ON AN ASYMPTOTICALLY LINEAR ELLIPTIC DIRICHLET PROBLEM

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Under very simple conditions, we prove the existence of one positive and one negative solution of an asymptotically linear elliptic boundary value problem. Even for the resonant case at infinity, we do not need to assume any more conditions to ensure the boundness of the (PS) sequence of the corresponding functional. Moreover, the proof is very simple.

1. Introduction

In this paper, we consider the existence of one-signed solutions for the following Dirichlet problem:

$$\begin{aligned} -\Delta u &= f(x,u), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$. The conditions imposed on $f(x,t)$ are as follows:

(f_1) $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$; $f(x,0) = 0$, for all $x \in \Omega$.

(f_2) $\lim_{|t|\to 0}(f(x,t)/t) = \mu$, $\lim_{|t|\to\infty}(f(x,t)/t) = \ell$ uniformly in $x \in \Omega$.

Since we assume (f_2), problem (1.1) is called *asymptotically linear* at both zero and infinity. This kind of problems have captured great interest since the pioneer work of [1]. For more information, see [2, 3, 4, 5, 6, 7, 8, 11, 12] and the references therein.

Obviously, the constant function $u = 0$ is a trivial solution of problem (1.1). Therefore, we are interested in finding nontrivial solutions. Let $F(x,u) = \int_0^u f(x,s)\,ds$. It follows from ($f_1$) and ($f_2$) that the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u)\,dx \tag{1.2}$$

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is of class C^1 on the Sobolev space $H_0^1 := H_0^1(\Omega)$ with norm

$$\|u\| := \left(\int_\Omega |\nabla u|^2 \right)^{1/2}, \tag{1.3}$$

and the critical points of J are solutions of (1.1). Thus we will try to find critical points of J. In doing so, we have to prove that the functional J satisfies the (PS) condition.

We denote by $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_i \le \cdots$ the eigenvalues of $(-\Delta, H_0^1)$ with eigenfunctions ϕ_i. If ℓ is an eigenvalue of $(-\Delta, H_0^1(\Omega))$, then the problem is *resonant* at infinity. This case is more delicate. To ensure that J satisfies the (PS) condition usually one needs to assume additional conditions, such as the well-known Landesman-Lazer condition, see, for example, [3, 4]; the angle condition at infinity, see [2].

Recently, in the case $0 \le \mu < \lambda_1 < \ell$, Zhou [12] obtained a positive solution of problem (1.1) under (f_2) and the following conditions:

(H_1) $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$; $f(x,t) \ge 0$, for all $t \ge 0$, $x \in \Omega$ and $f(x,t) \equiv f(x,0) \equiv 0$, for all $t \le 0$, $x \in \Omega$.

(H_2) $(f(x,t)/t)$ is nondecreasing with respect to $t \ge 0$, a.e. on $x \in \Omega$.

Note that our assumption (f_1) is weaker than (H_1). And condition (H_2) is a strong assumption.

In this paper, we prove that (f_1) and (f_2) are sufficient to obtain a positive solution and a negative solution of problem (1.1). Our main result is the following.

Theorem 1.1. *Assume that f satisfies (f_1) and (f_2). If $\mu < \lambda_1 < \ell$, then problem (1.1) has at least two nontrivial solutions, one is positive, the other is negative.*

Note that in Theorem 1.1, even in the resonant case, we do not need to assume any additional conditions to ensure that J satisfies the (PS) condition. Thus Theorem 1.1 greatly improves previous results, such as Zhou's [12]. This fact is interesting. The proof of Theorem 1.1 will be stated in Section 2.

We can also consider the asymptotically linear Dirichlet problem for the p-Laplacian

$$\begin{aligned} -\Delta_p u &= f(x,u), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.4}$$

where $1 < p < +\infty$. Let $0 < \lambda_1^p < \lambda_2^p \le \lambda_3^p \le \cdots$ be the sequence of variational eigenvalues of the eigenvalue problem

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{1.5}$$

It is known that $-\triangle_p$ has a smallest eigenvalue (see [5]), that is, the principle eigenvalue, λ_1^p, which is simple and has an associated eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega) \cap C^1(\Omega)$ that is strictly positive in Ω and $\int_\Omega \varphi_1^p = 1$. λ_1^p is defined as

$$\lambda_1^p = \min\left\{ \int_\Omega |\nabla u|^p : u \in W_0^{1,p}(\Omega), \int_\Omega |u|^p = 1 \right\}. \tag{1.6}$$

Assuming (f_1) and the following condition:

(f_2') $\lim_{|t|\to 0}(f(x,t)/|t|^{p-2}t) = \mu$, $\lim_{|t|\to\infty}(f(x,t)/|t|^{p-2}t) = \ell$ uniformly in $x \in \Omega$,

we obtain the following theorem.

Theorem 1.2. *Assume that f satisfies (f_1) and (f_2'). If $\mu < \lambda_1^p < \ell$, then problem (1.4) has at least two nontrivial solutions, one is positive, the other is negative.*

Remark 1.3. (1) The existence of a positive solution of problem (1.4) was obtained by Li and Zhou [7, Theorem 1.1], under (H_1), (f_2') with $\mu = 0$ and

(H_2') $(f(x,t)/t^{p-1})$ is nondecreasing in $t > 0$, for $x \in \Omega$.

Condition (H_2) is a strong assumption. Moreover, if ℓ is an eigenvalue of (1.5), they need another condition

(f F) $\lim_{t\to\infty}\{f(x,t)t - pF(x,t)\} = +\infty$ uniformly a.e. $x \in \Omega$

to produce a positive solution. Thus Theorem 1.2 extends [7, Theorem 1.1] greatly.

(2) Obviously, Theorem 1.1 is a special case of Theorem 1.2. But we would rather state the proof of Theorem 1.1 separately, because the proof is very simple and clear.

2. Proof of Theorem 1.1

In this section, we will always assume that (f_1) and (f_2) hold and give the proof of Theorem 1.1.

Consider the following problem:

$$\begin{aligned} -\Delta u &= f_+(x,u), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{2.1}$$

where

$$f_+(x,t) = \begin{cases} f(x,t), & t \geq 0, \\ 0, & t \leq 0. \end{cases} \tag{2.2}$$

Define a functional $J_+ : H_0^1 \to \mathbb{R}$,

$$J_+(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F_+(x,u)\,dx, \tag{2.3}$$

where $F_+(x,t) = \int_0^t f_+(x,s)\,ds$. We know $J_+ \in C^1(H_0^1, \mathbb{R})$.

Lemma 2.1. *J_+ satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset H_0^1$ be a sequence such that

$$|J_+(u_n)| \leq c, \quad J'_+(u_n) \longrightarrow 0. \tag{2.4}$$

It is easy to see that

$$|f_+(x,u)u| \leq C(1+|u|^2). \tag{2.5}$$

Now (2.4) implies that for all $\phi \in H_0^1$

$$\int_\Omega (\nabla u_n \nabla \phi - f_+(x,u_n)\phi)\,dx \longrightarrow 0. \tag{2.6}$$

Set $\phi = u_n$, we have

$$\begin{aligned} \|u_n\|^2 &= \int_\Omega f_+(x,u_n)u_n\,dx + \langle J'_+(u_n), u_n\rangle \\ &\leq \int_\Omega f_+(x,u_n)u_n\,dx + o(1)\|u_n\| \\ &\leq C + C\|u_n\|_2^2 + o(1)\|u_n\|, \end{aligned} \tag{2.7}$$

where $\|\cdot\|_2$ is the standard norm in $L^2 := L^2(\Omega)$. We claim that $\|u_n\|_2$ is bounded. For otherwise, we may assume that $\|u_n\|_2 \to +\infty$. Let $v_n = u_n/\|u_n\|_2$, then $\|v_n\|_2 = 1$. Moreover, from (2.7) we have

$$\|v_n\|^2 \leq o(1) + C + \frac{o(1)}{\|u_n\|_2} \cdot \frac{\|u_n\|}{\|u_n\|_2} = o(1) + C + o(1)\|v_n\|. \tag{2.8}$$

That is, $\|v_n\|$ is bounded. So, up to a subsequence, we have

$$v_n \rightharpoonup v \text{ in } H_0^1, \quad v_n \longrightarrow v \text{ in } L^2, \quad \text{for some } v \text{ with } \|v\|_2 = 1. \tag{2.9}$$

From (2.6) it follows that

$$\int_\Omega (\nabla v \nabla \phi - \ell v^+ \phi)\,dx = 0, \quad \forall \phi \in H_0^1, \tag{2.10}$$

where $v^+ = \max\{0, v\}$. From this and the regularity theory we have

$$\begin{aligned} -\Delta v &= \ell v^+, \quad x \in \Omega, \\ v &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{2.11}$$

The maximum principle implies that $v = v^+ \geq 0$. But $\ell > \lambda_1$ and hence $v \equiv 0$ which contradicts with $\|v\|_2 = 1$.

Since $\|u_n\|_2$ is bounded, from (2.7) we get the boundness of $\|u_n\|$. A standard argument shows that $\{u_n\}$ has a convergent subsequence. Therefore, J_+ satisfies the (PS) condition. □

LEMMA 2.2. *Let $\phi_1 > 0$ be a λ_1-eigenfunction of $(-\Delta, H_0^1)$ with $\|\phi_1\| = 1$, if $\mu < \lambda_1 < \ell$, then*

(a) *there exist $\rho, \beta > 0$ such that $J_+(u) \geq \beta$ for all $u \in H_0^1$ with $\|u\| = \rho$;*
(b) *$J_+(t\phi_1) \to -\infty$ as $t \to +\infty$.*

Proof. See the proof of [12, Lemma 2.5]. □

Now, we are in a position to state the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemmas 2.1, 2.2, and the Mountain Pass Theorem [9, Theorem 2.2], the functional J_+ has a critical point u_+ with $J_+(u_+) \geq \beta$. But $J_+(0) = 0$, that is, $u_+ \neq 0$. Then u_+ is a nontrivial solution of (2.1). From the strong maximum principle, $u_+ > 0$. Hence u_+ is also a positive solution of (1.1).

Similarly, we obtain a negative solution u_- of (1.1).

The proof is completed. □

Remark 2.3. If we assume further that $f \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$ and ℓ is not an eigenvalue of $(-\Delta, H_0^1)$, that is, $\ell \in (\lambda_i, \lambda_{i+1})$ for some $i \geq 2$. Then the functional J defined in Section 1 satisfies the (PS) condition. Using Morse theory, we can prove that problem (1.1) has one more nontrivial solution u with $C_i(J, u) \neq 0$, where $C_i(J, u)$ is the ith *critical group* of J at u.

Remark 2.4. If we assume that $\mu = \mu(x)$, $\ell = \ell(x)$, and $\mu(x) < \lambda_1$, $\ell(x) \in L^\infty(\Omega)$, $\ell(x) \geq \lambda_1$, $\operatorname{mes}\{x \in \Omega : \ell(x) > \lambda_1\} > 0$, then the conclusion of Theorem 1.1 is valid too. Since under this assumption, by (2.11) we can get $\lambda_1 \int_\Omega v\phi_1 = \int_\Omega \nabla v \nabla \phi_1 = \int_\Omega \ell(x) v \phi_1$, thus $v \equiv 0$.

3. Proof of Theorem 1.2 and final remarks

In this section, we sketch the proof of Theorem 1.2 and give some remarks. First, we recall the concept Fučik spectrum and a related result.

The *Fučik spectrum* of p-Laplacian with Dirichlet boundary condition is defined as the set Σ_p of those $(a, d) \in \mathbb{R}^2$ such that

$$
\begin{gathered}
-\Delta_p u = a(u_+)^{p-1} - d(u_-)^{p-1}, \quad x \in \Omega, \\
u = 0, \quad x \in \partial\Omega,
\end{gathered}
\tag{3.1}
$$

has a nontrivial solution, where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. By [5], we know that if $(a, d) \in \Sigma_p$ and $(a, d) \notin \mathbb{R} \times \lambda_1^p$, $(a, d) \notin \lambda_1^p \times \mathbb{R}$, then $a > \lambda_1^p$, $d > \lambda_1^p$. We will also need the following lemma, which is due to Zhang and Li [11, Lemma 3].

Lemma 3.1. *Assume that* $h \in C(\Omega \times \mathbb{R}, \mathbb{R})$,

$$\lim_{t\to+\infty} \frac{h(t)}{|t|^{p-2}t} = a, \qquad \lim_{t\to-\infty} \frac{h(t)}{|t|^{p-2}t} = d. \tag{3.2}$$

If $(a,d) \notin \Sigma_p$, *then the functional* $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$\varphi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega H(u)\,dx \tag{3.3}$$

satisfies the (PS) condition, where $H(u) = \int_0^u h(t)\,dt$.

Sketch of the proof of Theorem 1.2. As in Section 2, consider the trancated problem

$$\begin{aligned} -\Delta_p u &= f_+(x,u), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{3.4}$$

where f_+ is defined as in (2.2). Due to the maximum principle (see [10]), solutions of (3.4) are positive, thus are solutions of (1.4). We have

$$\lim_{t\to-\infty} \frac{f_+(x,t)}{|t|^{p-2}t} = 0, \qquad \lim_{t\to+\infty} \frac{f_+(x,t)}{|t|^{p-2}t} = \ell. \tag{3.5}$$

Since $\ell > \lambda_1^p$, one deduces directly from the definition of Fučik spectrum that $(\ell, 0) \notin \Sigma_p$, thus by Lemma 3.1, we deduce that the C^1-functional

$$J_+(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega F_+(x,u)\,dx \tag{3.6}$$

satisfies the (PS) condition on the Sobolev space $W_0^{1,p}(\Omega)$ with norm

$$\|u\|_{1,p} = \left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p}, \tag{3.7}$$

where $F_+(x,t) = \int_0^t f_+(x,s)\,ds$.

As [7, Lemma 2.3], the functional J_+ admits the "Mountain Pass Geometry." Thus J_+ has a nonzero critical point, which is a nontrivial solution of (3.4). From the strong maximum principle (see [10]), it is also a positive solution of (1.4).

Similarly, we obtain a negative solution of (1.4). □

Remark 3.2. Problems (1.1) and (1.4) can be resonant at infinity, this is the main difficulty in verifying the (PS) condition. But after trancating, the problems are not resonant with respect to the Fučik spectrum. Thus, from the Fučik spectrum point of view, the corresponding functionals of the trancated problems satisfies the (PS) condition naturally. And our limit conditions at zero allow us to use the trancation technique and apply the Mountain Pass Theorem.

These are the main ingredient of this work.

Remark 3.3. In fact, let $P := \{u \in H_0^1 : u(x) \geq 0, \text{ a.e.}\}$, the functional J does not satisfies the (PS) condition on the whole space H_0^1 whenever $\ell = \lambda_i$, $i > 1$, but from our proof J satisfies the (PS) condition on P. That is, the unbounded (PS) sequences do not belong to P. This idea may be used to weaken the compact conditions for other problems.

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