OBSTACLES TO BOUNDED RECOVERY

BORIS SHEKHTMAN

Received 10 September 2001

Let X be a Banach space, $V \subset X$ is its subspace and $U \subset X^*$. Given $x \in X$, we are looking for $v \in V$ such that $u(v) = u(x)$ for all $u \in U$ and $\|v\| \leq M\|x\|$. In this article, we study the restrictions placed on the constant M as a function of X, V, and U.

1. Introduction

In this article, we are concerned with the following problem: let X be a Banach space, over the field $\mathbb{F}$ ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$), $V \subset X$ is an n-dimensional subspace of X and $u_1, \dots, u_m$ are m linearly independent functionals on X. given $x \in X$ we want to recover x on the basis of the values $u_1(x), \dots, u_m(x) \in \mathbb{F}$.

Hence we are looking for a map $F : X \to V$ such that $u_j(Fx) = u_j(x)$ for all $j = 1, \dots, m$. Since we do not know x a priori we choose to look for a map F such that the norm of F

$$\|F\| = \sup\left\{\frac{\|Fx\|}{\|x\|} : 0 \neq x \in X\right\} \tag{1.1}$$

is as small as possible. We may also require additional properties on F such as linearity and idempotency.

To formalize these notions let X, V, $u_1, \dots, u_m$ be as before. Let $U = \operatorname{span}\{u_1, \dots, u_m\}$. The triple (X, U, V) is called a recovery triple. We consider three classes of operators

$$\begin{aligned}
\mathcal{F}(X,U,V) &:= \left\{F : X \longrightarrow V \mid u(Fx) = u(x)\ \forall u \in U\right\},\\
\mathcal{L}(X,U,V) &:= \left\{L : X \longrightarrow V \mid u(Lx) = u(x)\ \forall u \in U;\ L\text{-linear}\right\},\\
\mathcal{P}(X,U,V) &:= \left\{P \mid u(Px) = u(x)\ \forall u \in U\right\},
\end{aligned} \tag{1.2}$$

where P is a linear projection from X onto an m-dimensional subspace of V.

Abstract and Applied Analysis 6:7 (2001) 381–400
2000 Mathematics Subject Classification: 41A35, 46A32
URL: http://aaa.hindawi.com/volume-6/S1085337501000719.html

Respectively, we introduce three "recovery constants"

$$\begin{aligned} r(X,U,V) &:= \inf\big\{\|F\| : F \in \mathcal{F}(X,U,V)\big\},\\ lr(X,U,V) &:= \inf\big\{\|L\| : L \in \mathcal{L}(X,U,V)\big\},\\ pr(X,U,V) &:= \inf\big\{\|P\| : P \in \mathcal{P}(X,U,V)\big\}. \end{aligned} \tag{1.3}$$

Clearly

$$\begin{gathered} \mathcal{F}(X,U,V) \supset \mathcal{L}(X,U,V) \supset \mathcal{P}(X,U,V),\\ 1 \le r(X,U,V) \le lr(X,U,V) \le pr(X,U,V). \end{gathered} \tag{1.4}$$

The class $\mathcal{P}(X,U,V)$, and hence the rest of the classes are nonempty if and only if

$$\dim\big(U|_V\big) = m, \tag{1.5}$$

where $U|_V$ is the restrictions of functionals from U onto V.

In particular, we will always assume that $m \le n$. If $m = n$ and (1.5) holds then all three classes coincide and consist of uniquely defined linear projection. Hence the problem of estimating the recovery constants is reduced to estimating the norm of one projection. The problem of estimating $r(X,U,V)$ can also be considered as a local version of "SIN property" described in [1].

In this paper, we will characterize the recovery constants in terms of geometric relationships between Banach spaces X, U, V, and their duals.

In our setting U is an m-dimensional subspace of functionals on X. If we restrict U to be functionals on V, we obtain a *new* Banach space

$$\tilde{U} := U|_V. \tag{1.6}$$

Of course, algebraically it is the same space but the norm on $\tilde{U}$ is defined to be

$$\|u\|_{\tilde{U}} = \sup\left\{\frac{|u(x)|}{\|x\|} : 0 \neq x \in V\right\} \tag{1.7}$$

as opposed to

$$\|u\|_U = \sup\left\{\frac{|u(x)|}{\|x\|} : x \in X\right\} \tag{1.8}$$

and hence topologically these are two different spaces. In fact $\tilde{U} \subset V^*$ and may not even be isometric to any subspace of X^* (and in particular to U). It turns out that the recovery constants depend on how well U can be embedded in V^* and X^*, as well as how well U^* can be embedded into V. These results will be presented in Section 2.

In Section 3, we will construct examples of the triples (X,U,V) so that the different restriction constants coincide and also so that three of them are different from each other. Here we will use the Banach space theory to determine whether a given Banach space can or cannot be embedded into another Banach space. In particular, we will prove that $r(X,U,V) = lr(X,U,V)$ if $X = L_1$ and thus generalize some results of [8].

In the last section we will give some applications of the results when the space V consists of polynomials. We will reprove some known results and prove some new results on interpolation by polynomials by interpreting the norms of the interpolation operators as the recovery constants.

We will use the rest of this section to introduce some useful concepts from the local theory of Banach spaces. All of them can be found in the book [2].

Let E and V be two k-dimensional Banach spaces. The Banach-Mazur distance is defined to be

$$d(E,V) := \inf\left\{\|T\|\left\|T^{-1}\right\| \mid T \text{ is an isomorphism from } E \text{ onto } V\right\}. \quad (1.9)$$

Analytically $d(E,V) \le d_0$ for some $d_0 \ge 1$ if and only if there exists basis $e_1,\dots,e_k$ in E and $v_1,\dots,v_k$ in V and constants $C_1, C_2 > 0$ such that

$$C_2^{-1}\left\|\sum_{j=1}^{k}\alpha_j e_j\right\| \le \left\|\sum_{j=1}^{k}\alpha_j v_j\right\| \le C_1\left\|\sum_{j=1}^{k}\alpha_j e_j\right\| \quad (1.10)$$

holds for all $\alpha_1,\dots,\alpha_k \in \mathbb{F}$ and $C_1 \cdot C_2 \le d_0$.

By homogeneity, it is equivalent to finding basis $e'_1,\dots,e'_k \in E$ and $v'_1,\dots,v'_k \in V$ such that

$$\left\|\sum_{j=1}^{k}\alpha_j e'_j\right\| \le \left\|\sum_{j=1}^{k}\alpha_j v'_j\right\| \le d_0\left\|\sum\alpha_j e'_j\right\|. \quad (1.11)$$

The following properties are obvious:

$$\begin{gathered} 1 \le d(E,V) \le d(E,G)\cdot d(G,V), \\ d(E,V) = d\big(E^*,V^*\big). \end{gathered} \quad (1.12)$$

Next we will need the notion of projection constant. Let V be a subspace of X. Define a relative projectional constant $\lambda(V,X)$ to be

$$\lambda(V,X) = \inf\left\{\|P\| : P \text{ is a projection from } X \text{ onto } V\right\}. \quad (1.13)$$

Now the absolute projectional constant $\lambda(V)$ of an arbitrary space V is defined to be

$$\lambda(V) := \sup\left\{\lambda(V,X) : X \supset V\right\}. \quad (1.14)$$

Here are a few properties

$$1 \le \lambda(V) = \lambda(V,X) \quad (1.15)$$

if X is one of the following spaces $L_\infty(\mu)$, $l_\infty(\Gamma)$, $C(K)$.

$$\lambda(V) \leq d(E,V)\cdot\lambda(E), \tag{1.16}$$

this property shows that the absolute projectional constant is an isomorphic invariant.

$$\lambda(V) \leq d\big(V, l_\infty^k\big) \quad \text{where } k = \dim V \tag{1.17}$$

if V, E are subspaces of $L_1(\mu)$ space and $d(V,E) = 1$, then $\lambda(V, L_1(\mu)) = \lambda(E, L_1(\mu))$. Let E and X be Banach spaces and $a \geq 1$ be fixed. We say that E a-embedded into X

$$E \underset{a}{\hookrightarrow} X \tag{1.18}$$

if there exists a subspace $E_1 \subset X$ such that

$$d\big(E, E_1\big) \leq a. \tag{1.19}$$

An operator $J : E \to E_1$ such that $\|J\|\|J^{-1}\| \leq a$ is called an a-embedding.

We say that the embedding $E \underset{a}{\hookrightarrow} X$ is b-complemented if there exists a subspace $E_1 \subset X$ such that $d(E, E_1) \leq a$ and $\lambda(E_1, X) \leq b$.

The rest of the notions and results from the theory of Banach spaces will be introduced as needed.

2. General theorems

The following two theorems of Helly will play a fundamental role in this section (cf. [3]).

THEOREM 2.1. *Let X be a Banach space, $x_1, \ldots, x_k \in X$; $\alpha_1, \ldots, \alpha_k \in \mathbb{F}$. There exists a functional $u \in X^*$ with*

$$\|u\| \leq M; \quad u\big(x_j\big) = \alpha_j \tag{2.1}$$

if and only if for every sequence of numbers $a_1, \ldots, a_k \in \mathbb{F}$,

$$\left\| \sum_{j=1}^{k} a_j x_j \right\| \geq \frac{1}{M} \left| \sum_{j=1}^{k} a_j \alpha_j \right|. \tag{2.2}$$

THEOREM 2.2. *Let X be a Banach space, $u_1, \ldots, u_k \in X^*$, $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$. For every $\epsilon > 0$ there exists an $x \in X$ such that $\|x\| \leq M + \epsilon$, $u_j(x) = \alpha_j$ if and only if for every sequence $a_1, \ldots, a_k \in \mathbb{F}$,*

$$\left\| \sum_{j=1}^{k} a_j u_j \right\| \geq \frac{1}{M} \left| \sum_{j=1}^{k} a_j \alpha_j \right|. \tag{2.3}$$

We now turn our attention to the recovery constants.

Let (X, U, V) be a recovery triple. Let $\tilde{U} := U|_V$. For every $u \in U \subset X^*$ let $\tilde{u} = u \mid V \in \tilde{U} \subset V^*$.

THEOREM 2.3. *Let $r_0 \geq 1$, then*

$$r(X, U, V) \leq r_0 \tag{2.4}$$

if and only if the operator $J : \tilde{U} \to U$ defined by $J^{-1}u = \tilde{u}$ has the norm $\|J\| \leq r_0$. In other words,

$$r(X, U, V) = \sup \left\{ \frac{\|u\|}{\|\tilde{u}\|} : 0 \neq u \in U \right\}. \tag{2.5}$$

Proof. Let $u_1, \ldots, u_m$ be a basis in U. Then $\tilde{u}_1, \ldots, \tilde{u}_m$ is a basis $\tilde{U}$. Let $x \in X$, $\|x\| = 1$, $u_j(x) = \alpha_j$. Let $r(X, U, V) \leq r_0$. Then for every $\epsilon > 0$ there exists $F \in \mathbb{F}(X, U, V)$ such that $\|Fx\| \leq r_0$ for all $x \in X$ with $\|x\| \leq 1$. Hence for $v := F(x) \in V$ we have $\|v\| \leq r_0 + \epsilon$; $u_j(v) = \alpha_j = \tilde{u}_j(v)$. By Theorem 2.2

$$\left\| \sum_{j=1}^{m} a_j \tilde{u}_j \right\| \geq \frac{1}{r_0} \left| \sum_{j=1}^{m} a_j \alpha_j \right| \tag{2.6}$$

for all $a_1, \ldots, a_m \in \mathbb{F}$.

Hence for every $x \in X$ with $\|x\| \leq 1$ and every $a_1, \ldots, a_m \in \mathbb{F}$

$$\left\| \sum_{j=1}^{m} a_j \tilde{u}_j \right\| \geq \frac{1}{r_0} \left| \sum_{j=1}^{m} a_j u_j(x) \right|. \tag{2.7}$$

Passing to the supremum over all x with $\|x\| \leq 1$ we obtain

$$\left\| \sum_{j=1}^{m} a_j u_j \right\| \leq r_0 \left\| \sum_{j=1}^{m} a_j \tilde{u}_j \right\| \tag{2.8}$$

or equivalently

$$\left\| J \left(\sum_{j=1}^{m} a_j \tilde{u}_j \right) \right\| \leq r_0 \left\| \sum_{j=1}^{m} a_j \tilde{u}_j \right\|. \tag{2.9}$$

For the proof of the converse, assume that r_0 is such that (2.8) holds. Then for every fixed $x \in X$ with $\|x\| \leq 1$ and every $a_1, \ldots, a_m \in \mathbb{F}$

$$\left\| \sum a_j \tilde{u}_j \right\| \geq \frac{1}{r_0} \left\| \sum a_j u_j \right\| \geq \frac{1}{r_0} \left| \sum a_j u_j(x) \right|. \tag{2.10}$$

Now by Theorem 2.2, for every $\epsilon > 0$ there exists $v \in V$ such that $\|v\| \leq r_0 + \epsilon$; $u_j(x) = \tilde{u}_j(v)$. $\square$

COROLLARY 2.4. *The quantity* $r(X,U,V) = r_0$ *if and only if the operator* $J : \tilde{U} \to U$ *defined by* $\tilde{u} = J^{-1}u$ *realizes an* r_0*-embedding*

$$U \underset{r_0}{\hookrightarrow} V^*. \tag{2.11}$$

Proof. J is an isomorphism from U onto $\tilde{U} \subset V^*$. Since $\tilde{u}$ is a restriction of u we have $\|\tilde{u}\| \leq \|u\|$. Hence $\|J^{-1}u\| \leq \|u\|$ and $d(U,\tilde{U}) \leq \|J\|\,\|J^{-1}\| \leq r_0$. □

COROLLARY 2.5. *If* $r(X,U,V) \leq r_0$ *then there exists an embedding* $U \underset{r_0}{\hookrightarrow} V^*$.

This corollary is completely obvious and we stated it solely for the reason of future use.

At the end of this section, we will give an example that shows that the converse to Corollary 2.5 does not hold. It does not suffice to have some embedding $U \underset{r_0}{\hookrightarrow} V^*$ to obtain $r(X,U,V) \leq r_0$. It has to be a very specific embedding $J : \tilde{u} \to u$.

We will now deal with $pr(X,U,V) = \inf\{\|P\| : P \in \mathcal{P}(X,U,V)\}$. For the next theorem we fix the basis $u_1,\dots,u_m \in U$ and for any sequence $\alpha_1,\dots,\alpha_m \in \mathbb{F}$ define

$$|\|(\alpha_j)\|| := \sup\left\{\frac{|\sum_{j=1}^m a_j\alpha_j|}{\|\sum_{j=1}^m a_j u_j\|} : a_j \in \mathbb{F} : \sum_{j=1}^m |a_j| \neq 0\right\}. \tag{2.12}$$

THEOREM 2.6. *Let* $r_1 \geq 1$. *Then* $pr(X,U,V) \leq r_1$ *if and only if for every* $\epsilon > 0$, *there exist* $v_1,\dots,v_m \in V$ *such that* $u_j(v_k) = \delta_{jk}$ *and*

$$|\|(\alpha_j)\|| \leq \left\|\sum_{j=1}^m \alpha_j v_j\right\| \leq (r_1+\epsilon)|\|(\alpha_j)\|| \quad \forall \alpha_1,\dots,\alpha_m \in \mathbb{F}. \tag{2.13}$$

Proof. First, let $P \in \mathcal{P}(X,U,V)$. Then

$$Px = \sum_{j=1}^m u_j(x)v_j \tag{2.14}$$

for some $v_j \in V$ with $u_j(v_k) = \delta_{jk}$. We want to show that

$$|\|(\alpha_j)\|| \leq \left\|\sum_{j=1}^m \alpha_j v_j\right\| \leq \|P\| \cdot |\|(\alpha_j)\||. \tag{2.15}$$

Given a sequence $\alpha_1, \ldots, \alpha_m$, let $M = \inf\{\|x\| : u_j(x) = \alpha_j\}$. Then by Theorem 2.2

$$M = \sup\left\{ \frac{\left|\sum_{j=1}^m a_j\alpha_j\right|}{\left\|\sum a_j u_j\right\|} : a_j \in \mathbb{F};\ \sum_{j=1}^m |a_j| \neq 0 \right\} = \left|\left\|\left(\alpha_j\right)\right\|\right|. \tag{2.16}$$

For every $\epsilon > 0$ let $x_\epsilon \in X$ be such that $\|x_\epsilon\| \le M+\epsilon;\ u_j(x_\epsilon) = \alpha_j$. We have

$$\left\|\sum_{j=1}^m \alpha_j v_j\right\| = \left\|Px_\epsilon\right\| \le \|P\|(M+\epsilon). \tag{2.17}$$

Since this is true for all ϵ and in view of (2.16) we obtain the right-hand side inequality in (2.15).

For the left-hand side we have

$$\begin{aligned}\left\|\sum_{j=1}^m \alpha_j v_j\right\| &\ge \sup\left\{ \frac{\left\|\left(\sum_{k=1}^m a_k u_k\right)\left(\sum_{j=1}^m \alpha_j v_j\right)\right\|}{\left\|\sum a_k u_k\right\|} : \sum |a_k| \neq 0 \right\} \\ &= \sup\left\{ \frac{\left\|\sum_{k=1}^m a_k\alpha_k\right\|}{\left\|\sum a_k u_k\right\|} : \sum |a_k| \neq 0 \right\} = \left|\left\|\left(\alpha_k\right)\right\|\right|.\end{aligned} \tag{2.18}$$

To prove the converse, let $v_1, \ldots, v_m \in V$ with $u_k(v_j) = \delta_{kj}$ and let (2.13) holds for some arbitrary ϵ. Define $P \in \mathcal{P}(X, U, V)$ by $Px = \sum_{j=1}^m u_j(x)v_j$. We have

$$\begin{aligned}\|Px\| &= \left\|\sum_{j=1}^m u_j(x)v_j\right\| \le \left(r_1+\epsilon\right)\left|\left\|u_j(x)\right\|\right| \\ &\le \left(r_1+\epsilon\right)\sup\left\{ \frac{\left\|\sum_{j=1}^m a_j u_j(x)\right\|}{\left\|\sum_{j=1}^m a_j u_j\right\|} \sum |a_j| \neq 0 \right\} \le \left(r_1+\epsilon\right)\|x\|.\end{aligned} \tag{2.19}$$

□

Corollary 2.7. *For every $\epsilon > 0$ there exists a subspace $V_0 \subset V$ such that $d(V_0, U^*) \le pr(X, U, V)+\epsilon$; that is, for every $r_1 > pr(X, U, V)$ there exists an r_1-embedding*

$$U^* \underset{r_1}{\hookrightarrow} V. \tag{2.20}$$

Proof. Observe that the space $(\mathbb{F}^n, |\|\cdot\||)$ is isometric to the dual of U. Hence (2.13) defines a map

$$T : \left(\alpha_j\right) \longrightarrow \sum_{j=1}^m \alpha_j v_j; \qquad T : U^* \longrightarrow \operatorname{span}\left\{v_1, \ldots, v_m\right\} \subset V \tag{2.21}$$

such that $\|T\| \le r_1;\ \|T^{-1}\| \le 1$. □

Comparing Corollaries 2.5 and 2.7 we see that an operator $P \in \mathcal{P}(X, U, V)$ with a small norm forces a good embedding

$$T : U^* \hookrightarrow V \tag{2.22}$$

while having an operator $F \in \mathcal{F}(X, U, V)$ with a small norm implies a sort of a "dual embedding"

$$J : U \hookrightarrow V^*. \tag{2.23}$$

In general, (2.22) does not imply (2.23) and that is why (as we will see in the next section) $pr(X, U, V)$ may be much larger than $r(X, U, V)$.

However, there are cases when (2.22) and (2.23) are equivalent. This happens if there exist a projection from V onto TU^* or from V^* onto JU of small norms, that is, if

$$\lambda\big(TU^*, V\big) \qquad \text{or} \qquad \lambda\big(JU, V^*\big) \tag{2.24}$$

is small. To rephrase it: (2.22) and (2.23) are equivalent if one of the two embeddings is well complemented.

Proposition 2.8. *Let $r_0 = r(X, U, V)$ and let $a \geq 1$. Then $pr(X, U, V) \leq ar_0$ if there exists a projection Q from V^* onto $\tilde{U}$ with $\|Q\| \leq a$.*

Proof. For the proof it is convenient to consider the following diagram:

$$\begin{tikzcd}
V^* \arrow[d, phantom, "\cup"] \arrow[dr, "JQ"] & & \\
\tilde{U} \arrow[r, hook, "J", "r_0"'] & U & \subset \; X^*,
\end{tikzcd} \tag{2.25}$$

where Q is a projection from V^* onto $\tilde{u}$ with $\|Q\| \leq a$. Hence $\|JQ\| \leq ar_0$. The map $(JQ)^* = Q^*J^*$ maps X^{**} onto V. Furthermore $\dim \operatorname{Im} Q^*J^* \leq \dim \operatorname{Im} Q^* \leq m$. Observe that $u(Q^*J^*x) = \tilde{u}(Q^*J^*x) = (JQ\tilde{u})(x) = (J\tilde{u})(x) = u(x)$. Thus Q^*J^* is a projection from X^{**} into an m-dimensional subspace of V with $\|Q^*J^*\| \leq ar_0$. Let $P = Q^*J^* \mid X$. Then $P \in \mathcal{P}(X, U, V)$ and $\|P\| \leq ar_0$. □

The converse of Proposition 2.8 may not be true. The small change in wording, however, makes it true.

Corollary 2.9. *Let $r_0 = r(X, U, V)$ and let $a \geq 1$. Then $pr(X, U, V) \leq ar_0$ if and only if for every $\epsilon > 0$ there exists a projection Q from V^* onto $\tilde{U}$ such that $\|JQ\| \leq ar_0 + \epsilon$.*

Proof. The sufficiency follows from Proposition 2.8. Suppose that $pr(X, U, V) \leq ar_0$. Then there exists a projection $P \in \mathcal{P}(X, U, V)$ such that $\|P\| \leq$

$ar_0+\epsilon$. Since P maps X into V hence $P^*: V^* \to X^*$ and

$$\operatorname{Im} P^* = U. \tag{2.26}$$

Thus $Q := J^{-1}P^*$ projects V^* onto $\tilde{U}$ and

$$\|JQ\| = \left\|JJ^{-1}P^*\right\| = \left\|P^*\right\| = \|P\| \le ar_0. \tag{2.27}$$

□

We will now rephrase Corollary 2.9 in terms of the diagram

$$\begin{tikzcd}
V^* \arrow[ddr, "\hat{J}"] & & & \\
\cup & & & \\
\tilde{U} \arrow[r, hook, "J"] & U & \subset & X^*.
\end{tikzcd} \tag{2.28}$$

Corollary 2.10. *Let $r_1 \ge 1$. Then $pr(X,U,V) \le r_1$ if and only if for every $\epsilon > 0$ the operator J in (2.28) can be extended to an operator $\hat{J}$ from V^* onto U, that is, if and only if there exists an operator $\tilde{J}$ from V^* into U such that $\|\hat{J}\| \le r_1+\epsilon$ and $\hat{J}|_{\tilde{U}} = J$.*

Proof. If $pr(X,U,V) \le r_1$ then we conclude from Corollary 2.9 (cf. diagram (2.25)) that $\hat{J} := JQ$ is the desired extension of J. Conversely, let $\hat{J}$ be an extension with $\|\hat{J}\| \le r_1+\epsilon$. Then $Q := J^{-1}\hat{J}$ is a projection from V^* onto $\tilde{U}$ with $\|JQ\| = \|\hat{J}\| \le r_1+\epsilon$. □

Since $U \subset X^*$, we can view J as an embedding of $\tilde{U}$ into X^* and $\hat{J}$ to be an extension of J from V^* into all of X^*. However, there are other extensions of J to an operator from V^* into X^* with the range not limited to U. This subtle difference turns out to be the key to the linear recovery.

Theorem 2.11. *Let (X,U,V) be a recovery triple. Let $r_2 \ge 1$ and $J: \tilde{U} \hookrightarrow U \subset X^*$. Then $lr(X,U,V) \le r_2$ if and only if for every $\epsilon > 0$ there exists a linear extension $S: V^* \to X^*$ of an operator $J: \tilde{U} \hookrightarrow X^*$ such that*

$$\|S\| \le r_2+\epsilon. \tag{2.29}$$

Proof. We again illustrate it on the diagram

$$\begin{tikzcd}
V^* \arrow[ddrrr, "S"] & & & \\
\cup & & & \\
\tilde{U} \arrow[r, hook, "J"] & U & \subset & X^*.
\end{tikzcd} \tag{2.30}$$

Let S be such an extension with $\|S\| \le r_2 + \epsilon$. Then $S^* : X^* \to V$. Since S is an extension of J we have $S\tilde{u} = u$ for every $\tilde{u} \in \tilde{U} \subset V^*$. Therefore for every $x \in X^{**}$ and every $u \in U$

$$x(u) = x\big(S\tilde{u}\big) = \big(S^*x\big)\big(\tilde{u}\big). \tag{2.31}$$

In particular, if $x \in X \subset X^{**}$ we have $S^*x \in V$ and

$$u(x) = \tilde{u}\big(S^*x\big) = u\big(S^*x\big). \tag{2.32}$$

Thus $L := S^* \mid X$ defines a linear operator from X onto V such that $u(x) = u(Lx)$ and $\|L\| \le \|S^*\| = \|S\| \le r_2 + \epsilon$.

In the other direction, let $L \in \mathscr{L}(X, U, V)$ with $\|L\| \le r_2 + \epsilon$. Then L^* is map from V^* into X^* and for every $\tilde{u} \in \tilde{U} \subset V^*$

$$\big(L^*\tilde{u}\big)(x) = \tilde{u}(Lx) = u(Lx) = u(x). \tag{2.33}$$

Thus $L^*\tilde{u} = u$ for every $\tilde{u} \in \tilde{U}$ and $\|L^*\| \le r_2 + \epsilon$. Hence L^* is the desired extension of J. □

It is a little surprising that $r(X, U, V)$ and $pr(X, U, V)$ depend (at least explicitly) only on the relationship between U and V, yet $lr(X, U, V)$ which is squeezed in between those two constants depend explicitly on the space X as well as U and V.

We finish this discussion by demonstrating that the converse results to Corollaries 2.5 and 2.7 are false. Thus only the existence of specific embeddings of $U \hookrightarrow V^*$ and of $U^* \hookrightarrow V$ give the estimates for the recovery constants.

Example 2.12. Let $X = L_1[0,1]$, $V = \mathrm{span}[\chi_{[0,1/2]}, \chi_{[1/2,1]}]$. Let $U = \mathrm{span}\,\{r_1, r_2\} \subset L_\infty$ where $r_1 = 1$;

$$r_2(x) = \begin{cases} 1 & \text{if } 0 \le x \le \dfrac{1}{2}, \\ -1 & \text{if } \dfrac{1}{2} < x \le 1. \end{cases} \tag{2.34}$$

It is easy to check that

$$\big\|\alpha r_1 + \beta r_2\big\|_\infty = |\alpha| + |\beta| \tag{2.35}$$

and U is isometric to l_1^2. Similarly $\|\alpha\chi_{[0,1/2]} + \beta\chi_{[1/2,1]}\|_1 = |\alpha| + |\beta|$ and V is isometric to l_1^2. Let $e_1 = (1,0)$, $e_2 = (0,1)$ and consider a map $T : l_\infty^2 \to l_1^2$ defined by $Te_1 = (1/2)(e_1 + e_2)$, $Te_2 = (1/2)(e_1 - e_2)$. Then

$$\big\|\big(\alpha e_1 + \beta e_2\big)\big\|_\infty = \max\big\{|\alpha|, |\beta|\big\} = \frac{1}{2}|\alpha + \beta| + \frac{1}{2}|\alpha - \beta| = \big\|T\big(\alpha e_1 + \beta e_2\big)\big\|_1. \tag{2.36}$$

Hence l_1^2 is isometric to $l_\infty^2 = (l_1^2)^*$ and all the spaces U, V, U^*, V^* are isometric. Therefore $U^* \underset{1}{\hookrightarrow} V$ and $U \underset{1}{\hookrightarrow} V^*$ and since all the spaces are of the same dimension, the embeddings are 1-complemented. Thus all the conditions of Corollaries 2.5 and 2.7 are satisfied with $r_0 = r_2 = 1$. Yet we will show that $r(X, U, V) \geq 2$. Indeed let $\tilde{r}_1, \tilde{r}_2$ be the restrictions of r_1 and r_2 onto V. Then

$$\left\|\alpha\tilde{r}_1 + \beta\tilde{r}_2\right\| = \sup\left\{\frac{\left|\left(\int_0^1 \left(a\chi_{[0,1/2]} + b\chi_{[1/2,1]}\right)\left(\alpha r_1 + \beta r_2\right)\right)\right|}{\int_0^1 \left|a\chi_{[0,1/2]} + b\chi_{[1/2,1]}\right|}\right\},$$

$$\sup_{a,b} \frac{\left|(1/2)(\alpha a + \alpha b) + (1/2)(\beta a - \beta b)\right|}{(1/2)|a| + |b|} \tag{2.37}$$

$$= \sup_{a,b} \frac{\left|a(\alpha + \beta) + b(\alpha - \beta)\right|}{|a| + |b|} = \max\left\{|\alpha| + |\beta|, |\alpha| - |\beta|\right\}.$$

Choosing $\alpha = 1$, $\beta = 1$ we have

$$\left\|\alpha r_1 + \beta r_2\right\| = 2 = 2\left\|\alpha\tilde{r}_1 + \beta\tilde{r}_2\right\|. \tag{2.38}$$

Hence $\|J\| \geq 2$ and by Theorem 2.3, $r(X, U, V) \geq 2$.

3. Comparison of the recovery constants

In this section, we will establish some relationships between various recovery constants. Recall that for $E \subset X$ the notation $\lambda(E, X)$ stands for a relative projectional constant

$$\lambda(E, X) = \inf\left\{\|P\| : P \text{ is a projection from } X \text{ onto } E\right\}. \tag{3.1}$$

PROPOSITION 3.1. *Let (X, U, V) be a recovery triple. Let*

$$\dim U = m \leq n = \dim V. \tag{3.2}$$

Then

$$pr(X, U, V) \leq \lambda\left(U, X^*\right) lr(X, U, V) \leq \sqrt{m}\, lr(X, U, V), \tag{3.3}$$

$$\begin{aligned} pr(X, U, V) &\leq \lambda\left(\tilde{U}, V^*\right) r(X, U, V) \\ &\leq \min\left\{\sqrt{m}, \sqrt{n-m} + 1\right\} r(X, U, V). \end{aligned} \tag{3.4}$$

Proof. Let Q be a projection from X^* onto U and let S be an extension of J (cf. diagram (2.30)) to an operator from V^* into X^* with $\|S\| \leq lr(X, U, V) + \epsilon$. Then $\hat{J} := QS$ is the map from V^* onto U and it is an extension of J to an operator from V^* onto U. By Corollary 2.10, we have

$$pr(X, U, V) \leq \left\|\hat{J}\right\| \leq \|Q\|\,\|S\| \leq \|Q\|\left(r(X, U, V) + \epsilon\right). \tag{3.5}$$

Hence we proved the left-hand side of (3.3). The right-hand side follows from the standard estimate (cf. [4])

$$\lambda\big(U, X^*\big) \le \lambda(U) \le \sqrt{\dim U}. \tag{3.6}$$

The left-hand side of (3.4) is a reformulation of Proposition 2.8, and the right-hand side of (3.4) follows from another standard estimate (cf. [4])

$$\lambda\big(\tilde{U}, V^*\big) \le \min\Big\{\sqrt{\dim \tilde{U}}, \sqrt{\operatorname{codim} \tilde{U}} + 1\Big\}. \tag{3.7}$$

□

Remark 3.2. Using the estimate for relative projectional constant in [4] the right-hand side of (3.4) can be improved to $\lambda(\tilde{U}, V^*) r(X,U,V) \le f(n,k) r(X,U,V)$ where $f(n,k) := \sqrt{m}(\sqrt{m}/n + \sqrt{(n-1)(n-k)}/n)$.

It was observed in [8] that $r(X,U,V) = pr(X,U,V)$ if $X = L_1(\mu)$ and $U = \operatorname{span}[u_1,\dots,u_m] \subset L_\infty$ where $u_1,\dots,u_m$ are functions with disjoint support. In this case U is isometric to l_∞^m. We are now in a position to extend this observation in two different directions.

Proposition 3.3. *For any Banach space X*

$$pr(X,U,V) \le d\big(U, l_\infty^m\big) r(X,U,V). \tag{3.8}$$

Proof. Let T be an isomorphism from U onto l_∞^m with $\|T\|\,\|T^{-1}\| = d(U, l_\infty^m)$. Consider the diagram

$$\begin{tikzcd}
V^* \arrow[ddrr, "A"] & & & \\
\cup & & & \\
\tilde{U} \arrow[r, hook, "J"] & U \arrow[r, "T"] & l_\infty^m \arrow[r, "T^{-1}"] & U.
\end{tikzcd} \tag{3.9}$$

It is well known (cf. [10]) that every operator with the range in l_∞^m can be extended to an operator from a bigger space (in this case V^*) with the same norm. Let A be such an extension of the operator TJ. Then $\tilde{J} := T^{-1}A$ is an extension of J to an operator from V^* to U with

$$\begin{aligned}
\|\hat{J}\| &= \big\|T^{-1}A\big\| \le \big\|T^{-1}\big\|\,\|A\| \\
&= \big\|T^{-1}\big\|\,\|TJ\| \le \big\|T^{-1}\big\|\,\|T\|\,\|J\| \le d\big(u, l_\infty^m\big)\|J\|.
\end{aligned} \tag{3.10}$$

By Corollary 2.10, we obtain (3.8). □

Proposition 3.4. *Let $X = L_1(\mu)$. Then for any U, V*

$$lr(X,U,V) = r(X,U,V). \tag{3.11}$$

Proof. In this case $X^* = L_\infty(\mu)$ and hence the operator $J : \tilde{U} \hookrightarrow U$ can be considered as an operator from $\tilde{U}$ into $L_\infty(\mu)$. Using again the "projective property" of $L_\infty(\mu)$ (cf. [10]) we can extend J to an operator S from V^* to $L_\infty(\mu)$ so that $\|J\| = \|S\|$. By Theorem 2.11, we obtain the conclusion of the proposition. □

Example 3.7 will demonstrate that "lr" in this proposition cannot be replaced by "pr".

We now wish to demonstrate (by means of examples) that $r(X, U, V)$ can be arbitrarily large; that one can find a sequence (X, U_m, V_n) such that $r(X, U_m, V_n)$ is bounded, yet $lr(X, U_m, V_n)$ tends to infinity as $\sqrt{m}$; and that there exists a sequence (X, U_m, V_n) such that $lr(X, U_m, V_n)$ is bounded, yet $pr(X, U_m, V_n)$ tends to infinity as $\sqrt{m}$. Also the estimates (3.3) and (3.4) are asymptotically best possible. These examples also serve to demonstrate the usefulness of the results in Section 2 for estimating the recovery constants.

Example 3.5. For arbitrary $X, V, M > 0$ there exists $U \subset X^*$ such that $r(X, U, V) \geq M$.

Construction 3.6. Fixing $X, V, M > 0$, it is a matter of triviality to show that there exists a projection P from X onto V

$$Px = \sum_{j=1}^{n} u_j(x) v_j \tag{3.12}$$

such that $\|P\| \geq M$. Pick $U = \operatorname{span}[u_1, \ldots, u_n]$. Then

$$\mathcal{F}(X, U, V) = \mathcal{L}(X, U, V) = \mathcal{P}(X, U, V) = \{P\}. \tag{3.13}$$

Hence $r(X, U, V) = \|P\| \geq M$.

For the next two examples we will need the Rademacher function $r_j(t) := \operatorname{sign} \sin(2^{j-1}\pi t)$, $0 \leq t \leq 1$. It is well known (cf. [2]) that

$$\left\| \sum_{j=1}^{n} \alpha_j r_j \right\|_{L_\infty} = \sum_{j=1}^{n} |\alpha_j| \tag{3.14}$$

while

$$C \sqrt{\sum_{j=1}^{n} |\alpha_j|^2} \leq \left\| \sum_{j=1}^{n} \alpha_j r_j \right\|_{L_1} \leq \sqrt{\sum_{j=1}^{n} |\alpha_j|^2} \tag{3.15}$$

for some absolute constant $C > 0$.

Example 3.7. There exists a sequence of recovery triples (X, U_m, V_n) with $n = 2^m$ such that $r(X, U_m, V_n) = lr(X, U_m, V_n) = 1$ yet $pr(X, U_m, V_n) \geq C_1\sqrt{m}$ for some universal constant $C_1 > 0$.

Construction 3.8. Let $A_j = [(j-1)/2^m, j/2^m]$. And let $V \subset L_1[0,1]$ spanned by χ_{A_j}. Hence $X = L_1[0,1]$; $V \subset L_1[0,1]$ and $n = \dim V = 2^m$. Let $U = \operatorname{span}\{r_j\}_{j=0}^{m-1} \subset L_\infty[0,1] \subset \mathcal{M}[0,1]$. It is easy to see that $\|\sum \alpha_j \tilde{r}_j\| = \|\sum \alpha_j r_j\|_\infty = \sum |\alpha_j|$. Hence by Theorem 2.3, we have $r(X,U,V) = 1$. Since $X = L_1$ we use Proposition 3.4 to conclude that $lr(X,U,V) = 1$. Since U is isometric to $l_1^{(m)}$, U^* is isometric to $l_\infty^m V$ is isometric to l_1^n. It is a well-known fact (cf. [6]) that for every subspace $E \subset l_1^n$ with $\dim E = m$

$$d\left(E, l_\infty^m\right) \geq C_1\sqrt{m}, \tag{3.16}$$

where $C_1 > 0$ is some universal constant. Thus we conclude that for every subspace $V_0 \subset V$

$$d\left(V_0, U^*\right) \geq C_1\sqrt{m} \tag{3.17}$$

and by Corollary 2.7 we have

$$pr(X,U,V) \geq C_1\sqrt{m}. \tag{3.18}$$

Example 3.9. There exists a constant $C > 0$ such that for every integer m there exists a recovery triple (X,U,V) with $\dim U = m$, $\dim V = n = 2^{m-1}$ such that

$$r(X,U,V) = 1, \qquad lr(X,U,V) \geq C\sqrt{m}. \tag{3.19}$$

Construction 3.10. Pick $X = L_\infty[0,1]$, $V = \operatorname{span}\{r_1, \ldots, r_{2m}\} \subset L_\infty$. Next we partition $[0,1]$ into $2^{2^{m-1}}$ equal intervals and pick any m of them: $A_1, A_2, \ldots, A_m$. Let

$$u_j = 2^{2^{m-1}} \cdot \chi_{A_j} \quad j = 1, \ldots, m. \tag{3.20}$$

Let $U = \operatorname{span}\{u_1, \ldots, u_m\} \subset L_1[0,1] \subset (L_\infty[0,1])^*$.

Then $\|\sum_{j=1}^m \alpha_j u_j\|_{L_1} = \sum_{j=1}^m |\alpha_j|$ and U is *isometric* to l_1^n. It follows (cf. [6]) that $\lambda(U, (L_\infty[0,1])^*) = 1$. Hence by Proposition 3.1

$$lr(X,U,V) = pr(X,U,V). \tag{3.21}$$

U^* is isometric to l_∞^m while V is isometric to l_1^n.

As in the previous example we conclude that for every subspace $V_0 \subset V$ with $\dim V_0 = m$ we have

$$d\big(V_0, U^*\big) \geq C\sqrt{m} \tag{3.22}$$

and by Corollary 2.7, we obtain

$$lr(X, U, V) = pr(X, U, V) \geq C\sqrt{m}. \tag{3.23}$$

We will now choose intervals A_j so that $r(X, U, V) = 1$ or equivalently (by Theorem 2.3) so that

$$\begin{aligned} &\sup\left\{\int_0^1 \left(\sum_{j=1}^{m} a_j u_j\right)\left(\sum_{k=1}^{2^{m-1}} \alpha_k r_k\right) : \sum |\alpha_k| = 1\right\} \\ &\quad = \sum_{j=1}^{m} |a_j| = \left\|\sum_{j=1}^{m} a_j u_j\right\|_{L_1}. \end{aligned} \tag{3.24}$$

In order to do that recall that for every distribution of signs $\epsilon_1, \ldots, \epsilon_{2^m}$ where $\epsilon_1 = 1$; $\epsilon_j = \pm 1$ there exists a subinterval A in our partition such that $\operatorname{sign} r_j(t) = \epsilon_j$ for $t \in A$. Let $A_1 = [0, 2^{-2^{m-1}}]$, choose A_2 to be such that

$$\chi_{A_2}\left(\sum_{k=1}^{2^{m-1}} \alpha_k r_k\right) = \left(\sum_{k=1}^{2^{m-2}} \alpha_k - \sum_{k=2^{m-2}\neq 1}^{2^{m-1}} \alpha_k\right)\chi_{A_2}. \tag{3.25}$$

Choose A_3 to satisfy

$$\begin{aligned} \chi_{A_3}\left(\sum_{k=1}^{2^{m-1}} \alpha_k r_k\right) = \Bigg(&\sum_{k=1}^{2^{m-3}} \alpha_k - \sum_{k=2^{m-3}+1}^{2^{m-2}} \alpha_k \\ &+ \sum_{k=2^{m-2}+1}^{2^{m-2}+2^{m-3}} \alpha_k - \sum_{k=2^{m-2}+2^{m-3}+1}^{2^{m-1}} \alpha_k\Bigg)\chi_{A_3}, \end{aligned} \tag{3.26}$$

continuing this way we come down to choosing A_m so that

$$\chi_{A_m}\left(\sum_{k=1}^{2^{m-1}} \alpha_k r_k\right) = \big(\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \cdots + \alpha_{(2^{m-1}-1)} - \alpha_{2^{m-1}}\big)\chi_{A_m}. \tag{3.27}$$

Expanding the integral in (3.24) we obtain

$$\int\left(\sum_{j=1}^{m} a_j u_j\right)\left(\sum_{k=1}^{2^{m-1}} \alpha_k r_k\right)$$
$$= a_1\left(\sum_{k=1}^{2^{m-1}} \alpha_k\right) + a_2\left(\sum_{k=1}^{2^{m-2}} \alpha_k - \sum_{k=2^{m-2}+1}^{2^{m-1}} \alpha_k\right)$$
$$+\cdots+a_m\left(\sum_{k=1}^{2^{m-1}} (-1)^{k-1}\alpha_k\right)$$
$$= \alpha_1\left(\sum_{j=1}^{m} \epsilon_{1,j} a_j\right) + \alpha_2\left(\sum_{j=1}^{m} \epsilon_{2,j} a_j\right) + \cdots + \alpha_{2^{m-1}}\left(\sum_{j=1}^{m} \epsilon_{2^{m-1},j} a_j\right), \tag{3.28}$$

where $\epsilon_{k,j} = \pm 1$, and for each k the collection $(\epsilon_{k,1}, \ldots, \epsilon_{k,m})$ is distinct, with $\epsilon_{k,1} = 1$. Since there are precisely 2^{m-1} such choices, hence

$$\max\left\{\left|\sum_{j=1}^{m} \epsilon_{k,j} a_j\right| : k = 1, \ldots, 2^{m-1}\right\}$$
$$= \max\left\{\left|\sum_{j=1}^{m} \epsilon_j a_j\right| : \epsilon_j = \pm 1\right\} = \sum_{j=1}^{m} |a_j|. \tag{3.29}$$

Combining this with (3.28) we have

$$\max\left\{\int_0^1\left(\sum_{j=1}^{m} a_j u_j\right)\left(\sum_{k=1}^{2^{m-1}} \alpha_k r_k\right) : \sum_{k=1}^{2^{k-1}} |\alpha_k| = 1\right\}$$
$$= \max\left\{\left|\sum_{j=1}^{m} \epsilon_{k,j} a_j\right| : k = 1, \ldots, 2^{m-1}\right\} = \sum_{j=1}^{m} |a_j| \quad \text{(by (3.28))}. \tag{3.30}$$

This proves (3.24) and thus $r(X, U, V) = 1$.

Remark 3.11. In this example $\dim V = 2^{m-1}$ is much greater than the $\dim U = m$. I could not construct an example of triples (X, U_m, V_n) so that

(a) m is proportional to n (say $n = 10m$)
(b) $r(X, U_m, V_n)$ are uniformly bounded
(c) $lr(X, U_m, V_n) \to \infty$ as $m \to \infty$.

It would be interesting to know if such example is possible. In view of the next section it will also be interesting to find out if such example is possible with $n = m + o(m)$.

4. Applications to polynomial recovery

In this section, we will examine the situation where X is one of the following Banach spaces $C(\mathbb{T})$, $L_1(\mathbb{T})$, $H_1(\mathbb{T})$, $A(\mathbb{T})$ the last being the disk-algebra on the unit circle $\mathbb{T}$. Let H_n be the space of polynomials of degree at most $n-1$. Let U_m be an arbitrary subspace of X^* of dimension m.

THEOREM 4.1 (Faber). *If $n = m$, then there exists a constant $C > 0$ such that*

$$r\big(X, U_n, H_n\big) \geq C \log n \longrightarrow \infty. \tag{4.1}$$

Hence in each one of the spaces X there exists an obstacle to bounded recovery. It is interesting to observe that only in $C(\mathbb{T})$ this is the strong obstacle.

PROPOSITION 4.2. *Let H_n^p be the space of polynomials H_n equipped with the L_p-norm. Then*

(a) $(H_n^\infty)^*$ cannot *be embedded uniformly into $C(\mathbb{T})^*$;*
(b) $(H_n^\infty)^*$ can *be uniformly embedded into $A(\mathbb{T})^*$;*
(c) H_n^1 can *be embedded uniformly into $(H_1(\mathbb{T}))^*$ and $(L_1(\mathbb{T}))^*$.*

Proof. Part (a) was proved in [9], part (b) follows from an observation of Pelcinski and Bourgain (cf. [10, Proposition 3E15]), and part (c) follows from the fact that any sequence of finite-dimensional spaces can be uniformly embedded into $(H_1(\mathbb{T}))^*$ and $(L_1(\mathbb{T}))^*$. □

For the linear recovery there is a strengthening of Faber theorem (cf. [7, 8]).

THEOREM 4.3. *Under the notation in this section*

$$lr\big(X, U_m, H_n\big) \geq C \log \frac{n}{n-m+1}. \tag{4.2}$$

In particular if $n - m = o(n)$ then $lr(X, U_m, H_n) \to \infty$.

In [8], it was observed that $r(L_1, U_m, H_n) \to \infty$ under an additional condition that $d(U_m, C_\infty^m)$ is uniformly bounded. The following corollary follows immediately from Theorem 4.3 and Proposition 3.4.

COROLLARY 4.4. *For any m-dimensional subspace $U_m \subset L_\infty$*

$$r\big(L_1, U_m, H_n\big) \geq c \log \frac{n}{n-m+1}. \tag{4.3}$$

It is still an open problem whether $r(C(\mathbb{T}), U_m, H_n)$ is bounded if $n-m = o(n)$. Here is a partial result that uses Proposition 2.8.

PROPOSITION 4.5. *Let $n-m = o(\log n)^2$. Then*

$$r\big(L_1, U_m, H_n\big) \longrightarrow \infty \tag{4.4}$$

for any sequence of m-dimensional subspaces $U_m \subset C(\mathbb{T})^*$.

Proof. Let $n-m = o(\log n)^2$. Then codimension of $\widetilde{U}_m$ in (H_n^∞) is $n-m$. By [4] there exists a projection P from $(H_n^\infty)^*$ onto $\widetilde{U}_m$ such that $\|P\| \le \sqrt{n-m}+1$.

By Proposition 2.8,

$$pr\big(C(\mathbb{T}), U_m, H_n\big) \le \big(\sqrt{n-m}+1\big)\, r(m,n). \tag{4.5}$$

From Theorem 4.3, we have

$$r(m,n) \ge \frac{pr\big(C\mathbb{T}, U_m, H_n\big)}{\sqrt{n-m}+1} \ge C\frac{\log n}{o(\log n)} \longrightarrow \infty. \tag{4.6}$$

□

In the positive direction, Bernstein proved (cf. [5]) that for any constant $a > 1$ there exists a subspace $U_m \subset (C(\mathbb{T}))^*$ such that

$$\ell r\big(C(\mathbb{T}), U_m, H_n^\infty\big) \le \theta(1) \tag{4.7}$$

if $n \ge am$. The functionals in U_m are the linear span of point evaluation and thus U_m is isometric to ℓ_1^m. Hence we have the following corollary.

COROLLARY 4.6. *For any $a > 1$ there exists a constant $C(a)$ and a subspace $U_m \subset C(\mathbb{T})^*$ such that*

$$pr\big(C(\mathbb{T}), U_m, H_n^\infty\big) \le C(a) \tag{4.8}$$

if $n > am$.

Proof. Since U_m is isometric to ℓ_1^m the space U_m^* is isometric to ℓ_∞^m. Since every operator $\widetilde{U}_m$ into ℓ_∞^m can be extended to an operator from $(H_n^\infty)^*$ into ℓ_∞^m hence by Corollary 2.10 and from (4.8) we conclude

$$pr\big(C(\mathbb{T}), U_m, H_n^\infty\big) \le r\big(C(\mathbb{T}), U_m, H_n^\infty\big) \le O(1). \tag{4.9}$$

□

We will end this section (and this paper) with the discussion of a “dual version” of a problem of polynomial recovery. The exact relationship between this problem and the problem of bounded recovery is not known to me at the present time.

Let $t_1,\dots,t_m \in \mathbb{T}$ and this time $m \geq n$. Let $p \in H_n$. Can one bound a uniform norm of the polynomial p in terms of the bounds on the values $|p(t_j)|$? Just as in the case of polynomial recovery, the answer is "yes" if $m > an$ with $a > 1$.

THEOREM 4.7. *Let $a > 1$, let $m > an$. Let $t_1,\dots,t_m$ be uniform points on $\mathbb{T}$. Then there exists a constant $A = A(a)$ such that*

$$\left\| p(t) \right\| \leq A(a) \cdot \max \left\| p\left(t_j\right) \right\|. \tag{4.10}$$

Conjecture 4.8. Let $m = n + o(n)$. And let $t_1,\dots,t_m$ be arbitrary points in $\mathbb{T}$. Then there exist polynomials $p_n \in H_n$ such that $|p_n(t_j)|$; $j = 1,\dots,m$ and yet $\|p_n\|_\infty \to \infty$.

Here we will prove an analogue of Proposition 4.5 in this case.

THEOREM 4.9. *Let $t_1, H_n \dots, t_m \in \mathbb{T}$ and $m = n + o(\log^2 n)$. Then there exist polynomials $p_n \in H_n$ such that*

$$\left| p_n\left(t_j\right) \right| < 1 : j = 1,\dots,m, \qquad \left\| p_n \right\|_\infty \longrightarrow \infty. \tag{4.11}$$

Proof. Let $\widetilde{T}_n$ be a linear map from H_n^∞ onto ℓ_∞^m defined by

$$\widetilde{T}_n p = \left(p\left(t_j\right)\right) \in \ell_\infty^m. \tag{4.12}$$

Then $\|\widetilde{T}_n\| \leq 1$; $\widetilde{T}_n$ is one-to-one and thus $\widetilde{T}_n$ induces isomorphisms T_n from H_n^∞ onto $E_n := \widetilde{T}_n(H_\infty^n)$. It now follows from [4] that

$$\lambda\left(E_n\right) \leq \sqrt{m-n} + 1. \tag{4.13}$$

By (3.8) and Theorem 4.1 we have

$$\left\| T_n^{-1} \right\| = \left\| T_n \right\| \left\| T_n^{-1} \right\| \geq d\left(E_n, H_\infty^n\right) \geq \frac{\log n}{\sqrt{m-n+1}} \longrightarrow \infty \tag{4.14}$$

which is equivalent to the statement of the theorem. ☐

We hope to explore further similarities between this problem and recovery constants in a subsequent paper.

References

[1] F. Deutsch, *Simultaneous interpolation and norm-preservation*, Delay Equations, Approximation and Application (Mannheim, 1984), Internat. Schriftenreihe Numer. Math., vol. 74, Birkhäuser, Basel, 1985, pp. 122–132. MR 88f:41033. Zbl 0573.41030.

[2] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators*, Cambridge Studies in Advanced Mathematics, vol. 43, Cambridge University Press, Cambridge, 1995. MR 96i:46001. Zbl 0855.47016.

[3] E. Helly, *Uber Systeme Linearer Gleichungen mit Unendlich Vleden Unbekannten*, Monatsh. f. Math. u. Phys. **31** (1921), 60–91 (German).

[4] H. König, D. R. Lewis, and P. K. Liu, *Finite-dimensional constants*, Studia Math. **75** (1983), 341–358.

[5] I. P. Natanson, *Constructive Theory of Approximation*, Moscow, 1949.

[6] A. Pełczyński, *Geometry of finite-dimensional Banach spaces and operator ideals*, Notes in Banach Spaces, Univ. Texas Press, Texas, 1980, pp. 81–181. MR 82j:46020. Zbl 0461.46014.

[7] B. Shekhtman, *On the norms of interpolating operators*, Israel J. Math. **64** (1988), no. 1, 39–48. MR 90c:41005. Zbl 0667.41017.

[8] ———, *On polynomial "interpolation" in* L_1, J. Approx. Theory **66** (1991), no. 1, 24–28. MR 92d:41010. Zbl 0733.42002.

[9] ———, *On the strong form of the Faber theorem*, Stochastic Processes and Functional Analysis (Riverside, CA, 1994) (J. A. Goldstein et al., eds.), Lecture Notes in Pure and Appl. Math., vol. 186, Dekker, New York, 1997, pp. 215–218. MR 97m:46035. Zbl 0896.46023.

[10] P. Wojtaszyk, *Banach Spaces for Analysts*, Cambridge University Press, Cambridge, 1991.

Boris Shekhtman: Department of Mathematics, University of South Florida, Tampa, FL 33620-5700, USA

E-mail address: boris@math.usf.edu