# ON THE STRONGLY DAMPED WAVE EQUATION AND THE HEAT EQUATION WITH MIXED BOUNDARY CONDITIONS

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We study two one-dimensional equations: the strongly damped wave equation and the heat equation, both with mixed boundary conditions. We prove the existence of global strong solutions and the existence of compact global attractors for these equations in two different spaces.

## 1. Introduction

In this paper, we study existence of strong solutions and existence of global compact attractors for the following one-dimensional problems.

The strongly damped wave equation,

$$\begin{aligned} &u_{tt} - u_{xx} - u_{txx} = g(t), \quad 0 < x < \ell,\ 0 < t < T; \\ &u(t,0) = 0, \qquad u_x(t,\ell) + u_{tx}(t,\ell) = \rho\big(u_t(t,\ell)\big), \end{aligned} \tag{1.1}$$

and the heat equation

$$z_t - z_{xx} + G(z) = h(t), \qquad z(t,0) = 0, \qquad z_x(t,\ell) = \rho\big(z(t,\ell)\big). \tag{1.2}$$

Here $\ell$ and $T$ are positive constants, $\rho : \mathbb{R} \to \mathbb{R}$ is a nonincreasing and bounded function, $g, h \in L^1(0,T; L^2(0,\ell))$, and $G$ is an operator from a subspace of $H^1$ into $L^2$. In the case where $\rho$ is not continuous, we will understand $\rho(x_0)$, at a point of discontinuity $x_0$, as being the whole interval $[\rho(x_0+0),\ \rho(x_0-0)]$. In this case $\rho$ will be a multi-valued function, and the "equal signs" in the last equations of (1.1) and (1.2) will be changed to "belong signs." So, the boundary conditions at $x = \ell$ will be written, respectively, as

$$u_x(t,\ell) + u_{tx}(t,\ell) \in \rho\big(u_t(t,\ell)\big), \qquad z_x(t,\ell) \in \rho\big(z(t,\ell)\big), \tag{1.3}$$

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or equivalently,

$$\big(u_t(t,\ell), u_x(t,\ell)+u_{tx}(t,\ell)\big) \in \Gamma, \qquad \big(z(t,\ell), z_x(t,\ell)\big) \in \Gamma, \tag{1.4}$$

where $\Gamma$ is the graph of the multi-valued function $\rho$.

The existence of global solutions for these two problems can be obtained using the theory of monotone operators. The problem (1.2) gives rise to a maximal monotone operator $A$ that is of subdifferential type, $A = \partial\varphi$, where $\varphi$ is a lower semicontinuous and convex functional. This problem was studied in [1] under some conditions on $G$, in particular the existence of strong solutions was proved.

Our goal is to obtain existence of global compact attractor. To reach this goal, first of all, we will obtain a relation between the solutions of the two problems. With this relation we can use one problem to get the properties of the other, in particular this relation will be used to prove the existence of strong solutions for the problem (1.1). Once we have existence of solutions, we will start working in order to get the existence of the attractors. For our purpose, we will study the problem (1.2) in two different spaces $L^2$ and $H^1$ and using the relation between the solutions we will prove the existence of attractors for the problems. More specifically, setting $u_t = v$, where $u(t)$ is solution operator given by (1.1), we will study the evolution of three operators, $z(t)$ given by (1.2), in the spaces $L^2$ and $H^1$, $u(t)+v(t)$ in the space $H^1$ and $v(t)$ in the space $L^2$.

In order to obtain the results we will use the following procedures: to prove the bounded dissipativeness of the problem (1.1) we will construct an appropriate equivalent norm in the space. The bounded dissipativeness of (1.2) in $H^1$ will be obtained using the uniform Gronwall lemma with some appropriate estimates. The proof of the compactness of the operators will be done using arguments of Aubin-Lion's type.

Asymptotic behavior of parabolic equations with monotone principal part was recently studied by Carvalho and Gentile in [3], the main difference with our case, problem (1.2), is that our functional $\varphi$ is not equivalent to the norm of the space.

## 2. Abstract formulation and existence of solutions

As usual in wave equations context, setting $v = u_t$, (1.1) can be seen as a system:

$$\begin{gathered} u_t = v, \quad v_t = (u+v)_{xx} + g, \quad 0 < x < \ell,\ 0 < t < T; \\ u(t,0) = 0, \qquad u_x(t,\ell) + v_x(t,\ell) = \rho\big(v(t,\ell)\big). \end{gathered} \tag{2.1}$$

Therefore, our problem (1.1) can be viewed as an evolution equation

$$\dot{w} + Aw = f(t) \tag{2.2}$$

in the Hilbert space

$$\mathcal{H} = H_{1,0} \times L^2(0,\ell), \quad H_{1,0} = \big\{u \in H^1(0,\ell) : u(0) = 0\big\}, \tag{2.3}$$

with the inner product

$$\big\langle (u_1,v_1),(u_2,v_2)\big\rangle_{\mathcal{H}} = \int_0^\ell \big(u_1' u_2' + v_1 v_2\big)\,dx, \tag{2.4}$$

where

$$f(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}, \qquad A : \mathfrak{D}(A) \subset \mathcal{H} :\longrightarrow \mathcal{H}, \tag{2.5}$$

is given by

$$A(u,v) = \big(-v, -(u+v)''\big) \tag{2.6}$$

on the domain

$$\mathfrak{D}(A) = \big\{(u,v) \in H_{1,0} \times H_{1,0} : (u+v) \in H^2(0,\ell) \text{ and } (u+v)'(\ell) \in \rho\big(v(\ell)\big)\big\}. \tag{2.7}$$

Throughout the paper we denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the usual inner product and norm of $L^2$, respectively. We use the terminology of Brézis [2] and Hale [4]

Lemma 2.1. *The operator $A$ is maximal monotone.*

*Proof.* If $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ are in $\mathfrak{D}(A)$, we have by integrating by parts that

$$\begin{aligned} &\big\langle w_1 - w_2, Aw_1 - Aw_2 \big\rangle \\ &\qquad = -\big(v_1(\ell) - v_2(\ell)\big)\big[\big(u_1+v_1\big)'(\ell) - \big(u_2+v_2\big)'(\ell)\big] + \int_0^\ell \big(v_1' - v_2'\big)^2 dx. \end{aligned} \tag{2.8}$$

Since $\rho$ is nonincreasing and $(u_i + v_i)'(\ell) \in \rho(v_i(\ell))$, $i = 1, 2$, we have

$$\big\langle w_1 - w_2, Aw_1 - Aw_2 \big\rangle \geq 0, \tag{2.9}$$

therefore, $A$ is a monotone operator.

We prove that $A$ is maximal by showing that $R(I + A) = \mathcal{H}$. In fact, if $(f, g) \in \mathcal{H}$ we consider $z$ as being the unique solution of the ODE problem:

$$z - 2z'' = f + 2g := h \in L^2(0,\ell), \qquad z(0) = 0, \qquad z'(0) = a \in \mathbb{R}, \tag{2.10}$$

where $a$ is chosen conveniently. Since $z \in H^2(0,\ell) \cap H_{1,0}$ and $f \in H_{1,0}$, setting

$$u = \frac{1}{2}(z + f), \qquad v = \frac{1}{2}(z - f), \tag{2.11}$$

we have that $u, v \in H_{1,0}$, $u + v = z \in H^2(0,\ell)$, and

$$u - v = f, \qquad v - (u+v)'' = g. \tag{2.12}$$

Therefore, it remains to be proved that $(u+v)'(\ell) \in \rho(v(\ell))$ or equivalently $z'(\ell) \in \tilde{\rho}(z(\ell))$, where

$$\tilde{\rho}(x) = \rho\bigg(\frac{1}{2}\big(x - f(\ell)\big)\bigg). \tag{2.13}$$

We obtain that condition by choosing the constant $a$ appropriately. Setting

$$M = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}, \tag{2.14}$$

we have from the variation constant formula

$$\begin{pmatrix} z(\ell) \\ z'(\ell) \end{pmatrix} = ae^{\ell M}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2}\int_0^\ell e^{(\ell-s)M}\begin{pmatrix} 0 \\ h(s) \end{pmatrix} ds. \tag{2.15}$$

Since

$$e^{\ell M}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}\sinh\left(\frac{\ell}{\sqrt{2}}\right) \\ \cosh\left(\frac{\ell}{\sqrt{2}}\right) \end{pmatrix}, \tag{2.16}$$

we have that the right-hand side of (2.15) is a straight line in plane, parametrized by $a$, with positive slope. Therefore, there will be a unique $a$ that gives the intersection with the nonincreasing graph of $\tilde{\rho}$. The lemma is proved. □

Solutions of abstract evolution equations will be considered in the sense of Brézis [2], that is we have the following definition.

*Definition 2.2.* Let $f$ be in $L^1(0,T;\mathcal{H})$. A continuous function $w:[0,T]\to\mathcal{H}$ is a solution (or strong solution) of

$$\dot{w}(t) + Aw(t) = f(t) \tag{2.17}$$

if $w$ satisfies

(i) $w(t)\in\mathcal{D}(A)$, $\forall t\in(0,T)$,

(ii) $w(t)$ is absolutely continuous (AC) on every compact set $K\subset(0,T)$ (therefore $\dot{w}(t)$ exists a.e. in $(0,T)$),

(iii) $\dot{w}(t)+A(w(t)) = f(t)$, a.e. in $(0,T)$.

Moreover, $w\in C([0,T];\mathcal{H})$ is a weak solution of (2.17) if there exist sequences $(f_n)\in L^1(0,T;\mathcal{H})$ and $(w_n)\in C([0,T];\mathcal{H})$ such that $w_n$ are strong solutions of

$$\dot{w_n}(t) + A\big(w_n(t)\big) = f_n(t), \tag{2.18}$$

$f_n\to f$ in $L^1(0,T;\mathcal{H})$, and $w_n\to w$ uniformly in $[0,T]$.

We have from Theorem 3.4 of [2] the existence of weak solution for the problem (2.1).

In order to prove that this weak solution is in fact strong, we will look for a relation between the solutions of (2.1) and the solutions of (1.2).

The problem (1.2) was studied in [1], where $G$ is an operator

$$G : H_{1,0} \longrightarrow L^2(0,\ell) \tag{2.19}$$

not necessarily local and $h \in L^2(0,T;L^2(0,\ell))$. The problem can be written as the abstract evolution problem in $L^2(0,\ell)$

$$\dot{z} + \mathcal{A}z = F(t,z), \tag{2.20}$$

where $F(t,z) = -G(z) + h(t)$ and $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2(0,\ell) \to L^2(0,\ell)$ is the operator given by

$$\mathcal{A}z = -z'' \tag{2.21}$$

on the domain

$$\mathcal{D}(\mathcal{A}) = \{z \in H_{1,0} \cap H^2(0,\ell) : z'(\ell) \in \rho(z(\ell))\}. \tag{2.22}$$

From Lemmas 2.1 and 2.2 of [1] we have that the operator $\mathcal{A}$ is strongly monotone, that is,

$$\langle \mathcal{A}z_1 - \mathcal{A}z_2,\ z_1 - z_2 \rangle \geq |z_1' - z_2'|^2 \tag{2.23}$$

and of subdifferential type, $\mathcal{A} = \partial\varphi$, where $\varphi : L^2(0,\ell) \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, and lower semicontinuous function defined by

$$\varphi(z) = \begin{cases} p(z(\ell)) + \dfrac{1}{2}\displaystyle\int_0^\ell z'(x)^2\,dx & \text{if } z \in H_{1,0}, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.24}$$

where $p$ is given by

$$p(z) = \int_0^z -\rho(s)\,ds. \tag{2.25}$$

We should observe that $\varphi$ may assume negative values, but the following estimate is true:

$$|z'|^2 \leq k_1\varphi(z) + k_2, \quad \forall z \in H_{1,0}, \tag{2.26}$$

where $k_1,\ k_2$ are constants; in particular $\varphi$ is bounded below.

Indeed, since $|\rho(s)|$ is bounded (by a constant $k$), we have for $z \in H_{1,0}$

$$p(z(\ell)) \geq -k|z(\ell)| = -k\left|\int_0^\ell z'(x)\,dx\right| \geq -k\int_0^\ell |z'(x)|\,dx, \tag{2.27}$$

and then

$$\int_0^\ell \left(\frac{1}{4}z'(x)^2 - k^2\right)dx \leq \int_0^\ell \left(\frac{1}{2}z'(x)^2 - k|z'(x)|\right)dx \leq \varphi(z) \tag{2.28}$$

implies the estimate (2.26).

When $G$ is Lipschitz continuous and $h \in L^2(0,T;L^2(0,\ell))$, it was proved, [1, Theorems 3.2 and 4.1], that the solutions of (1.2) are strong, in particular $z(t) \in \mathcal{D}(\mathcal{A})$, for every $t \in (0,T)$. Moreover, from Theorem 3.6 of [2] the solution $z$ satisfies

$$\sqrt{t}\frac{dz}{dt}(t) \in L^2(0,T;L^2(0,\ell)) \tag{2.29}$$

and when $z(0) \in \mathcal{D}(\varphi) = H_{1,0}$,

$$\frac{dz}{dt}(t) \in L^2\big(0,T;L^2(0,\ell)\big). \tag{2.30}$$

Consider the following relations between the problems (2.1) and (1.2):

$$z(t,x) = u(t,x) + v(t,x) - u(t,\ell)\xi(x), \tag{2.31}$$

$$G(z) = z(\ell)\xi, \tag{2.32}$$

$$h(t,x) = g(t,x) + v(t,x) + u(t,\ell)\xi''(x), \tag{2.33}$$

where $\xi : [0,\ell] \to \mathbb{R}$ is a smooth function satisfying $\xi(0) = 0$, $\xi(\ell) = 1$, and $\xi'(\ell) = 0$.

The operator $G$, given in (2.32), can be considered as an operator from $H^1(0,\ell)$ with values in $L^2(0,\ell)$, and also with values in $H_{1,0}$. In both of these cases $G$ is Lipschitz continuous and satisfies

$$|G(z)| \le c\,|z'|, \tag{2.34}$$

since

$$|z(\ell)| = \left|\int_0^\ell z'(x)\,dx\right| \le \|z'\|_{L^1}. \tag{2.35}$$

It is easy to see that if $(u,v)$ is a solution of (2.1) then $z$, given by (2.31), is a solution of (1.2) with $h$ given by (2.33) and with initial condition $z(0) = u(0) + Sv(0)$.

Conversely, if $z$ is a solution of (1.2), we consider the problem in $H_{1,0}$ given by

$$\frac{du}{dt}(t) + u(t) - J(t)u(t) = 0, \quad u(0) = 0, \tag{2.36}$$

where $J(t)u(t) = G(u(t)) + z(t)$.

Since $J(t) : H_{1,0} \to H_{1,0}$, for $t > 0$, is globally Lipschitz, this problem has existence and uniqueness of solutions, see [2, Theorem 1.4]. If $u(t)$ is this unique solution, then considering $v(t)$ given by the relation (2.31) and $g$ by the relation (2.33) we have that $(u,v)$ satisfies the problem (2.1) with $u(0) = 0$ and $v(0) = z(0)$.

Under these conditions we can prove the following result.

THEOREM 2.3. *If $g \in L^2(0,T;L^2(0,\ell))$, then for every $w_0 = (u_0,v_0) \in \mathcal{H}$ there exists a unique strong solution $w = (u,v) \in C([0,T];\mathcal{H})$ of (2.1) such that $w(0) = w_0$. Moreover, the solution $w = (u,v)$ satisfies*

$$\sqrt{t}\,\frac{d}{dt}(u+v)(t) \in L^2\big(0,T;L^2(0,\ell)\big) \tag{2.37}$$

*and, for $v(0) \in H_{1,0}$,*

$$\frac{d}{dt}(u+v)(t) \in L^2\big(0,T;L^2(0,\ell)\big). \tag{2.38}$$

*Proof.* Since $z$, given by (2.31), is a strong solution of (1.2), in particular $z(t) \in \mathfrak{D}(\mathcal{A})$, $\mathfrak{D}(\mathcal{A})$ given in (2.22), for every $t \in (0,T)$. It is easy to see that $(u,v)$ is a strong solution of (2.1).

From (2.31)

$$\frac{d}{dt}(u+v) = \frac{dz}{dt} + v(t,\ell)\xi, \quad v(t,\ell) = z(t,\ell), \tag{2.39}$$

and $z(t,\ell) \in L^2(0,T)$, according to the trace theorem for Lipschitz domain, [7, page 15], therefore (2.37) and (2.38) follow, respectively, from (2.29) and (2.30). The proof is complete. □

Although we are interested in studying the influence of the nonlinear boundary condition in the problems, we should observe that we have existence of strong solution in more general situations. In fact, we can consider

$$\begin{gathered} u_{tt} - u_{xx} - u_{txx} + q\big(t,x,u,u_t\big) = 0, \quad 0 < x < \ell,\ 0 < t < T; \\ u(t,0) = 0, \qquad u_x(t,\ell) + u_{tx}(t,\ell) = \rho\big(u_t(t,\ell)\big), \end{gathered} \tag{2.40}$$

where

($q_1$) the application $(t,x) \to q(t,x,w)$ belongs to $L^2(0,T;L^2(0,\ell))$, for every fixed $w \in \mathcal{H}$;

($q_2$) there exists $k > 0$, such that

$$\big|q\big(t,x,w_1\big) - q\big(t,x,w_2\big)\big|_{L^2(0,\ell)} \le k\big\|w_1 - w_2\big\|_{\mathcal{H}}, \quad \forall t \in [0,T],\ \forall w_1,w_2 \in \mathcal{H}. \tag{2.41}$$

This problem can be viewed as an abstract evolution equation in the Hilbert space $\mathcal{H}$

$$\dot{w} + Aw + B(t,w) = 0, \tag{2.42}$$

where $B : [0,T] \times \mathcal{H} \to \mathcal{H}$ is given by

$$B = (0,q). \tag{2.43}$$

From the assumptions ($q_1$) and ($q_2$), we have that $B$ satisfies

($B_1$) for every $w \in \mathcal{H}$, the application $t \to B(t,w)$ belongs to $L^2(0,T;\mathcal{H})$;

($B_2$) there exists $k > 0$, such that

$$\big\|B\big(t,w_1\big) - B\big(t,w_2\big)\big\| \le k\big\|w_1 - w_2\big\|, \quad \forall t \in [0,T],\ \forall w_1,w_2 \in \mathcal{H}. \tag{2.44}$$

Under the above assumptions we have the following result.

THEOREM 2.4. *For every $w_0 \in \mathcal{H}$ there exists a unique strong solution $w \in C([0,T];\mathcal{H})$ of (2.42) satisfying $w(0) = w_0$.*

*Proof.* We use the method of Brézis [2]. Since, for every $w \in C([0,T];\mathcal{H})$, $B(t,w(t)) \in L^2(0,T;\mathcal{H})$, we can consider the sequence $w_n$ in $C([0,T];\mathcal{H})$, defined by $w_0(t) = w_0$ and $w_{n+1}$ is the weak solution of

$$\dot{w}_{n+1}(t) + A\big(w_{n+1}(t)\big) = -B\big(t,w_n(t)\big), \quad w_{n+1}(0) = w_0 \tag{2.45}$$

which exists by Theorem 2.3. Using the first inequality of Lemma 3.1 of [2], we obtain

$$\begin{aligned}\left\|w_{n+1}(t)-w_n(t)\right\| &\le \int_0^t \left\|B\big(\sigma, w_n(\sigma)\big)-B\big(\sigma, w_{n-1}(\sigma)\big)\right\| d\sigma \\ &\le k\int_0^t \left\|w_n(\sigma)-w_{n-1}(\sigma)\right\| d\sigma,\end{aligned} \tag{2.46}$$

therefore

$$\left\|w_{n+1}(t)-w_n(t)\right\| \le \frac{(kt)^n}{n!}\left\|w_1-w_0\right\|_{L^\infty}. \tag{2.47}$$

Thus, the sequence $w_n$ converges uniformly to $w$ in $[0,T]$, so $w$ is a weak solution of

$$\dot{w}(t)+A\big(w(t)\big)=-B\big(t,w(t)\big), \quad w(0)=w_0. \tag{2.48}$$

Now, since $B(t,w(t))=(0,q(t,\cdot,w(t)))$ and it is easy to see that $q(t,\cdot,w(t))\in L^2(0,T;L^2(0,\ell))$, we have from Theorem 2.3 that $w$ is a strong solution of (2.42). The proof is complete. □

It is not difficult to see that the strong solutions, given by this theorem, depend continuously on the initial data. More specifically, we have that there exists a positive constant $c$ such that

$$\|w(t)-\tilde{w}(t)\|_{L^\infty([0,T];\mathcal{H})} \le c\left\|w_0-\tilde{w}_0\right\|_{\mathcal{H}}, \tag{2.49}$$

where $w(t)$ and $\tilde{w}(t)$ are solutions of (2.42) with initial conditions $w_0$ and $\tilde{w}_0$, respectively.

## 3. Existence of attractors in $L^2$

We start by constructing an equivalent norm in the space $\mathcal{H}$.

Lemma 3.1. *If $W(w)$ is given by*

$$W(w)=W(u,v)=\int_0^\ell\left[\frac{1}{2}\big(u'\big)^2+\frac{1}{2}v^2+2\beta uv\right]dx, \tag{3.1}$$

*where*

$$0<\beta<\frac{2}{2\ell^2+1}, \tag{3.2}$$

*then $W^{1/2}$ is an equivalent norm in $\mathcal{H}$.*

*Moreover, there exists a positive constant $\lambda$ such that*

$$\int_0^\ell\left[\big(\beta\ell^2-1\big)f^2-2\beta g^2-2\beta fg\right]dx \le -\lambda\left(|f|^2+|g|^2\right) \quad \forall f,g\in L^2(0,\ell). \tag{3.3}$$

*Proof.* Using Poincaré ($|u| \le (\ell/\sqrt{2})|u'|$) and Schwarz inequalities, we have

$$-\frac{\beta\ell}{\sqrt{2}}\left(\left|u'\right|^2+|v|^2\right) \le \int_0^\ell 2\beta u v\,dx \le \frac{\beta\ell}{\sqrt{2}}\left(\left|u'\right|^2+|v|^2\right). \tag{3.4}$$

Using (3.2) we can see that

$$\frac{\beta\ell}{\sqrt{2}} < \frac{1}{2}. \tag{3.5}$$

Therefore, if $\eta = 1/2-\beta\ell/\sqrt{2}$, we have

$$\eta\left(\left|u'\right|^2+|v|^2\right) \le W(u,v) \le \left(\left|u'\right|^2+|v|^2\right), \tag{3.6}$$

then $W^{1/2}$ is an equivalent norm in $\mathcal{H}$.

The second part of the lemma follows by noticing that

$$\int_0^\ell \left[\left(\beta\ell^2-1\right)f^2-2\beta g^2-2\beta f g\right]dx \le \left(\beta\ell^2-1\right)|f|^2-2\beta|g|^2+2\beta|f||g| \tag{3.7}$$

and, for $\beta$ satisfying (3.2), the right-hand side of this inequality is a negative definite form. □

THEOREM 3.2. *If $g,h \in L^\infty(\mathbb{R}^+;L^2(0,\ell))$, then the problems (1.2) and (2.1) are bounded dissipative. More precisely, if $(u,v)$ and $z$ are the solutions of (1.2) and (2.1), with initial conditions $(u_0,v_0)$ and $z_0$, respectively, then there exist positive constants $c_1, c_2$, and $\mu$ such that*

$$\left\|\left(u(t),\ v(t)\right)\right\|_{\mathcal{H}} \le c_1\left\|\left(u_0,\ v_0\right)\right\|_{\mathcal{H}} e^{-\mu t}+c_2, \tag{3.8}$$

$$|z(t)| \le c_1\left|z_0\right| e^{-\mu t}+c_2. \tag{3.9}$$

*Moreover, for $z_0 \in H_{1,0}$ and $r$ positive, there exist positive constants $a, b$, with $b=b(r)$ depending on $r$, such that*

$$\int_t^{t+r} \varphi\left(z(s)\right)ds \le a\left|z_0\right|^2 e^{-\mu t}+b, \quad t \ge 0. \tag{3.10}$$

*Proof.* From the relation between the two problems the estimate (3.9) follows from (3.8). To prove (3.8) it is enough to consider initial data in the domain $\mathfrak{D}(A)$. Using (2.1) and Poincaré inequality ($|v|^2 \le (\ell^2/2)|v'|^2$), we obtain after an integration by parts that for almost every $t$

$$\begin{aligned}
\dot W(t) &= \frac{d}{dt}W\left(u(t),v(t)\right)\\
&\le \int_0^\ell \left[\left(\beta\ell^2-1\right)\left(v'\right)^2-2\beta\left(u'\right)^2-2\beta u'v'\right]dx && (3.11)\\
&\quad+\left[2\beta u(\ell)+v(\ell)\right](u+v)'(\ell)+\int_0^\ell [2\beta u+v]g(t)\,dx. && (3.12)
\end{aligned}$$

The first integral, line (3.11), can be estimated using Lemma 3.1

$$\int_0^\ell \left[ \left(\beta \ell^2 - 1\right)\left(v'\right)^2 - 2\beta\left(u'\right)^2 - 2\beta u'v' \right] dx \leq -\lambda \left( \left|u'\right|^2 + \left|v'\right|^2 \right). \tag{3.13}$$

To estimate the terms in line (3.12), we observe that $(u+v)'(\ell)$ satisfies the boundary condition, so it is bounded by some constant $M$, then using (2.35) we can show that there exists a positive constant $c$, such that, for every $\delta > 0$

$$\left[2\beta u(\ell) + v(\ell)\right](u+v)'(\ell) \leq c\left( \delta\left( \left|u'\right|^2 + \left|v'\right|^2 \right) + \frac{1}{\delta} M^2 \right). \tag{3.14}$$

Using Poincaré inequality we also obtain

$$\int_0^\ell (2\beta u + v) g(t)\, dx \leq c\left( \delta\left( \left|u'\right|^2 + \left|v'\right|^2 \right) + \frac{1}{\delta} \|g\|^2 \right). \tag{3.15}$$

Choosing $\delta$ sufficiently small, we obtain positive constants $\mu_i = 1, 2$, and $K$, such that

$$\dot{W}(t) \leq -\mu_1\left( \left|u'\right|^2 + \left|v'\right|^2 \right) + K \leq -\mu_2 W(t) + K. \tag{3.16}$$

Solving this differential inequality, we obtain

$$W(t) \leq e^{-\mu_2 t} W(0) + \frac{K}{\mu_2} \tag{3.17}$$

that implies (3.8).

In order to prove inequality (3.10) we have that $\mathscr{A}$ is the subdifferential of the functional $\varphi$ and $\varphi(0) = 0$, therefore $\varphi(z) \leq \langle \mathscr{A}z,\ z\rangle$. So, multiplying (1.2) by $z$ we obtain

$$\frac{1}{2}\frac{d}{dt}|z|^2 + \varphi(z) \leq -\langle G(z),\ z\rangle + \langle h,\ z\rangle. \tag{3.18}$$

The operator $G$ satisfies (2.34), then, using (2.26), we obtain for every $\delta > 0$ a constant $M$ depending on $\delta$ such that

$$\left| \left\langle G\left(z(t)\right), z(t) \right\rangle \right| \leq \delta \varphi\left(z(t)\right) + M\left( |z(t)|^2 + 1 \right) \tag{3.19}$$

and, since

$$|\langle h(t), z(t)\rangle| \leq c\left( |z(t)|^2 + 1 \right), \tag{3.20}$$

we obtain by grouping the equivalent terms and choosing a convenient small value for $\delta$ that

$$\frac{d}{dt}|z(t)|^2 + \varphi\left(z(t)\right) \leq a_1 + a_2 |z(t)|^2, \tag{3.21}$$

for some positive constants $a_1$, $a_2$. Integrating this inequality from $t$ to $t+r$ we obtain

$$\int_t^{t+r} \varphi\left(z(s)\right) ds \leq |z(t)|^2 + a_1 r + a_2 \int_t^{t+r} |z(s)|^2\, ds. \tag{3.22}$$

This inequality and (3.9) imply (3.10). $\square$

THEOREM 3.3. *If* $h \in L^\infty(\mathbb{R}^+; L^2(0,\ell))$*, then the solution operator* $T_h(t) : L^2(0,\ell) \to L^2(0,\ell)$*, associated to the solution of (1.2), is a compact operator for each* $t > 0$.

*Proof.* Multiplying (1.2) by $\phi \in H_{1,0}$, we obtain

$$\langle z_t, \phi \rangle = z_x(t,\ell)\phi(\ell) - \langle z_x, \phi_x \rangle - \langle G(z), \phi \rangle + \langle h, \phi \rangle, \tag{3.23}$$

therefore (3.10) and (3.23) imply that $z_t \in L^2(0,T; H'_{1,0})$ and

$$\int_0^T \|z_t\|^2_{H'_{1,0}} dt \le C\big(|z(0)|, T\big). \tag{3.24}$$

To prove the compactness it is enough to consider initial data in a dense subset of $L^2(0,\ell)$. Let $B$ be the bounded set $B = B(r) \cap H_{1,0}$, where $B(r)$ the ball of $L^2(0,\ell)$ with center at zero and radius $r$, and $T_h(t)(z_0)$ the solution of (1.2) with initial condition $z_0$.

From (3.10) and (3.24),

$$\bar{B} = \big\{T_h(\cdot)\big(z_0\big); z_0 \in B\big\} \tag{3.25}$$

is a bounded set in the Banach space

$$W = \left\{ v \in L^2\big(0,T; H_{1,0}\big); v_t = \frac{dv}{dt} \in L^2\big(0,T; H'_{1,0}\big)\right\}. \tag{3.26}$$

Therefore, from Theorem 5.1 of [6], $\bar{B}$ is a precompact set in $L^2(0,T; L^2(0,\ell))$. Then, if $(z_n)$ is a sequence in $B$, taking subsequences if necessary, we can suppose that $(T_h(\cdot)(z_n))$ converges to some function $z(\cdot) \in L^2(0,T; L^2(0,\ell))$, and also, for almost every $\tau \in (0,T)$,

$$T_h(\tau)\big(z_n\big) \longrightarrow z(\tau) \quad \text{as } n \longrightarrow \infty. \tag{3.27}$$

Consider now the evolution operator $S(\cdot)(z,h)$ given by

$$S(t)(z,h) = \big(T_h(t)z, h_t\big), \tag{3.28}$$

where $h_t$ is the translation of $h$, $h_t(\tau) = h(t+\tau)$. From [8], $S(t) : t \ge 0$ is a dynamical system. Therefore, for $t > 0$, there exists $\tau \in (0,t)$ such that (3.27) is true, then

$$\begin{aligned}
\big(T_h(t)z_n, h_t\big) &= S(t)\big(z_n, h\big) = S(t-\tau)S(\tau)\big(z_n, h\big) \\
&= S(t-\tau)\big(T_h(\tau)z_n, h_\tau\big) \longrightarrow S(t-\tau)\big(z(\tau), h_\tau\big) \\
&= \big(T_{h_\tau}(t-\tau)z(\tau), h_t\big)
\end{aligned} \tag{3.29}$$

implies the compactness of $T_h(t)$.

Denoting by $v_{u_0}(t)$ the dynamical system given by the problem (2.1), when the initial condition $u(0) = u_0 \in H_{1,0}$ is fixed. Using Theorems 3.2 and 3.3 and the relation (2.31), we can state the next result that is a consequence of Theorem 2.2 of Ladyzhenskaya [5]. □

THEOREM 3.4. *Under the above conditions the two dynamical systems* $z(t)$ *and* $v_{u_0}(t)$ *have compact global attractors in* $L^2(0,\ell)$.

## 4. Existence of attractors in $H_{1,0}$

We start doing some estimates of the solution $z(t)$ of (1.2) when the initial condition $z(0) \in H_{1,0}$. Using Theorem 3.6 of [2], we have that $t \to \varphi(z(t))$ is absolutely continuous and

$$\frac{d}{dt}\varphi\big(z(t)\big) = \big\langle \mathcal{A}z(t), z_t(t)\big\rangle, \quad \text{a.e.}, \tag{4.1}$$

then

$$\frac{d}{dt}\varphi\big(z(t)\big) = -|\mathcal{A}z(t)|^2 + \langle \mathcal{A}z(t), -G(z)+h\rangle, \tag{4.2}$$

and integrating on $t$ we obtain

$$\begin{aligned}
&\int_0^t |\mathcal{A}z(s)|^2\,ds + \varphi\big(z(t)\big) \\
&\quad \le \varphi\big(z(0)\big) + \int_0^t |\mathcal{A}z(s)|\,\big|h(s)-G\big(z(s)\big)\big|\,ds \\
&\quad \le \varphi\big(z(0)\big) + \int_0^t \frac{1}{2}|\mathcal{A}z(s)|^2\,ds + \int_0^t \big|G\big(z(s)\big)\big|^2\,ds + \int_0^t |h(s)|^2\,ds.
\end{aligned} \tag{4.3}$$

Using (2.26) and (2.34), we obtain, for $t \in [0,T]$, that

$$\frac{1}{2}\int_0^t |\mathcal{A}z(s)|^2\,ds + \varphi\big(z(t)\big) \le \varphi\big(z(0)\big) + c_1 + c_2\int_0^t \varphi\big(z(s)\big)\,ds \tag{4.4}$$

for some constants $c_1, c_2$.

Thus, from Gronwall inequality, there exists a constant $C(\varphi(z(0)),T)$ depending on $\varphi(z(0))$ and $T$ such that

$$\varphi\big(z(t)\big) \le C\big(\varphi\big(z(0)\big), T\big), \tag{4.5}$$

$$\int_0^t |\mathcal{A}z(s)|^2\,ds \le C\big(\varphi\big(z(0)\big), T\big), \tag{4.6}$$

in particular, we have $z \in L^\infty(0,T;H_{1,0}) \cap L^2(0,T;H^2(0,\ell))$.

Moreover, if $z_1(t)$ and $z_2(t)$ are solutions with initial condition on $H_{1,0}$ we have, using (2.23),

$$\begin{aligned}
\big|\big(z_1(t)\big)' - \big(z_2(t)\big)'\big|^2 &\le \big\langle \mathcal{A}z_1(t) - \mathcal{A}z_2(t), z_1(t)-z_2(t)\big\rangle \\
&= -\frac{1}{2}\frac{d}{dt}\big|z_1(t)-z_2(t)\big|^2 - \big\langle G\big(z_1(t)\big) - G\big(z_2(t)\big), z_1(t)-z_2(t)\big\rangle.
\end{aligned} \tag{4.7}$$

Since $G$ is Lipschitz, we obtain after an integration on $t$

$$\begin{aligned}
&\frac{1}{2}\int_0^t \big|\big(z_1(s)\big)' - \big(z_2(s)\big)'\big|^2\,ds + \frac{1}{2}\big|z_1(t)-z_2(t)\big|^2 \\
&\qquad \le \frac{1}{2}\big|z_1(0)-z_2(0)\big|^2 + c\int_0^t \big|z_1(s)-z_2(s)\big|^2\,ds,
\end{aligned} \tag{4.8}$$

therefore, from Gronwall inequality, there exists a constant $C$ depending on $T$, such that

$$\big|z_1(t) - z_2(t)\big| \le C\,\big|z_1(0) - z_2(0)\big|, \tag{4.9}$$

$$\int_0^t \big|\big(z_1(s)\big)' - \big(z_2(s)\big)'\big|^2\,ds \le C\,\big|z_1(0) - z_2(0)\big|^2 \tag{4.10}$$

for $t \in [0, T]$.

Now we study the evolution of the problem (1.2) in $H_{1,0}$. Our first result is concerned with continuity with respect to time and initial data.

LEMMA 4.1. *The solution operator, $z(t) = T(t)z_0$, of the problem (1.2) is continuous in the variables $t$ and $z_0$ in the $H_{1,0}$-norm. More precisely, the operator*

$$\mathbb{R}^+ \times H_{1,0} \longrightarrow H_{1,0}, \qquad \big(t, z_0\big) \longrightarrow T(t)z_0, \tag{4.11}$$

*is continuous separately in each variable.*

*Proof.* Fix $z_0 \in H_{1,0}$ and let $(t_n)$ be a sequence in $\mathbb{R}^+$ converging to $t$, we know that the solution $(z(t_n))$ converges to $z(t)$ in $L^2(0,\ell)$ and, using Lemma 3.6 of [2], $(\varphi(z(t_n)))$ converges to $\varphi(z(t))$. Then, from (2.26), $|(z(t_n))'|$ is bounded, therefore, there exists a subsequence of $(z(t_n))$, that we keep denoting by $(z(t_n))$, that converges weakly to $z(t)$ in $H_{1,0}$.

First of all, we claim that the weak convergence implies the convergence of $(z(t_n,\ell))$. In fact considering a smooth function $\phi$ such that $\phi(0) = 0$ and $\phi(\ell) \ne 0$, we obtain by integrating by parts

$$\begin{aligned} \int_0^\ell z'\big(t_n, x\big)\phi(x)\,dx &= z\big(t_n, \ell\big)\phi(\ell) - \int_0^\ell z\big(t_n, x\big)\phi'(x)\,dx, \\ \int_0^\ell z'(t, x)\phi(x)\,dx &= z(t, \ell)\phi(\ell) - \int_0^\ell z(t, x)\phi'(x)\,dx. \end{aligned} \tag{4.12}$$

Thus, passing to the limit, $z(t_n,\ell) \to z(t,\ell)$, what proves our claim. Next, since $p$ is continuous and

$$\big\|z\big(t_n\big)\big\|_{H_{1,0}}^2 = 2\big[\varphi\big(z\big(t_n\big)\big) - p\big(z\big(t_n, \ell\big)\big)\big], \tag{4.13}$$

we have $\|z(t_n)\|_{H_{1,0}} \to \|z(t)\|_{H_{1,0}}$ that implies the strong convergence of $(z(t_n))$ to $z(t)$ and the continuity of the operator in the variable $t$. $\square$

Now we prove the continuity of the operator in the second variable. In fact, what we have is a stronger result:

THEOREM 4.2. *If $(z_{0_n})$ is a bounded sequence in $H_{1,0}$ and converges to $z_0$ in the $L^2(0,\ell)$-norm, then the corresponding solutions of (1.2) $z_n(t) = T(t)z_{0_n}$ converges to $z(t) = T(t)z_0$ in $H_{1,0}$, for fixed $t > 0$, as $n \to \infty$. In particular, for $t > 0$, the operator $T(t) : H_{1,0} \to H_{1,0}$ is compact.*

*Proof.* We have $(\varphi(z_{0_n}))$ bounded, then from (4.5) and (2.26) both sequences $(\varphi(z_n(t)))$ and $(|(z_n(t))'|)$ are uniformly bounded for $t \in [0,T]$. The convergence $z_{0_n} \to z_0$ in $L^2(0,\ell)$ and (4.10) imply the convergence

$$z_n \longrightarrow z \quad \text{in } L^2\big(0,T;H_{1,0}\big), \tag{4.14}$$

therefore $z_n(\tau) \to z(\tau)$ in $H_{1,0}$ for almost every $\tau \in [0,T]$.

For $t \in [0,T]$,

$$\begin{aligned}\big|\varphi\big(z_n(t)\big)-\varphi\big(z(t)\big)\big| \le \big|\varphi\big(z_n(t)\big)-\varphi\big(z_n(\tau)\big)\big|+\big|\varphi\big(z_n(\tau)\big)-\varphi\big(z(\tau)\big)\big| \\ +\big|\varphi\big(z(\tau)\big)-\varphi\big(z(t)\big)\big|.\end{aligned} \tag{4.15}$$

The first term in the right-hand side satisfies

$$\varphi\big(z_n(t)\big)-\varphi\big(z_n(\tau)\big)=\int_\tau^t \frac{d}{ds}\varphi\big(z_n(s)\big)\,ds \tag{4.16}$$

and from (4.2)

$$\frac{d}{dt}\varphi\big(z(t)\big) \le \big|G\big(z(t)\big)\big|^2+|h(t)|^2, \tag{4.17}$$

therefore the sequences $(d/dt(\varphi(z_n(t))))$ are uniformly bounded in $L^2(0,\ell)$ for every $t \in [0,T]$. Then (4.15) implies $\varphi(z_n(t)) \to \varphi(z(t))$ for every $t \in [0,T]$, as $n \to \infty$. Therefore, the same argument we have just used in the first part of the theorem implies that $z_n(t) \to z(t)$ in $H_{1,0}$-norm, as $n \to \infty$. □

THEOREM 4.3. *If $h \in L^\infty(0,\infty;L^2(0,\ell))$, then there exists a bounded set in $H_{1,0}$ that attracts all the solutions of the problem (1.2) with initial condition in a subset of $H_{1,0}$ that it is bounded in $L^2(0,\ell)$. In particular, the problem (1.2) is bounded dissipative in $H_{1,0}$.*

*Proof.* If $z(t)$ is a solution of the problem (1.2) with initial condition in $H_{1,0}$ we have, using (4.17), that $\varphi(z(t))$ satisfies the differential inequality

$$\frac{d}{dt}\varphi\big(z(t)\big) \le a_1\varphi\big(z(t)\big)+a_2+|h(t)|^2, \quad t>0, \tag{4.18}$$

where $a_1, a_2$ are constants.

For solution with initial conditions in $H_{1,0}$ and bounded in $L^2(0,\ell)$, (3.10) implies that $\int_t^{t+r}\varphi(z(s))ds$ is less than a fixed constant for $t$ sufficiently large, then we can use the uniform Gronwall lemma, see [9, page 89], to obtain the result of the theorem. □

As a consequence of the two previous theorems and the relation (2.31) we have the following theorem.

THEOREM 4.4. *Under the above conditions, the dynamical system $z(t)$ given by (1.2) has a compact global attractor in $H_{1,0}$. Moreover, for $v(0) \in H_{1,0}$, $u(t)+v(t)$ given by (2.1) has also a compact global attractor in $H_{1,0}$.*

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