TIME AVERAGING FOR RANDOM NONLINEAR ABSTRACT PARABOLIC EQUATIONS

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On the basis of G-convergence we prove an averaging result for nonlinear abstract parabolic equations, the operator coefficient of which is a stationary stochastic process.

1. Introduction

It is well known that the averaging principle is a powerful tool of investigation of ordinary differential equations, containing high frequency time oscillations, and a vast work was done in this direction (cf. [1]). This principle was extended to many other problems, like ordinary differential equations in Banach spaces, delayed differential equations, and so forth (for the simplest result of such kind we refer to [2]). It seems to be very natural to apply such an approach to the case of parabolic equations, either partial differential, or abstract ones. However, only a few papers deal with such equations. Most of them deal with linear and quasilinear equations in the case when high oscillations in coefficients and/or forcing term are of periodic or almost periodic nature [4, 6, 9, 8, 13, 14, 18]. Moreover, many applications give rise naturally to parabolic equations with highly oscillating random coefficients. For linear equations of such kind the averaging principle was studied in [15, 16, 17]. Note that, in [17] the so-called spatial and space-time averaging (homogenization) is investigated, while the time averaging is also considered.

In the present paper, we study the averaging problem for an abstract monotone parabolic equation, the operator coefficient of which is a stationary (operator valued) stochastic process. We prove that in this case the averaging takes place almost surely, that is, with probability 1. As a consequence, we get an averaging result for the case of almost periodic coefficients (almost periodic functions may be regarded as a particular case of a stationary process). This result is, so to speak, individual, in contrast to the main theorem which is statistical in its nature. Our approach differs from those used in the references we pointed out above, except [17], and is based on the theory of G-convergence of abstract parabolic operators. The last theory was developed in [7] in

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connection with homogenization of nonlinear parabolic equations (see [11] for detailed presentation). Note that, in [7, 11] a simple result on time averaging in the periodic case is obtained as well.

We point out that in this paper we make use of a characterization of stationary processes from the point of view of dynamical systems, which is equivalent to the standard definition [5], but seems to be more analytical.

The paper is organized as follows. Section 2 is devoted to the precise statement of the problem and the formulation of the main result. In Section 3, we present some preliminaries on G-convergence of abstract parabolic operators. Most of them are borrowed from [11]. The proof of the main result is contained in Section 4. In Section 5, we prove an averaging result for almost periodic parabolic equations. In Sections 6 and 7, we present a simple example and discuss some immediate extensions of our results, respectively.

2. Statement of the problem and the main result

Let Ω be a probability space, with a probability measure P. Assume that on Ω it is given an action of a measure preserving dynamical system $T(t)$, that is, for each $t \in \mathbb{R}$ a self-map $T(t) : \Omega \to \Omega$ is defined such that

(1) $T(t_1+t_2) = T(t_1)T(t_2)$ $(t_1, t_2 \in \mathbb{R})$ and $T(0) = I$, where I is the identity map,
(2) the map $\Omega \times \mathbb{R} \to \Omega$, defined by $(\omega, t) \mapsto T(t)\omega$ is measurable,
(3) $P(T(t)\mathcal{U}) = P(\mathcal{U})$ $(t \in \mathbb{R})$ for every measurable set $\mathcal{U} \subset \Omega$.

In addition, we always assume the dynamical system $T(t)$ to be ergodic. Recall that $T(t)$ is called *ergodic* if for each measurable function $f(\omega)$ on Ω such that $f(T(t)\omega) = f(\omega)$ almost everywhere (a.e.) one has $f(\omega) = \text{const.}$ a.e. In what follows we use standard notations for the Lebesgue spaces, as well as for the space of continuous functions. Moreover, $\langle f \rangle$ stands for a *mean value* of measurable function f on Ω:

$$\langle f \rangle = \int_\Omega f(\omega)\, dP(\omega). \tag{2.1}$$

Let V be a separable reflexive Banach space over the field $\mathbb{R}$ of reals, and let V^* be its dual space and H a Hilbert space identified with its dual, $H^* = H$. It is assumed that $V \subset H \subset V^*$ and all the embeddings here are dense and compact. We denote by $\|\cdot\|, |\cdot|$, and $\|\cdot\|_*$ the norms in V, H, and V^*, respectively, and $(\cdot,\cdot)$ stands for the inner product in H and the canonical bilinear form on $V^* \times V$ (the duality pairing).

Let $p > 1$ and $1/p + 1/p' = 1$. We fix nonnegative constants m, m_1, and m_2, positive constants c_1, c_2, c_3, and c_4, and reals α, β such that

$$0 < \alpha \le \min\left\{\frac{p}{2}, p-1\right\}, \qquad \beta \ge \max\{p, 2\}. \tag{2.2}$$

Consider a family $A(\omega) : V \to V^*$ $(\omega \in \Omega)$ of operators satisfying the *Carathéodory condition*

(C) *for almost all $\omega \in \Omega$ the operator $A(\omega) : V \to V^*$ is continuous, while $A(\omega)u$ is a measurable V^*-valued function for every $u \in V$,* and the following inequalities

$$\|A(\omega)u\|_*^{p'} \le m_1 + c_1\big(A(\omega)u, u\big), \tag{2.3}$$

$$\big(A(\omega)u, u\big) \ge c_2\|u\|^p - m_2, \tag{2.4}$$

$$\big\|A(\omega)u_1 - A(\omega)u_2\big\|_* \le c_3\Phi^{(p-1-\alpha)/p}\big(A(\omega)u_1 - A(\omega)u_2, u_1 - u_2\big)^{\alpha/\beta}, \tag{2.5}$$

$$\big(A(\omega)u_1 - A(\omega)u_2, u_1 - u_2\big) \ge c_4\Phi^{(p-\beta)/p}\big\|u_1 - u_2\big\|^\beta, \tag{2.6}$$

for every $u, u_1, u_2 \in V$ and almost all (a.a.) $\omega \in \Omega$, where

$$\Phi = \Phi\big(u_1, u_2\big) = m + \big(A(\omega)u_1, u_1\big) + \big(A(\omega)u_2, u_2\big). \tag{2.7}$$

It is always assumed that $m \ge 2m_2$ which implies $\Phi(u_1, u_2) > 0$ provided $\|u_1\| + \|u_2\| > 0$.

Now, we introduce a family $A_\omega(t)$ ($\omega \in \Omega$) of operator valued functions defined by

$$A_\omega(t) = A\big(T(t)\omega\big), \quad t \in \mathbb{R}. \tag{2.8}$$

It is not difficult to verify (cf. [11]) that for a.a. $\omega \in \Omega$ the operator function $A_\omega(t)$ is well defined, and satisfies the Carathéodory condition (on the real line now) and inequalities (3.1), (3.2), (3.3), and (3.4) below which are similar to (2.3), (2.4), (2.5), and (2.6). In particular, the operator $A_\omega(t)$ is bounded, coercive, and strictly monotone uniformly with respect to ω and t. Therefore, due to standard results on abstract monotone parabolic equations (cf. [10]), for a.a. $\omega \in \Omega$ the following Cauchy problem:

$$u' + A_\omega\left(\frac{t}{\varepsilon}\right)u = f \in L^{p'}\big(0, \tau; V^*\big), \tag{2.9}$$

$$u(0) = u_0 \in H \tag{2.10}$$

has a unique solution

$$u = u_{\omega,\varepsilon} \in L^p(0, \tau; V) \cap C\big([0, \tau]; H\big) \tag{2.11}$$

such that $u' = u'_{\omega,\varepsilon} \in L^{p'}(0, \tau; V^*)$. Here $\tau > 0$ is an arbitrary, but fixed, real number. We remark that at this point the whole set of assumptions (2.3), (2.4), (2.5), and (2.6) is not needed. We use them only to apply the results on G-convergence [11].

Let $\hat{A} : V \to V^*$ be an operator defined by

$$\hat{A}u = \langle Au\rangle = \int_\Omega A(\omega)u\,dP(\omega), \tag{2.12}$$

the mean value of $A(\omega)u$. It is easily seen that $\hat{A}$ acts continuously from V into V^* and satisfies inequalities (3.1), (3.2), (3.3), and (3.4). By the Birkhoff ergodic theorem (cf. [3]), for a.a. $\omega \in \Omega$ one has

$$\hat{A}u = \lim_{s\to\infty}\frac{1}{s}\int_0^s A\big(T(t)\omega\big)u\,dt = \lim_{s\to\infty}\frac{1}{2s}\int_{-s}^s A\big(T(t)\omega\big)u\,dt. \tag{2.13}$$

The following result justifies in the case we consider the principle of averaging.

THEOREM 2.1. *For a.a.* $\omega \in \Omega$, $u_{\omega,\varepsilon} \to \hat{u}$ *weakly in* $L^p(0,\tau;V)$, *strongly in* $C([0,\tau];H)$, *and* $u'_{\omega,\varepsilon} \to \hat{u}'$ *weakly in* $L^{p'}(0,\tau;V^*)$ *as* $\varepsilon \to 0$, *where* $\hat{u}$ *is the unique solution of the problem*

$$\hat{u}' + \hat{A}\hat{u} = f, \tag{2.14}$$

$$\hat{u}(0) = u_0. \tag{2.15}$$

3. G-convergence of abstract parabolic operators

To prove Theorem 2.1, we need certain preliminary results on G-convergence (we refer to [11] for more details). First, we recall some definitions.

Let $A_k(t)$, $t \in [0,\tau]$, $(k = 0,1,\ldots)$ be operators acting from V into V^*. Assume that they satisfy the Carathéodory condition on $[0,\tau]$ and inequalities

$$\|A_k(t)u\|_*^{p'} \le m_1 + c_1\big(A_k(t)u,u\big), \tag{3.1}$$

$$\big(A_k(t)u,u\big) \ge c_2\|u\|^p - m_2, \tag{3.2}$$

$$\big\|A_k(t)u_1 - A_k(t)u_2\big\|_* \le c_3\Phi_k^{(p-1-\alpha)/p}\big(A_k(t)u_1 - A_k(t)u_2, u_1-u_2\big)^{\alpha/\beta}, \tag{3.3}$$

$$\big(A_k(t)u_1 - A_k(t)u_2, u_1-u_2\big) \ge c_4\Phi_k^{(p-\beta)/p}\big\|u_1-u_2\big\|^{\beta}, \tag{3.4}$$

for all $u,u_1,u_2 \in V$ and a.a. $t \in [0,\tau]$, where

$$\Phi_k = m + \big(A_k(t)u_1,u_1\big) + \big(A_k(t)u_2,u_2\big). \tag{3.5}$$

Consider parabolic operators

$$L_k u = u' + A_k(t)u, \quad (k = 0,1,\ldots), \tag{3.6}$$

acting from the space

$$W_0 = \big\{u \in L^p\big(0,\tau;V^*\big) \mid u' \in L^{p'}\big(0,\tau;V^*\big),\ u(0) = 0\big\} \tag{3.7}$$

into $L^{p'}(0,\tau;V^*)$. Endowed with the graph norm

$$\|u\|_{W_0} = \|u\|_{L^p(0,\tau;V)} + \big\|u'\big\|_{L^{p'}(0,\tau;V^*)}, \tag{3.8}$$

W_0 becomes a reflexive Banach space. As it was already mentioned, due to our assumptions the operators L_k are invertible. One says that L_0 is a *G-limit* of L_k, $k = 0,1,\ldots$, (in symbols, $L_k \xrightarrow{G} L_0$) if $L_k^{-1}f \to L_0^{-1}f$ weakly in W_0 for all $f \in L^{p'}(0,\tau;V^*)$.

We have the following results [11].

THEOREM 3.1. *Let* L_k $(k = 0,1,\ldots)$ *be a sequence of parabolic operators satisfying (3.1), (3.2), (3.3), and (3.4). Then there exists a subsequence* $L_{k'}$ *and a parabolic operator* L *satisfying (3.1), (3.2), (3.3), and (3.4), with possibly different values of* $m, m_1, m_2, c_1, c_2, c_3$, *and* c_4, *such that* $L_k \xrightarrow{G} L$.

We now point out that, in fact, our parabolic operators act on a larger space consisting of all functions from $L^p(0,\tau;V)$ which have first derivative in $L^{p'}(0,\tau;V^*)$. Such functions are not necessarily vanishing at 0.

THEOREM 3.2. *Let $L_k \xrightarrow{G} L$, $u_k \in L^p(0,\tau;V)$ with $u_k' \in L^{p'}(0,\tau;V^*)$. Assume that $L_k u_k \to f$ strongly in $L^{p'}(0,\tau;V^*)$, $u_k \to u$ weakly in $L^{p'}(0,\tau;V)$, and $u_k' \to u'$ weakly in $L^{p'}(0,\tau;V^*)$. Then $Lu = f$ and $A_k(t)u_k \to A(t)u$ weakly in $L^{p'}(0,\tau;V^*)$.*

PROPOSITION 3.3. *Assume that $L_k \xrightarrow{G} L$, $f \in L^{p'}(0,\tau;V^*)$, and $u_0 \in H$. Let $u_k \in L^p(0,\tau;V)$ be a (unique) solution of the Cauchy problem*

$$L_k u_k = u_k' + A_k(t)u_k = f, \qquad u_k(0) = u_0, \tag{3.9}$$

such that $u_k' \in L^{p'}(0,\tau;V^*)$. Then $u_k \to u$ weakly in $L^p(0,\tau;V)$ and strongly in $C([0,\tau];H)$, $u_k' \to u'$ weakly in $L^{p'}(0,\tau;V^*)$, where u is a (unique) solution of the Cauchy problem for L with the same initial data u_0.

Proof. Multiplying (3.9) by u_k and integrating, we obtain

$$\frac{1}{2}|u_k(t)|^2 - \frac{1}{2}|u_0|^2 + \int_0^t \big(A_k(s)u_k(s), u_k(s)\big)\,ds = \int_0^t \big(f(s), u_k(s)\big)\,ds. \tag{3.10}$$

Now due to assumption (3.2), we see that u_k is a bounded sequence in $L^p(0,\tau;V)$ and $C([0,\tau];H)$. Using (3.1) and (3.9), we obtain from the last observation the boundedness of u_k' in $L^{p'}(0,\tau;V^*)$. Since $L^p(0,\tau;V)$ and $L^{p'}(0,\tau;V^*)$ are reflexive spaces, passing to a subsequence, we can assume that $u_k \to u$ weakly in $L^p(0,\tau;V)$ and $u_k' \to u'$ weakly in $L^{p'}(0,\tau;V^*)$. In addition, due to Lemma 1.3.4 of [11], we can also assume that $u_k \to u$ strongly in $C([0,\tau];H)$. (In fact, this lemma is stated in [11] only under a stronger assumption $u_0 = 0$. However, the proof works equally well if we assume only that $u_k(0) = u_0 \in H$.) By Theorem 3.1, u is a solution of $Lu = f$, while $u(0) = u_0$ due to convergence of u_k in $C([0,\tau];H)$. Since such a solution u is unique, the passage to a subsequence above is unnecessary and the proof is complete. □

In Section 5, we also use the following result (see [11, Corollary 1.3.1]).

PROPOSITION 3.4. *Let*

$$L_k^n u = u' + A_k^n(t)u \quad (n,k = 0,1,2,\ldots) \tag{3.11}$$

be a double sequence of parabolic operators. Assume that $L_k^n \xrightarrow{G} L_0^n$ as $k \to \infty$ for all $n = 1,2,\ldots$, and

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{t\in[0,\tau]} \sup_{u\in V} \frac{\left\| A_k^n(t)u - A_k^o(t)u \right\|_*}{1 + \|u\|^{p-1}} = 0 \tag{3.12}$$

uniformly with respect to $k = 0,1,2,\ldots$. Then $L_k^0 \xrightarrow{G} L_0^0$.

4. Proof of Theorem 2.1

Consider parabolic operators $L_{\omega,\varepsilon}$ and $\hat{L}$ generated by the left-hand sides of (2.9) and (2.14), respectively. First, we point out that for a.a. $\omega \in \Omega$ the operators $L_{\omega,\varepsilon}$ satisfy all the assumptions of Section 3.

THEOREM 4.1. *For each $\tau > 0$ and for a.a. $\omega \in \Omega$, we have*

$$L_{\omega,\varepsilon} \xrightarrow{G} \hat{L} \quad \text{as } \varepsilon \longrightarrow 0. \tag{4.1}$$

Theorem 4.1 together with Proposition 3.3 imply obviously Theorem 2.1. To prove Theorem 4.1 we need to introduce an operator of "differentiation" along trajectories of our dynamical system $T(t)$ (see [11, Section 3.1], for more details). Associated to $T(t)$, there exists a one-parameter groups of operators $G(t)$ acting in all the spaces $L^r(\Omega, E)$, where $E = V, H$ or V^*, $1 \le r \le \infty$.

The operator $G(t)$ is defined by

$$\big(G(t)f\big)(\omega) = f\big(T(t)\omega\big), \quad t \in \mathbb{R},\ \omega \in \Omega. \tag{4.2}$$

It is easily seen that $G(t)$ is an isometric operator in each space under consideration. Moreover,

$$G^*(t) = G(-t), \quad t \in \mathbb{R}. \tag{4.3}$$

Now $G(t)$ is considered as an operator in $L^r(\Omega; E)$ $(1 < r < \infty)$, hence, $G^*(t)$ acts in $L^{r'}(\Omega; E^*)$. In particular, $G(t)$ is a group of unitary operator in $L^2(\Omega; H)$.

The group $G(t)$ is strongly continuous in $L^r(\Omega; E)$, with $1 \le r < \infty$. The generator ∂ of this group is a closed linear operator in $L^r(\Omega; E)$. Due to (4.3), ∂ is skew-symmetric:

$$\big\langle(\partial f, g)\big\rangle = -\big\langle(f, \partial g)\big\rangle, \quad \forall f \in D\big(\partial, L^r(\Omega; E)\big), \forall g \in D\big(\partial, L^{r'}\big(\Omega; E^*\big)\big), \tag{4.4}$$

where $D(\partial, L^r(\Omega; E))$ is the domain of ∂ in $L^r(\Omega; E)$, $1 < r < \infty$.

However, for our purpose we need to consider ∂ as an (unbounded) operator from $L^p(\Omega; V)$ into $L^{p'}(\Omega; V^*)$. Denote by $\mathcal{W}(\Omega)$ the completion of

$$D\big(\partial; L^p(\Omega; V)\big) \cap D\big(\partial; L^{p'}\big(\Omega; V^*\big)\big) \tag{4.5}$$

with respect to the norm

$$\|f\|_{\mathcal{W}} = \|f\|_{L^p(\Omega;V)} + \|\partial f\|_{L^{p'}(\Omega;V^*)}. \tag{4.6}$$

This is a reflexive Banach space densely embedded into $L^p(\Omega; V)$. Now the action of ∂ can be extended to $\mathcal{W}(\Omega)$ and we get the desired operator from $L^p(\Omega; V)$ into $L^{p'}(\Omega; V^*)$, with the domain $\mathcal{W}(\Omega)$. Making use of the same smoothing arguments in [11, Section 3.1], we see that this operator, still denoted by ∂, is skew-symmetric: $\partial^* = -\partial$. Moreover, if $f \in \mathcal{W}(\Omega)$, then, for a.a. $\omega \in \Omega$, $f(T(t)\omega) \in L^p_{\text{loc}}(\mathbb{R}; V)$. For its distributional derivative we have

$$\big[f\big(T(t)\omega\big)\big]' = (\partial f)\big(T(t)\omega\big) \in L^{p'}_{\text{loc}}\big(\mathbb{R}; V^*\big). \tag{4.7}$$

We also remark that, due to ergodicity assumption, the kernel $\ker \partial$ consists of constant functions on Ω.

Proof of Theorem 4.1. Independently of τ, for a.a. $\omega \in \Omega$ the operators $L_{\omega,\varepsilon}$ satisfy the assumptions of Theorem 3.1. Hence, for any sequence of ε's converging to 0, there exists a subsequence, still denoted by ε, and a parabolic operator

$$L_0 u = u' + A_0(t)u \tag{4.8}$$

such that $L_{\omega,\varepsilon} \xrightarrow{G} L_0$, $\omega \in \Omega_0$, where Ω_0 is a set of measure 1. To prove the theorem it suffices now to show that $A_0(t) = \hat{A}$ for a.a. $t \in [0,\tau]$. In particular, this means that the passage to a subsequence above is superfluous.

Fix $u \in V$ and consider the following identity:

$$\left(u + \varepsilon w_\delta\left(\frac{t}{\varepsilon}\right)\right)' + A_\omega\left(\frac{t}{\varepsilon}\right)\left(u + \varepsilon w_\delta\left(\frac{t}{\varepsilon}\right)\right) = \hat{A}u + \phi_{\varepsilon,\delta} + \psi_{\varepsilon,\delta}, \tag{4.9}$$

where

$$\begin{aligned} \phi_{\varepsilon,\delta} &= \varepsilon\left[w_\delta\left(\frac{t}{\varepsilon}\right)\right]' + A_\omega\left(\frac{t}{\varepsilon}\right) - \hat{A}u, \\ \psi_{\varepsilon,\delta} &= A_\omega\left(\frac{t}{\varepsilon}\right)\left(u + \varepsilon w_\delta\left(\frac{t}{\varepsilon}\right)\right) - A_\omega\left(\frac{t}{\varepsilon}\right)u. \end{aligned} \tag{4.10}$$

Now we specify the function w_δ. Since ∂ is skew self-adjoint and $\ker\partial$ is just the space of constant functions, the image of ∂ is dense in the subspace

$$\{f \in L^{p'}(\Omega; V^*) : \langle f\rangle = 0\}. \tag{4.11}$$

Therefore, for every $\delta > 0$ there exist $W_\delta \in \mathcal{W}(\Omega)$, $b^\delta, c^\delta \in L^{p'}(\Omega; V^*)$ such that

$$\begin{gathered} \langle b^\delta\rangle = \langle c^\delta\rangle = 0, \quad \hat{A}u - A(\omega)u = b^\delta(\omega) - c^\delta(\omega), \quad \partial W_\delta = b^\delta, \\ \|c^\delta\|_{L^{p'}(\Omega;V^*)} \le \delta. \end{gathered} \tag{4.12}$$

Moreover, one can assume that $\langle W_\delta\rangle = 0$.

We set

$$w_\delta(t) = W_\delta(T(t)\omega). \tag{4.13}$$

Now

$$\begin{aligned} \left\|w_\delta\left(\frac{t}{\varepsilon}\right)\right\|^p_{L^p(0,\tau;V)} &= \int_0^\tau \left\|w_\delta\left(\frac{t}{\varepsilon}\right)\right\|^p_V dt = \varepsilon\int_0^{\tau/\varepsilon} \|w_\delta(t)\|^p_V\, dt \\ &= \varepsilon\int_0^{\tau/\varepsilon} \|W_\delta(T(t)\omega)\|^p_V\, dt. \end{aligned} \tag{4.14}$$

Hence, by the Birkhoff ergodic theorem,

$$\left\|w_\delta\left(\frac{t}{\varepsilon}\right)\right\|^p_{L^p(0,\tau;V)} \longrightarrow \tau\|W_\delta\|^p_{L^p(\Omega;V)} \tag{4.15}$$

as $\varepsilon \to 0$. Therefore,

$$\left\| w_\delta\left(\frac{t}{\varepsilon}\right)\right\|^p_{L^p(0,\tau;V)} \le C\|W_\delta\|^p_{L^p(\Omega;V)}. \tag{4.16}$$

Due to (4.7),

$$\left[\varepsilon w_\delta\left(\frac{t}{\varepsilon}\right)\right]' = \left(\partial W_\delta\right)\left(T\left(\frac{t}{\varepsilon}\right)\omega\right). \tag{4.17}$$

Hence, as above

$$\left\|\left[\varepsilon w_\delta\left(\frac{t}{\varepsilon}\right)\right]'\right\|^{p'}_{L^{p'}(0,\tau;V^*)} \le C\|b^\delta\|_{L^{p'}(\Omega;V^*)}. \tag{4.18}$$

Thus, by (4.16), $\varepsilon w_\delta(t/\varepsilon) \to 0$ strongly in $L^p(0,\tau;V)$ for any fixed $\delta > 0$.

Now choose a sequence of δ's converging to 0. Then, to each such δ one can assign $\varepsilon = \varepsilon(\delta)$ such that $\varepsilon \to 0$ and $\varepsilon w_\delta(t/\varepsilon) \to 0$ strongly in $L^p(0,\tau;V)$ as $\delta \to 0$. Since, due to (4.18), $[\varepsilon w_\delta(t/\varepsilon)]'$ remains bounded in $L^{p'}(0,\tau;V^*)$ we conclude that $[\varepsilon w_\delta(t/\varepsilon)]' \to 0$ weakly in this space. At the same time, inequality (2.5) implies that $\psi_{\varepsilon,\delta} \to 0$ strongly in $L^{p'}(0,\tau;V^*)$. Finally, we have, evidently, $\phi_{\varepsilon,\delta} = c^\delta(T(t/\varepsilon)\omega)$. Using again the Birkhoff ergodic theorem, we see that $\|\phi_{\varepsilon,\delta}\|_{L^{p'}(0,\tau;V^*)} \to 0$ as $\delta \to 0$, uniformly with respect to ε.

Now, applying Theorem 3.2, we deduce from (4.9)

$$u' + A_0(t)u = \hat{A}u. \tag{4.19}$$

Since u is independent of t, we complete the proof. □

5. Almost periodic averaging

We now consider the averaging problem for the equation

$$u' + A\left(\frac{t}{\varepsilon}\right)u = f \in L^{p'}\left(0,\tau;V^*\right). \tag{5.1}$$

We assume that the operator function $A(t) : V \to V^*$ satisfies inequalities (3.1), (3.2), (3.3), and (3.4), and the function

$$\frac{A(t)v}{1+\|v\|^{p-1}}, \quad v \in V, \tag{5.2}$$

is almost periodic, in the sense of Bohr, in $t \in \mathbb{R}$ uniformly with respect to $v \in V$ [12]. More precisely, continuous operators from V into V^*, having power growth of order $p-1$, form a metric space, with the metric

$$d\left(A_1,A_2\right) = \sup_{v\in V}\frac{\left\|A_1v - A_2v\right\|}{1+\|v\|^{p-1}}. \tag{5.3}$$

Thus, we assume that $A(t)$ is an almost periodic function with values in this metric space, that is, for every sequence $t_k \to \infty$ there exist a subsequence $t_{k'}$ and an operator function $A'(t)$ such that

$$\lim_{k'\to\infty} \sup_{t\in\mathbb{R}} d\big(A(t+t_{k'}), A'(t)\big) = 0. \tag{5.4}$$

To apply Theorem 2.1, we recall the notion of Bohr compactification $\mathbb{R}_B$ of $\mathbb{R}$ [12]. There exist a compact abelian group $\mathbb{R}_B$ and a dense continuous embedding $\mathbb{R} \subset \mathbb{R}_B$ of abelian groups such that every almost periodic function on $\mathbb{R}$ is, in fact, a restriction to $\mathbb{R}$ of a continuous function on $\mathbb{R}_B$. Moreover, each continuous function on $\mathbb{R}_B$ restricted to $\mathbb{R}$ gives rise to an almost periodic function. We refer to [12] for detailed presentation of the theory of almost periodic functions from this point of view.

Now we set $\Omega = \mathbb{R}_B$ and denote by P the normalized Haar measure on $\mathbb{R}_B$. We define the dynamical system $T(t)$ by

$$T(t)\omega = \omega + t, \quad \omega \in \Omega = \mathbb{R}_B,\ t \in \mathbb{R} \subset \mathbb{R}_B. \tag{5.5}$$

Denote by $A(\omega)$ a (unique) extension of $A(t)$ to $\mathbb{R}_B$. Then (5.1) results from (2.9) after a substitution $\omega = 0$. Theorem 2.1 implies averaging for a.a. $\omega \in \mathbb{R}_B$, but not for $\omega = 0$, in general. Nevertheless, we have the following theorem.

THEOREM 5.1. *Let u_ε be a solution of Cauchy problem (5.1), (2.10), and $\hat{u}$ a solution of (2.14), (2.15), where*

$$\hat{A}v = \lim_{S\to\infty} \frac{1}{S}\int_0^S A(t)v\,dt = \lim_{S\to\infty} \frac{1}{2S}\int_{-S}^S A(t)v\,dt. \tag{5.6}$$

Then $u_\varepsilon \to \hat{u}$ weakly in $L^p(0,\tau;V)$ and strongly in $C([0,\tau];H)$, $u'_\varepsilon \to \hat{u}'$ weakly in $L^{p'}(0,\tau;V^*)$.

Proof. By Theorem 4.1, there exists a measurable set $\Omega_0 \subset \mathbb{R}_B$ of measure 1 such that $L_{\omega,\varepsilon} \xrightarrow{G} \hat{L}$ for all $\omega \in \Omega_0$. However, each set of measure 1 in $\mathbb{R}_B$ is dense. Therefore, there exists a sequence $\omega_n \in \Omega_0$ such that $\omega_n \to 0$. Moreover,

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} d\big(A_{\omega_n}(t), A(t)\big) = 0. \tag{5.7}$$

Due to Proposition 3.4, we have $L_{0,\varepsilon} \xrightarrow{G} \hat{L}$. Applying Proposition 3.3, we obtain the result. □

6. An example

Now we consider a simple example. Let $Q \subset \mathbb{R}^n$ be a bounded open set and $a(\omega,t)$ a stationary stochastic process a.a. realizations of which are contained between two positive constants. The last assumption may be expressed as follows: $a(\omega,t) = a(T(t)\omega)$, where $a(\omega) \in L^\infty(\Omega)$ and $a(\omega) \geq \alpha_0 > 0$. The equation

$$u' - \nabla\left(a\left(\omega, \frac{t}{\varepsilon}\right)|\nabla|^{p-2}\nabla u\right) = f, \tag{6.1}$$

together with the homogeneous Dirichlet condition on ∂Q, reduces to (2.9), with $V = W_0^{1,p}(Q)$ (the Sobolev space) and $H = L^2(Q)$, provided $p \geq 2$. All assumptions (2.3), (2.4), (2.5), and (2.6) are easy to verify. The averaged equation is

$$\hat{u}' - \hat{a}\nabla\big(|\nabla\hat{u}|^{p-2}\nabla\hat{u}\big) = f, \tag{6.2}$$

where $\hat{a}$ is the mean value of the process a.

7. Some generalizations

First of all, we note that in (2.9) we can consider the forcing term f of the form $f_0(t) + f_1(T(t/\varepsilon)\omega)$, where $f_0 \in L^{p'}(0,\tau;V^*)$ and $f_1 \in L^{p'}(\Omega;V^*)$. This situation reduces immediately to the case of Theorem 2.1 if we replace the operator $A(\omega)$ by a new operator $\tilde{A}(\omega) = A(\omega) - f_1(\omega)$. It is easily seen that $\tilde{A}(\omega)$ satisfies all the assumptions of Section 2 whenever $A(\omega)$ does.

Moreover, one can extend Theorem 2.1 to the case when the equation under consideration contains the slow variable t as well as the fast one t/ε, that is, is of the form

$$u' + A_\omega\left(t, \frac{t}{\varepsilon}\right)u = f, \tag{7.1}$$

where $f = f(t)$, or even $f = f_\omega(t, t/\varepsilon)$. To do this we need only to consider instead of $A(\omega)$ an operator function $A(t,\omega)$ defined on $[0,\tau] \times \Omega$ and satisfying the same assumption as in Section 2, with Ω replaced by $[0,\tau] \times \Omega$. Certainly, in this case

$$A_\omega\left(t, \frac{t}{\varepsilon}\right) = A\left(t, T\left(\frac{t}{\varepsilon}\right)\omega\right). \tag{7.2}$$

A similar remark concerns with $f = f_\omega(t, t/\varepsilon)$.

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