot) = I(y_0) f(\cdot), y \in \Theta_F^T(y_0), y_0 \in B\}$ is bounded in $L_2(0,T;X)$. Indeed, there exists N for which $G(t,B) \subset B_N$, $\forall t \geq 0$, and then $\max_{t\in[0,T]} \|y(t)\| \leq N$, $\forall y(\cdot) \in \Theta_F^T(B)$. By virtue of (G2), $\|f(t)\| \leq D_1 + D_2\|y(t)\| \leq D_1 + D_2 N$, $\forall f(\cdot) \in M(B,T)$. Thus $M(B,T)$ is bounded in $L_2(0,T;X)$. So, repeating the proof of [14, Theorem 8] we obtain that $\forall t > 0$, $\exists R = R(t,B)$ such that $G(t,B) \subset M_R$. Therefore $G(t,B)$ is precompact in X.

Hence, it follows from Theorem 1.1 that there exists the global compact attractor Ξ. Moreover, by [14, Remarks 5 and 8], $\Xi = G(t,\Xi)$, $\forall t \geq 0$, and the minimality property holds. □

Remark 2.4. Theorem 2.3 generalizes Theorem 9 from [14], in which F is supposed to be Lipschitz in the multivalued sense.

Consider the application of the previous result to the problem

$$\begin{gathered} \frac{\partial y}{\partial t} \in \triangle y + f(y) + h, \quad \text{on } \Omega \times (0,T), \\ y|_{\partial\Omega} = 0, \\ y(x,0) = y_0(x), \quad x \in \Omega, \end{gathered} \tag{2.4}$$

where $h \in L_2(\Omega)$, $\Omega \subset \mathbb{R}^n$ is a bounded open domain with smooth boundary $\partial\Omega$ and $f : \mathbb{R} \to 2^{\mathbb{R}}$ satisfies:

(H1) $f : \mathbb{R} \to C_v(\mathbb{R})$;

(H2) $\exists D_1, D_2 \geq 0$ such that $\sup_{y \in f(s)} |y| \leq D_1 + D_2|s|$, $\forall s \in \mathbb{R}$;
(H3) f is w-upper semicontinuous;
(H4) $\exists M \geq 0$, $\alpha > 0$ such that $\forall s \in \mathbb{R}$, $\forall y \in f(s)$, $ys \leq (\lambda_1 - \alpha)|s|^2 + M$, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

To come to problem (2.1), we define $F : H \to 2^H$, $H = L_2(\Omega)$,

$$F(y) = \{\xi + h \mid \xi \in H, \xi(x) \in f(y(x)) \text{ a.e. } x \in \Omega\}. \tag{2.5}$$

It is well known that $-\Delta$ is the subdifferential of the proper convex lower semicontinuous function $\phi(u) = \int_\Omega (1/2)|\nabla u|^2 dx$ with $D(\phi) = H_0^1(\Omega)$ and (G5) holds [6].

Proposition 2.5. *The map F satisfies (G1)–(G4).*

Proof. Condition (H4) in a standard way [14, Theorem 10] provides that (G4) holds. The map f has compact values and then it is upper semicontinuous, so that it is measurable [2, Proposition 8.2.1]. Hence, there exists a measurable selection $g(s) \in f(s)$, $s \in \mathbb{R}$ [2, Theorem 8.1.3]. Then for any $y \in H$, $g(y(x))$ is a measurable selection of $f(y(x))$. In view of (H2), we have that $\forall y \in H$, $\forall (\xi + h) \in F(y)$, $\|\xi + h\| \leq \sqrt{\int_\Omega |\xi(x)|^2 dx} + \|h\| \leq \sqrt{\int_\Omega (D_1 + D_2|y(x)|)^2 dx} + \|h\| \leq \tilde{D}_1 + \tilde{D}_2\|y\|$, so that $F(y) \neq \emptyset$, $\forall y \in H$, and (G2) holds. Following [14, Lemma 11] we obtain that $F : H \to C_v(H)$.

Now we prove that if $f : \mathbb{R} \to C_v(\mathbb{R})$ is upper semicontinuous and satisfies (H2) then F is upper semicontinuous on H. Since the map f is upper semicontinuous, is upper hemicontinuous [1, page 60]. We prove that F is also hemicontinuous, that is, from $u_n \to u$ in H and $\sigma_n(p) := \sigma(F(u_n), p) = \sup_{v \in F(u_n)} (p, v) \to \sigma_0(p), \forall p \in H$, it follows that $\sigma(F(u), p) \geq \sigma_0(p)$. Indeed, $\forall p \in H$, $\forall n \geq 1$ $\exists v_n \in F(u_n)$ such that $(p, v_n) > \sigma_n(p) - 1/n$. Moreover, by virtue of (G2) with accuracy to a subsequence $v_n \to v$ weakly in H. Now we can use [16, Chapter 3, Theorem 6], taking $X = Y = \mathbb{R}$, $p = q = 2$. Since $(u_n(x), v_n(x)) \in \text{graph}\,(f)$ for a.e. $x \in \Omega$, $u_n \to u$ in H, $v_n \to v$ weakly in H, all the conditions of the mentioned theorem hold and we have $v(x) \in f(u(x))$ for a.e. $x \in \Omega$. Then passing to the limit in the last inequality we have $(p, v) \geq \sigma_0(p)$, $v \in F(u)$. Thus, $\sup_{v \in F(u)} (p, v) = \sigma(F(u), p) \geq \sigma_0(p)$ and hence $F : H \to C_v(H)$ is hemicontinuous. For arbitrary $u_0 \in H$ conditions (G1)–(G2) hold, so that $F(u_0)$ is weakly compact and convex in H and hence according to [16, Chapter 3, Theorem 10] F is upper semicontinuous at u_0. Therefore, G3 is satisfied. □

Now, Theorem 2.3 implies the following theorem.

Theorem 2.6. *Let (H1)–(H4) hold. The semiflow generated by (2.4) has the global compact invariant attractor Ξ, which is the minimal closed set attracting all bounded sets.*

Example 2.7. A model of regional economic growth.

Consider a closed economy on a bounded domain $\Omega \subset \mathbb{R}^n$ and the following variables: $y(x, t)$ is the stock of available capital; $u(x, t)$ is the rate of investment. From the local conservation of capital it follows, as a particular case, that the equation (see

[17, page 603]):

$$\begin{aligned} \frac{\partial y}{\partial t} &= \Delta y+\omega(y)+g(y)+u, \quad \text{on } \Omega\times(0,T),\\ y|_{\partial\Omega} &= 0,\\ y(x,0) &= y_0(x), \quad x\in\Omega,\\ 0\le u(x,t) &\le \theta\big(y(x,t)\big), \quad \text{on } \Omega\times(0,T), \end{aligned} \tag{2.6}$$

where $-\omega(y)$, ω being non-decreasing, represents a recursive depreciation of capital and $-g(y)$ is the nonlinear rate of demand. The Dirichlet boundary conditions imply the fact that the economy is closed. We assume that the functions $\omega, g:\mathbb{R}\to\mathbb{R}$, $\theta:\mathbb{R}\to\mathbb{R}_+$ are continuous and have at most linear growth.

Define the multivalued map $f:\mathbb{R}\to 2^{\mathbb{R}}$,

$$f(s)=\big\{\omega(s)+g(s)+\xi \mid 0\le\xi\le\theta(s)\big\}. \tag{2.7}$$

It is straightforward to check that (H1)–(H3) hold. If we assume that

$$\begin{aligned} \big(\omega(s)+g(s)+\theta(s)\big)s &\le \big(\lambda_1-\alpha\big)s^2+M, \quad \forall s\ge 0,\\ \big(\omega(s)+g(s)\big)s &\le \big(\lambda_1-\alpha\big)s^2+M, \quad \forall s\le 0, \end{aligned} \tag{2.8}$$

then (H4) is also satisfied. Therefore, equation (2.6) is a particular case of (2.4) and Theorem 2.6 holds.

3. Approximation of the attractor

Now we are interested in the possibility of the approximation of the attractor Ξ. For this we assume that the following stronger conditions hold instead of (G2) and (G4):

(G2*) $\exists C>0$ such that $\sup_{u\in F(v)}\|u\|\le C$, $\forall v\in H$;

(G4*) $\exists\gamma>0$ such that $(\partial\varphi(y),y)\ge\gamma\|y\|^2$, $\forall y\in D(\partial\varphi)$.

Conditions (G2*), (G4*) imply (G2), (G4). Indeed, for any $\xi\in-\partial\varphi(y)+F(y)$ we have $(\xi,y)\le-\gamma\|y\|^2+\sup_{u\in F(y)}\|u\|\|y\|\le-\gamma\|y\|^2+C\|y\|$. Hence $(\xi,y)\le\|y\|(-\gamma\|y\|+C)$ and condition (G4) holds for $\delta=M=(1/\gamma)(C+1)$. Due to condition (G2*) we can use [20, Theorem 1.1] and construct the sequence $\{F_n: H\mapsto C_v(H)\}$ such that $\forall u\in H$, $F(u)=\bigcap_{n=1}^{\infty}F_n(u)$, $F_{n+1}(u)\subset F_n(u)$, F_n are locally Lipschitz (in the multivalued sense) and have locally Lipschitz selections and for each F_n condition (G2*) holds with the same constant C. Moreover, $\operatorname{dist}(F_n(u),F(u))\to 0$, $\forall u\in H$. By F_n we construct in the same way as before the m-semiflows G_n, since (G1)–(G4) are satisfied for the maps F_n. From Theorem 2.3 it follows the existence of the compact global invariant attractor Ξ_n for each G_n, $n\ge 1$. The maps F_n are more regular than F, so it is interesting to consider whether the attractors Ξ_n converge to Ξ in the Hausdorff metric.

Theorem 3.1. *Let (G1), (G2*), (G3), (G4*) hold. Then* $\operatorname{dist}_H(\Xi,\Xi_n)\to 0$, *as* $n\to\infty$.

Proof. We note that $\Xi=G(t,\Xi)\subset G_n(t,\Xi)\subset B_\epsilon(\Xi_n)$, $\forall\epsilon>0$, $t\ge T(\epsilon)$, and since the sets Ξ_n are compact, we have $\Xi\subset\Xi_n$, $\forall n\ge 1$. Analogously, $\Xi_{n+1}\subset\Xi_n$. Hence,

$\bigcup_{n=1}^{\infty} \Xi_n \cup \Xi = \bigcup_{n=1}^{\infty} \Xi_n = \Xi_1$. We must show that $\forall \epsilon > 0$, $\exists N$ such that $\Xi_n \subset B_\epsilon(\Xi)$, $\forall n \geq N$. In view of Theorem 1.2 we have to prove that $\bigcup_{n=1}^{\infty} \Xi_n \in \beta(H)$ (but we have already shown that such a set is compact) and for large t the next property holds: $\forall \epsilon > 0$, $\exists N$ such that $G_n(t, \Xi_1) \subset B_\epsilon(G(t, \Xi_1))$, $\forall n \geq N$. Now we prove it. On the set $\Lambda = \{n, n \geq 1, +\infty\}$ we introduce the metric $\rho(m,n) = |1/m - 1/n|$ $(1/\infty = 0)$. Hence (Λ, ρ) is a metric compact space. Let $\lambda_0 := +\infty$. Now it is sufficient to verify that the map $\Lambda \ni \lambda \mapsto G_\lambda(t, \Xi_1)$ is upper semicontinuous at λ_0. Since $G_\lambda(t, \Xi_1)$ is compact for any $\lambda \in \Lambda$ (this follows from the fact that the map $G_\lambda(t, \cdot)$ is upper semicontinuous and have compact values [1, page 42]), (Λ, ρ) is a compact metric space and $G_\lambda \subset G_1$, $\forall \lambda \in \Lambda$, it is sufficient to prove that its graph on Λ is compact in $\Lambda \times H$ [2, Proposition 1.4.8], that is, the set $D = \{(\lambda, u) \mid \lambda \in \Lambda,\ u \in G_\lambda(t, \Xi_1)\}$ is compact in $\Lambda \times H$. Let $\{(\lambda_n, u_n)\} \subset D$. Hence $\lambda_n \to \lambda_0$ and we have to prove that there exists $u_1 \in G_{\lambda_0}(t, \Xi_1)$ such that $u_n \to u_1$ in H (with accuracy to a subsequence). We have $u_n = u_n(t)$, $u_n(\cdot) = I(\eta_n) f_n(\cdot)$, $u_n(0) = \eta_n \in \Xi_1$. Hence, there exists $\eta_0 \in \Xi_1$ and a subsequence such that $\eta_n \to \eta_0$. We consider $z_n(\cdot) = I(\eta_0) f_n(\cdot)$. Let $\sigma - L_1(0,T;H)$ be the space $L_1(0,T;H)$ endowed with the weak topology. In view of the inequality $\|f_n(\tau)\| \leq C$, a.e. $\tau \in (0,T)$, for a subsequence $f_n \to f$ in $\sigma - L_1(0,T;H)$. Since $\{f_n\}$ are uniformly integrable and the semigroup $S(t,\cdot)$ generated by $-\partial\phi$ is compact (this follows from (G5) [10, page 1398]), there exist a subsequence $\{z_n(\cdot)\}$ such that $z_n \to z$ in $C(0,T;H)$ [9, Theorem 2.3]. Hence, $z_n \to z$ in $C(0,T;H)$, $f_n \to f$ in $\sigma - L_1(0,T;H)$ and $z(\cdot) = I(\eta_0) f(\cdot)$ [19, Lemma 1.3]. Therefore $\max_{t\in[0,T]} \|u_n(t) - z(t)\| \leq \max_{t\in[0,T]} \|I(\eta_n) f_n(t) - I(\eta_0) f_n(t)\| + \max_{t\in[0,T]} \|I(\eta_0) f_n(t) - I(\eta_0) f(t)\| \leq \|\eta_n - \eta_0\| + \max_{t\in[0,T]} \|z_n(t) - z(t)\| \to 0$, $n \to \infty$. Thus $u_n(t) \to z(t)$, $\forall t \in [0,T]$, $z(0) = \eta_0 \in \Xi_1$. We prove the fact that $f(t) \in F(z(t))$ for a.e. $t \in [0,T]$. First we note that $f_n(t) \in F_n(z_n(t))$, a.e. $t \in [0,T]$. We prove that $\exists N$ such that $\forall n \geq N$, $f(t) \in B_{1/n}(F_n(z(t)))$, a.e. on $(0,T)$. Indeed, let it not be so. Then $\forall N \geq 1$, $\exists\ n \geq N$ such that $f(t) \notin B_{1/n}(F_n(z(t)))$. On the other hand, from the w-semicontinuity and the facts proved above $\forall n \geq 1$, $\exists m(n) \geq n$ such that $F_n(z_k(t)) \subset B_{1/2n}(F_n(z(t)))$, $\forall k \geq m(n)$. So $\bigcup_{k\geq m(n)} F_n(z_k(t)) \subset B_{1/2n}(F_n(z(t)))$. As $k \geq m(n) \geq n$, so $\bigcup_{k\geq m(n)} F_k(z_k(t)) \subset B_{1/2n}(F_n(z(t)))$. Hence, by virtue of the convexity of $F_n(z)$ we have $\overline{\mathrm{co}} \bigcup_{k\geq m(n)} f_k(t) \subset B_{1/n}(F_n(z(t)))$ and therefore $f(t) \notin \overline{\mathrm{co}} \bigcup_{k\geq m(n)} f_k(t)$. From [19, Proposition 1.1] we obtain a contradiction. Thus $\forall n \geq N$, $\exists g_n \in F_n(z(t))$ such that $\|g_n - f(t)\| \leq 1/n$. Hence $g_n \to f(t)$ in H and from $F_{n+1}(z(t)) \subset F_n(z(t))$ it follows that $f(t) \in F_n(z(t))$, $\forall n \geq N$. Thus $f(t) \in F(z(t))$, a.e. on $(0,T)$, and $u_n = u_n(t) \to z(t) = u_1 \in G_{\lambda_0}(t, \Xi_1)$. □

Remark 3.2. Theorem 3.1 holds for inclusion (2.4) if we assume that $D_2 = 0$ in condition (H2). (G2*) and (G4*) will be satisfied with $C = D_1(\mu(\Omega))^{1/2}$ and $\gamma = \lambda_1$.

4. Dependence on a parameter

Now we are interested in the continuous dependence on a parameter. Consider the sequence of problems (2.1) with right-hand sides F_n satisfying:

(R1) $F_n : H \to C_v(H)$; $\forall u \in H, \forall n \geq 1$;

(R2) $F_{n+1}(u) \subset F_n(u)$, $\forall n \geq 1$;

(R3) $\exists D_1, D_2 \geq 0$ such that $\sup_{v\in F_1(u)} \|v\| \leq D_1 + D_2\|u\|, \forall u \in H$;
(R4) F_n are w-upper semicontinuous $\forall n \geq 1$;
(R5) $\forall u \in H$, $\bigcap_{n=1}^{\infty} F_n(u) \neq \emptyset$ and $F(u) = \bigcap_{n=1}^{\infty} F_n(u)$ is w-upper semicontinuous;
(R6) $\exists \delta > 0$, $M > 0$ such that $\forall u \in D(\partial\phi)$, $\|u\| > M$, $\forall n \geq 1$, $\forall y \in -\partial\phi(u) + F_n(u)$,

$$(y, u) \leq -\delta. \tag{4.1}$$

As before we assume that (G5) holds. Since $F(u) \subset F_n(u)$, $F_{n+1}(u) \subset F_n(u)$, $\forall u \in H$, $\forall n \geq 1$, conditions (G1)–(G4) hold for all F_n, F (with the same constants D_1, D_2). Let G_n, G be the semiflows corresponding to F_n, F. Then in view of Theorem 2.3 there exist the global compact attractors Ξ_n, Ξ corresponding to G_n, G, respectively.

THEOREM 4.1. *Let (R1)–(R6) and (G5) hold. Then* $\text{dist}_H(\Xi_n, \Xi) \to 0$, *as* $n \to \infty$.

Proof. As in Theorem 1.2, $\Xi \subset \cdots \subset \Xi_{n+1} \subset \Xi_n \subset \cdots \subset \Xi_1$, $\forall n \geq 1$, and the desired result will be obtained if we show that for any sequence $u_n \in G_n(t, \Xi_1)$ there exists $u_1 \in G(t, \Xi_1)$ such that $u_n \to u_1$ in H (with accuracy to a subsequence). From the proof of Theorem 2.3 it follows that $G(t, B_N) \subset B_N$, $G_n(t, B_N) \subset B_N$, $\forall n \geq 1$, $\forall N > M$. Let $u_n = u_n(t)$, $u_n(\cdot) = I(\eta_n) f_n(\cdot)$, $u_n(0) = \eta_n \in \Xi_1$, $\|\eta_n\| \leq N$, $\forall n \geq 1$, where $N > M$. Then $\max_{t\in[0,T]} \|u_n(t)\| \leq N$. Hence, $\|f_n(t)\| \leq D_1 + D_2 N$, a.e. on $(0, T)$, and we can use the same arguments as in the final part of the proof of Theorem 1.2. □

Remark 4.2. We note that conditions (R1)–(R5) do not imply that $\text{dist}(F_n(u), F(u)) \to 0$, as $n \to \infty$.

Proof. Consider the space $H = l_2 = \{y = (y_1, y_2, \ldots) \mid \sum_{i=1}^{\infty} |y_i|^2 < \infty\}$ and the sequence of constant maps $F_n(u) \equiv Y_n = \{y \in l_2 \mid y_1 = \cdots = y_n = 0, \|y\| \leq 1\}$, $n \geq 1$. The sets Y_n are nonempty, bounded, closed and convex and $F(u) = \cap_{n=1}^{\infty} F_n(u) = \{0\}$. It is obvious that the maps F_n, F are w-upper semicontinuous and satisfy (R2)–(R3) (with $D_1 = 1, D_2 = 0$). We take $\xi_n = (\underbrace{0, \ldots, 0}^{n \text{ times}}, 1, 0, \ldots) \in F_n$. Since $\|\xi_n - 0\| = 1$, we have $\text{dist}(F_n(u), F(u)) \geq 1$, $\forall n \geq 1$. □

Consider the sequence of problems

$$\begin{gathered} \frac{\partial y}{\partial t} \in \triangle y + f_n(y) + h, \quad \Omega \times (0, T), \\ y|_{\partial\Omega} = 0, \\ y(x, 0) = y_0(x), \quad x \in \Omega, \end{gathered} \tag{4.2}$$

where $h \in L_2(\Omega)$, $\Omega \subset \mathbb{R}^n$ is a bounded open domain with smooth boundary $\partial\Omega$ and $f_n : \mathbb{R} \to 2^{\mathbb{R}}$ satisfy:
(L1) $f_n : \mathbb{R} \to C_v(\mathbb{R})$, $f_{n+1}(t) \subset f_n(t)$, $\forall t \in \mathbb{R}$, $\forall n \geq 1$;
(L2) $\exists D_1, D_2 \geq 0$ such that $\sup_{y\in f_1(s)} |y| \leq D_1 + D_2|s|$, $\forall s \in \mathbb{R}$;

(L3) f_n are w-upper semicontinuous $\forall n \geq 1$;
(L4) $\exists M \geq 0$, $\alpha > 0$ such that $\forall s \in \mathbb{R}$, $\forall n \geq 1$, $\forall y \in f_n(s)$, $ys \leq (\lambda_1 - \alpha)|s|^2 + M$.
Define $F_n, F : H \to 2^H$, $H = L_2(\Omega)$,

$$\begin{aligned} F_n(y) &= \{\xi + h \mid \xi \in H,\ \xi(x) \in f_n(y(x)) \text{ a.e. } x \in \Omega\}, \\ F(y) &= \{\xi + h \mid \xi \in H,\ \xi(x) \in \cap_{n=1}^{\infty} f_n(y(x)) \text{ a.e. } x \in \Omega\}. \end{aligned} \tag{4.3}$$

Proposition 4.3. *The maps F, F_n satisfy (R1)–(R6).*

Proof. Condition (L4) in a standard way [14, Theorem 10] provides that (R6) holds. It follows from (L1)–(L4) and Proposition 2.5 that the maps F_n satisfy (R1)–(R4).

(L1)–(L3) imply that all f_n are upper semicontinuous (because they are compact-valued) and for any $t \in \mathbb{R}$, $a > 0$, map the ball $B_a(t)$ into subsets of some compact set in $\mathbb{R}$. As $\{f_n(t)\}$ is a centered family of compacts, so $\bigcap_{n=1}^{\infty} f_n(t) \neq \emptyset$ and in view of [12, page 60] $f(\cdot) = \bigcap_{n=1}^{\infty} f_n(\cdot)$ is upper semicontinuous at t. It follows now from (L1)–(L4) that f satisfies (H1)–(H4). Then using again Proposition 2.5 we obtain that (R5) holds. □

Let Ξ_n, Ξ be the global attractors corresponding to f_n, f, respectively. As a consequence of Theorem 4.1 we have the following theorem.

Theorem 4.4. *Let (L1)–(L4) hold. Then* $\operatorname{dist}_H(\Xi_n, \Xi) \to 0$, *as* $n \to \infty$.

Example 4.5. A model of regional economic growth.

Consider in (2.6) a sequence of functions θ_n such that $\theta_{n+1}(s) \leq \theta_n(s)$, $\forall n \geq 1, \forall s \in \mathbb{R}$, and θ_1 satisfies (2.8). Then (L1)–(L4) hold and Theorem 4.4 takes place.

5. Perturbed differential inclusions

We are now interested in the upper semicontinuity of the global attractor for inclusion (2.1) under small perturbations. Consider the family of differential inclusions

$$\begin{gathered} \frac{du}{dt} \in -\partial\varphi(u) + F(u) + \epsilon S(u), \\ u(0) = u_0, \end{gathered} \tag{5.1}$$

where $\epsilon \geq 0$ is a small parameter and $S, F : H \to 2^H$ are multivalued maps satisfying (G1)–(G3) and

(G4**) there exist $\epsilon_0 > 0$, $\delta > 0$, $M > 0$ such that $\forall \epsilon \leq \epsilon_0$, $\forall u \in D(\partial\varphi)$, $\|u\| > M$, $\forall y \in -\partial\varphi(u) + F(u) + \epsilon S(u)$,

$$(y, u) \leq -\delta. \tag{5.2}$$

Lemma 5.1. *The maps $S_\epsilon(u) = F(u) + \epsilon S(u)$ are w-upper semicontinuous.*

Proof. Let $\eta > 0$ be arbitrary and $\gamma > 0$ be such that $\gamma + \epsilon\gamma \leq \eta$. In view of the w-upper semicontinuous of F, S there exists $\delta > 0$ such that if $\|u - u_0\| \leq \delta$ then

$$\operatorname{dist}\big(S_\epsilon(u), S_\epsilon\big(u_0\big)\big) \leq \operatorname{dist}\big(F(u), F\big(u_0\big)\big) + \epsilon \operatorname{dist}\big(S(u), S\big(u_0\big)\big) \leq \eta. \tag{5.3}$$

□

On the other hand, it is evident that S_ϵ satisfy (G1) and (G2) with $D_{1\epsilon} = \epsilon D_1^S + D_1^F$, $D_{2\epsilon} = \epsilon D_2^S + D_2^F$, where D_i^S, D_i^F are the constants in condition (G2) corresponding to S and F, respectively. If condition (G5) is also satisfied then in view of Theorem 2.3 for each $\epsilon \leq \epsilon_0$ inclusion (5.1) generates the multivalued semiflow $G_\epsilon : \mathbb{R}_+ \times \overline{D(\varphi)} \to \operatorname{Comp}(\overline{D(\varphi)})$ which has the global compact invariant attractor Ξ_ϵ.

Define the set-valued map $R(u) = \cup_{0 \leq \epsilon \leq \epsilon_0} \epsilon S(u)$.

Lemma 5.2. *The map R satisfies (G1)–(G3) and (G4**) replacing ϵS by R.*

Proof. It is clear that the set $R(u)$ is nonempty and bounded. Let $y_n \in R(u)$, $y_n \xrightarrow{n\to\infty} y$. Then $y_n = \epsilon_n z_n$, $z_n \in S(u)$. If there exists a subsequence $\epsilon_{n'} \to 0$ then $y = 0 \in R(u)$. In another case there exists n_0 such that $\epsilon_n \in [\delta, \epsilon_0]$, $\forall n \geq n_0$, for some $\delta > 0$. Take a converging subsequence $\epsilon_{n'} \to \epsilon_1 \in [\delta, \epsilon_0]$. It follows that $z_{n'} = y_{n'}/\epsilon_{n'} \to y/\epsilon_1 = z \in S(u)$, since $S(u)$ is closed. Hence, $y = \epsilon_1 z \in R(u)$, so that $R(u)$ is closed. Further, let $\epsilon y, \epsilon_1 z \in R(u)$ be arbitrary. Suppose that $\epsilon \leq \epsilon_1$. Then for any $\alpha \in [0,1]$,

$$\alpha\epsilon y + (1-\alpha)\epsilon_1 z = \epsilon_2\big(\alpha' y + (1-\alpha')z\big) = \epsilon_2 v, \tag{5.4}$$

where $\epsilon_2 = \alpha\epsilon + (1-\alpha)\epsilon_1$, $\alpha' = \alpha(\epsilon/\epsilon_2) \in [0,1]$. Since $S(u)$ is convex, $v \in S(u)$ and then $R(u)$ is convex. Therefore, $R(u) \in C_v(H)$ and (G1) holds.

Let us check (G3). Let u be arbitrary. Since S is w-upper semicontinuous, for any $\gamma > 0$ there exists $\delta > 0$ such that if $\|u - v\| \leq \delta$, then $S(v) \subset O_\gamma(S(u))$. Let $\epsilon y \in R(v)$ be arbitrary. We take $h \in S(u)$ such that $\operatorname{dist}(y, R(u)) = \|y - h\|$. Then

$$\operatorname{dist}\big(\epsilon y, R(u)\big) \leq \|\epsilon y - \epsilon h\| \leq \epsilon_0 \gamma. \tag{5.5}$$

It follows that $\operatorname{dist}(R(v), R(u)) \leq \epsilon_0\gamma$, if $\|u - v\| \leq \delta$, so that R is w-upper semicontinuous.

Finally, it is evident that R satisfies (G2) with $D_1^R = \epsilon_0 D_1^S$, $D_2^R = \epsilon_0 D_2^S$, and also that (G4**) holds. □

Theorem 5.3. *Let the maps F, S satisfy (G1)–(G3), (G4**) and (G5) hold. Then* $\operatorname{dist}(\Xi_\epsilon, \Xi_0) \to 0$, *as* $\epsilon \to 0^+$.

Proof. From Theorem 1.2 it follows that it is sufficient to check that $\cup_{\epsilon \leq \epsilon_0} \Xi_\epsilon \in \beta(\overline{D(\varphi)})$ and that the map $\epsilon \mapsto G_\epsilon(t, \overline{\cup_{\epsilon \leq \epsilon_0} \Xi_\epsilon})$ is w-upper semicontinuous at $\epsilon = 0$ for any $t \geq 0$.

First, we note that for any $\epsilon \leq \epsilon_0$, Ξ_ϵ belongs to the ball $B^\alpha = \{u \in H \mid \|u\| \leq M + \alpha\}$, where $\alpha > 0$. To prove this fact we shall use that for any $\gamma > 0$ and $u \in \overline{D(\varphi)}$ there exists $T(u, \epsilon)$ such that $G_\epsilon(T, u) \in B^\gamma$ and also that $G_\epsilon(t, B^\gamma) \subset B^\gamma$,

$\forall t \geq 0, \forall \epsilon \leq \epsilon_0$ (see the proof of Theorem 2.3). Let $\gamma < \alpha$. Since $G_\epsilon(T, \cdot)$ is upper semicontinuous (see again Theorem 2.3), for any $u \in \Xi_\epsilon$ we can find a neighborhood $O(u)$ such that $G_\epsilon(T, O(u)) \subset B^\alpha$. Since Ξ_ϵ is compact, from the covering $\cup_{u\in\Xi_\epsilon} O(u)$ we can obtain a finite subcovering $\cup_{i=1}^n O(u_i)$. Hence, $\Xi_\epsilon \subset G_\epsilon(t, \Xi_\epsilon) \subset B^\alpha$ (we take $t \geq \max_i\{T(u_i, \epsilon)\}$), as required. Hence, $\cup_{\epsilon\leq\epsilon_0} \Xi_\epsilon \in \beta(\overline{D(\varphi)})$.

In order to check the second property we shall prove first that the set $K_0 = \overline{\cup_{\epsilon\leq\epsilon_0} \Xi_\epsilon}$ is compact. Let G_R be the semiflow generated by inclusion (5.1) if we replace the map ϵS by R. Since $\epsilon S(u) \subset R(u)$, $\forall \epsilon \leq \epsilon_0$, it is clear that $G_\epsilon(u) \subset G_R(u)$, $\forall u \in \overline{D(\varphi)}$, $\forall \epsilon \leq \epsilon_0$. From Theorem 2.3 and Lemma 5.2 it follows that G_R has a compact global attractor Ξ_R. Obviously, Ξ_R is a globally attracting set for each G_ϵ, $\epsilon \leq \epsilon_0$. Hence, since Ξ_ϵ is the minimal closed set that attracts any bounded set for G_ϵ, it follows that $\Xi_\epsilon \subset \Xi_R$, $\forall \epsilon \leq \epsilon_0$. Therefore, K_0 is compact.

Suppose that the map $\epsilon \mapsto G_\epsilon(t, \overline{\cup_{\epsilon\leq\epsilon_0} \Xi_\epsilon})$ is not w-upper semicontinuous at $\epsilon = 0$ for some $t > 0$. Then there exists a γ-neighborhood O_γ of $G_0(t, K_0)$ and a sequence $u^{\epsilon_n} \in G_{\epsilon_n}(t, K_0)$, $\epsilon_n \to 0^+$, such that $u^{\epsilon_n} \notin O_\gamma$. Then $u^{\epsilon_n} = u_{\epsilon_n}(t)$, where $u_{\epsilon_n}(\cdot) = I(u^0_{\epsilon_n}) f_{\epsilon_n}(\cdot)$, $u^0_{\epsilon_n} \in K_0$, and $f_{\epsilon_n}(\tau) \in F(u_{\epsilon_n}(\tau)) + \epsilon_n S(u_{\epsilon_n}(\tau))$, a.e. $\tau \in (0, t)$. Arguing as in Theorems 3.1, 4.1 we obtain the existence of a subsequence (denoted again by ϵ_n) and functions f, u such that $f_{\epsilon_n} \to f$ in $\sigma - L_1([0, t], H)$, $u^0_{\epsilon_n} \to u_0 \in K_0$, $u_{\epsilon_n} \to u$ in $C([0, t], H)$ and $u(\cdot) = I(u_0) f(\cdot)$. We have to prove that $f(\tau) \in F(u(\tau))$, a.e. $\tau \in (0, t)$.

In view of [19, Proposition 1.1] for a.a. $\tau \in (0, t)$, $f(\tau) \in \cap_{m=1}^\infty \overline{\text{co}} \cup_{n\geq m} f_{\varepsilon_n}(\tau)$. Fix $\tau \in (0, t)$. Since F is w-upper semicontinuous and using condition (G2) for the map S, we obtain that for any $\delta > 0$ there exists $n > 0$ such that $\forall k \geq n$,

$$\begin{aligned}
&\operatorname{dist}\left(F\left(u_{\epsilon_k}(\tau)\right) + \epsilon_k S\left(u_{\epsilon_k}(\tau)\right), F\left(u(\tau)\right)\right) \\
&\quad \leq \operatorname{dist}\left(F\left(u_{\epsilon_k}(\tau)\right) + \epsilon_k S\left(u_{\epsilon_k}(\tau)\right), F\left(u_{\epsilon_k}(\tau)\right)\right) + \operatorname{dist}\left(F\left(u_{\epsilon_k}(\tau)\right), F\left(u(\tau)\right)\right) \\
&\quad \leq \epsilon_k\left(D_1^S + D_2^S \left\|u_{\epsilon_k}(\tau)\right\|\right) + \frac{\delta}{2} \leq \delta.
\end{aligned} \tag{5.6}$$

Since $F(u(\tau))$ is convex, this implies that $\overline{\text{co}} \cup_{k\geq n} f_{\varepsilon_k}(\tau) \subset O_\delta(F(u(\tau)))$. Hence, since $F(u(\tau))$ is closed, $f(\tau) \in F(u(\tau))$, a.e. $t \in (0, t)$. Then $u^{\varepsilon_n} \to u(t) \in G_0(t, K_0)$, which is a contradiction. □

Consider now the family of boundary value problems

$$\begin{gathered}
\frac{\partial u}{\partial t} \in \Delta u + f(u) + \epsilon j(u) + h, \quad \text{on } \Omega \times (0, T), \\
u\,|_{\partial\Omega} = 0, \\
u(0) = u_0,
\end{gathered} \tag{5.7}$$

where $h \in L_2(\Omega)$, $\epsilon \geq 0$ is small, $f, j : \mathbb{R} \to 2^{\mathbb{R}}$ satisfy (H1)–(H3) and f satisfies (H4).

Define the maps $F, S : H \to 2^H$, $H = L_2(\Omega)$, by

$$\begin{aligned}
F(u) &= \left\{y \in H \mid y(x) \in f\left(u(x)\right) + h(x), \text{ a.e. on } \Omega\right\}, \\
S(u) &= \left\{y \in H \mid y(x) \in j\left(u(x)\right), \text{ a.e. on } \Omega\right\}.
\end{aligned} \tag{5.8}$$

It follows from Proposition 2.5 that the maps F,S satisfy (G1)–(G3).

LEMMA 5.4. *Condition (G4**) holds.*

Proof. Since f satisfies (H4) and (G2) holds for S, we have that $\forall u \in D(\partial\varphi)$, $\forall y \in -\partial\varphi(u)+F(u)+\epsilon S(u)$,

$$\begin{aligned}(y,u) &\le -\lambda_1\|u\|^2+\left(\lambda_1-\alpha\right)\|u\|^2+M\mu(\Omega)+\epsilon\left(D_1+D_2\|u\|\right)\|u\|+\|u\|\|h\| \\ &\le \left(-\frac{\alpha}{2}+\epsilon D_2\right)\|u\|^2+M\mu(\Omega)+\frac{\epsilon^2 D_1^2}{\alpha}+\frac{1}{\alpha}\|h\|^2.\end{aligned} \tag{5.9}$$

Taking $\epsilon_0=\alpha/4D_2$ the last inequality implies that condition (G4**) holds. □

Since (G5) is also satisfied, we have obtained a particular case of inclusion (5.1), so that Theorem 5.3 implies the following result.

THEOREM 5.5. *Let f,j satisfy (H1)–(H3) and f satisfy (H4). Then* $\operatorname{dist}(\Xi_\epsilon,\Xi_0)\to 0$, *as* $\epsilon\to 0^+$.

Example 5.6. A model of regional economic growth.

Consider in (2.6) the family of functions $g_\epsilon=g_1+\epsilon g_2$, $\theta_\epsilon=\theta_1+\epsilon\theta_2$, where g_1,θ_1 satisfy the same conditions as g,θ and g_2,θ_2 are continuous and have at most linear growth. Define the multivalued maps $f,j:\mathbb{R}\to 2^{\mathbb{R}}$,

$$\begin{aligned}f(s)&=\left\{\omega(s)+g_1(s)+\xi \mid 0\le\xi\le\theta_1(s)\right\},\\ j(s)&=\left\{g_2(s)+\xi \mid 0\le\xi\le\theta_2(s)\right\}.\end{aligned} \tag{5.10}$$

Then we obtain a particular case of inclusion (5.7), so that Theorem 5.5 holds.

Finally, we remark that if in problems (2.4), (4.2), and (5.7) we replace the operator $-\Delta$ by $A(u)=-\sum_{i=1}^n(\partial/\partial x_i)(|\partial y/\partial x_i|^{p-2}(\partial y/\partial x_i))$, $p>2$, then all the results remain valid. In this case, conditions (H4), (L4) are not necessary. Indeed, we prove that (G4**) holds ((G4) and (R6) can be proved in a similar way). It follows from Poincaré inequality that $\langle Au,u\rangle=\|\nabla u\|_{L_p}^p\ge D\|u\|_{L_p}^p$ for some $D>0$. Let $\epsilon_0>0$ be arbitrary but fixed. Then using the Young inequality we have that $\forall u\in D(\partial\varphi)$, $\forall\epsilon\le\epsilon_0$, $\forall y\in -A(u)+F(u)+\epsilon S(u)$,

$$\begin{aligned}(y,u)&\le -D\|u\|_{L_p}^p+\epsilon\left(D_1+D_2\|u\|\right)\|u\|+\left(D_3+D_4\|u\|\right)\|u\|+\|u\|\|h\|\\ &\le -\widetilde{D}\|u\|^p+K,\end{aligned} \tag{5.11}$$

where $\widetilde{D}>0$, so that (G4**) holds.

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