ON NEUMANN HEMIVARIATIONAL INEQUALITIES

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We derive a nontrivial solution for a Neumann noncoercive hemivariational inequality using the critical point theory for locally Lipschitz functionals. We use the Mountain-Pass theorem due to Chang (1981).

1. Introduction

The problem under consideration is a hemivariational inequality of Neumann type. Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^1-boundary Γ. We have

$$\begin{aligned} &-\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) \in \partial j_1\left(z,x(z)\right) && \text{a.e. on } Z, \\ &-\frac{\partial x}{\partial n_p} \in \partial j_2\left(z,\tau\left(x(z)\right)\right) && \text{a.e. on } \Gamma,\ 2 \leq p < \infty. \end{aligned} \tag{1.1}$$

Here the boundary condition is in the sense of Kenmochi [7] and τ is the trace operator (see Kenmochi [7, page 123]).

The study of hemivariational inequalities has been initiated and developed by Panagiotopoulos [8]. Such inequalities arise in physics when we have nonconvex, nonsmooth energy functionals. For applications, one can see [9].

Many authors studied Dirichlet hemivariational inequalities (cf. Gasiński and Papageorgiou [5], Goeleven et al. [6], and others). Here we are interested in finding nontrivial solutions for Neumann hemivariational inequalities.

In Section 2, we recall some facts and definitions from the critical point theory for locally Lipschitz functionals and the subdifferential of Clarke.

2. Preliminaries

Let X be a Banach space and let Y be a subset of X. A function $f : Y \to \mathbb{R}$ is said to satisfy a Lipschitz condition (on Y) provided that, for some nonnegative

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scalar K, we have

$$\left|f(y)-f(x)\right| \leq K\|y-x\| \tag{2.1}$$

for all points $x, y \in Y$. Let f be a Lipschitz function near a given point x, and let v be any other vector in X. The generalized directional derivative of f at x in the direction v, denoted by $f^o(x;v)$ is defined as follows:

$$f^o(x;v) = \limsup_{\substack{y\to x\\ t\downarrow 0}} \frac{f(y+tv)-f(y)}{t}, \tag{2.2}$$

where y is a vector in X and t a positive scalar. If f is a Lipschitz function of rank K near x, then the function $v \to f^o(x;v)$ is finite, positively homogeneous, subadditive, and satisfies $|f^o(x;v)| \leq K\|v\|$. In addition, f^o satisfies $f^o(x;-v) = (-f)^o(x;v)$. Now we are ready to introduce the generalized gradient which is denoted by $\partial f(x)$ as follows:

$$\partial f(x) = \left\{w \in X^* : f^o(x;v) \geq \langle w, v\rangle \ \forall v \in X\right\}. \tag{2.3}$$

Some basic properties of the generalized gradient of locally Lipschitz functionals are the following:

(a) $\partial f(x)$ is a nonempty, convex, weakly compact subset of X^* and $\|w\|_* \leq K$ for every w in $\partial f(x)$;
(b) for every v in X, we have

$$f^o(x;v) = \max\left\{\langle w, v\rangle : w \in \partial f(x)\right\}. \tag{2.4}$$

If f_1, f_2 are locally Lipschitz functions then

$$\partial(f_1+f_2) \subseteq \partial f_1 + \partial f_2. \tag{2.5}$$

Recall the Palais-Smale condition ((PS)-condition) introduced by Chang [2].

Definition 2.1. We say that a Lipschitz function f satisfies the (PS)-condition if for any sequence $\{x_n\}$, $|f(x_n)|$ is bounded and $\lambda(x_n) = \min_{w\in\partial f(x_n)} \|w\|_{X^*} \to 0$ possesses a convergent subsequence.

The (PS)-condition can also be formulated as follows (see [4]).

$(\mathrm{PS})^*_{c,+}$: whenever $(x_n) \subseteq X$, $(\varepsilon_n), (\delta_n) \subseteq \mathbb{R}_+$ are sequences with $\varepsilon_n \to 0$, $\delta_n \to 0$, and such that

$$f(x_n) \longrightarrow c, \quad f(x_n) \leq f(x) + \varepsilon_n \left\|x - x_n\right\| \quad \text{if } \left\|x - x_n\right\| \leq \delta_n, \tag{2.6}$$

then (x_n) possesses a convergent subsequence $x_{n'} \to \hat{x}$.

Similarly, we define the $(\mathrm{PS})_c^*$ condition from below, $(\mathrm{PS})_{c,-}^*$, by interchanging x and x_n in inequality (2.6). And finally we say that f satisfies $(\mathrm{PS})_c^*$ provided that it satisfies $(\mathrm{PS})_{c,+}^*$ and $(\mathrm{PS})_{c,-}^*$. Note that these two definitions are equivalent when f is a locally Lipschitz functional.

The following theorem is the Mountain-Pass theorem for locally Lipschitz functionals.

Theorem 2.2. *If a locally Lipschitz functional $f : X \to \mathbb{R}$ on the reflexive Banach space X satisfies the* (PS)*-condition and the hypotheses:*

(i) *there exist positive constants ρ and a such that*

$$f(u) \geq a \quad \forall x \in X \text{ with } \|x\| = \rho; \tag{2.7}$$

(ii) $f(0) = 0$ *and there exists a point $e \in X$ such that*

$$\|e\| > \rho, \quad f(e) \leq 0, \tag{2.8}$$

then there exists a critical value $c \geq a$ of f determined by

$$c = \inf_{g \in G} \max_{t \in [0,1]} f\big(g(t)\big), \tag{2.9}$$

where

$$G = \big\{g \in C\big([0,1], X\big) : g(0) = 0,\ g(1) = e\big\}. \tag{2.10}$$

In what follows, we will use the well-known inequality

$$\sum_{j=1}^{N} \big(a_j(\eta) - a_j(\eta')\big)\big(\eta_j - \eta'_j\big) \geq C|\eta - \eta'|^p \tag{2.11}$$

for $\eta, \eta' \in \mathbb{R}^N$, with $a_j(\eta) = |\eta|^{p-2}\eta_j$.

3. Existence theorem

Let $X = W^{1,p}(Z)$. Our hypotheses on j_1, j_2 are the following:

H(j_1): the map $j_1 : Z \times \mathbb{R} \to \mathbb{R}$ is such that $z \to j_1(z,x)$ is measurable and $x \to j_1(z,x)$ is locally Lipschitz;

(i) for almost all $z \in Z$, all $x \in \mathbb{R}$, and all $v \in \partial j_1(z,x)$, we have $|v(z)| \leq c_1|x|^{p-1} + c_2|x|^{p^*-1}$;

(ii) there exists $\theta > p$ and $r_o > 0$ such that for all $|x| \geq r_o$, and $v \in \partial j_1(z,x)$, we have $0 < \theta j_1(z,x) \leq vx$, and moreover, there exists some $a \in L^1(Z)$ such that $j_1(z,x) \geq c_3|x|^{\theta} - a(z)$ for every $x \in \mathbb{R}$;

(iii) uniformly for almost all $z \in Z$, we have

$$\limsup_{x \to 0} \frac{j_1(z,x)}{|x|^p} \leq \theta(z) \tag{3.1}$$

with $\theta(z) \in L^\infty$ and $\theta(z) \leq 0$ with strict inequality in a set of positive measure.

H(j_2): the map $j_2(z,x)$ is such that $z \to j_2(z,x)$ is measurable and $j_2(z,\cdot)$ is a locally Lipschitz function such that for almost all $z \in Z$, $x \in \mathbb{R}$, and $v \in \partial j_2(z,x)$ we have $|v(z)| \le \alpha_1(z) + c_1|x|^\mu$, $0 \le \mu < p-1$ with $\alpha_1 \in L^\infty$, $c_1 > 0$, $j_2(\cdot,0) \in L^\infty(Z)$, and finally $j_2(z,\cdot) \ge 0$ for almost all $z \in Z$.

Theorem 3.1. *If hypotheses* H(j_1) *and* H(j_2) *hold, then problem (1.1) has a nontrivial solution* $x \in W^{1,p}(Z)$.

Proof. Let $\Phi : W^{1,p}(Z) \to \mathbb{R}$ and $\psi : W^{1,p}(Z) \to \mathbb{R}_+$ be defined by

$$\Phi(x) = -\int_Z j_1\big(z,x(z)\big)\,dz, \qquad \psi(x) = \frac{1}{p}\|Dx\|_p^p + \int_\Gamma j_2\big(z,\tau\big(x(z)\big)\big)\,d\sigma. \tag{3.2}$$

Clearly, Φ is locally Lipschitz (see Chang [2]), while we can check that ψ is locally Lipschitz too. Set $R = \Phi + \psi$.

Claim 3.2. *The function* $R(\cdot)$ *satisfies the (PS)-condition (in the sense of Costa and Gonçalves).*

We start with $(\text{PS})_{c,+}$ first. Let $\{x_n\}_{n\ge 1} \subseteq W^{1,p}(Z)$ such that $R(x_n) \to c$ when $n \to \infty$ and

$$R\big(x_n\big) \le R(x) + \varepsilon_n\big\|x - x_n\big\| \quad \text{with} \quad \big\|x - x_n\big\| \le \delta_n. \tag{3.3}$$

The above inequality is equivalent to the following:

$$R(x) - R\big(x_n\big) \ge -\varepsilon_n\big\|x - x_n\big\| \quad \text{with } \big\|x - x_n\big\| \le \delta_n \tag{3.4}$$

where $\varepsilon_n, \delta_n \to 0$. Choose $x = x_n + \delta x_n$ with $\delta\|x_n\| \le \delta_n$. Divide by δ. So, if $\delta \to 0$ we have

$$\lim_{\delta\to 0}\frac{R\big(x_n + \delta x_n\big) - R\big(x_n\big)}{\delta} \le R^o\big(x_n; x_n\big). \tag{3.5}$$

Then we obtain

$$R^o\big(x_n; x_n\big) \ge -\varepsilon_n\big\|x_n\big\|. \tag{3.6}$$

For the $(\text{PS})_{c,-}$ we have the following: let $\{x_n\}_{n\ge 1} \subseteq W^{1,p}(Z)$ such that $R(x_n) \to c$ when $n \to \infty$ and

$$R(x) \le R\big(x_n\big) + \varepsilon_n\big\|x - x_n\big\| \quad \text{with} \quad \big\|x - x_n\big\| \le \delta_n. \tag{3.7}$$

The above inequality is equivalent to the following:

$$0 \le (-R)(x) - (-R)\big(x_n\big) + \varepsilon_n\big\|x - x_n\big\| \quad \text{with} \quad \big\|x - x_n\big\| \le \delta_n. \tag{3.8}$$

Choose here $x = x_n - \delta x_n$ with $\delta\|x_n\| \le \delta_n$. We obtain

$$0 \le (-R)\big(x_n + \delta\big(-x_n\big)\big) - (-R)\big(x_n\big) + \varepsilon_n\delta\big\|x_n\big\|. \tag{3.9}$$

Divide this by δ. In the limit, we have

$$0 \leq \lim_{\delta \to 0} \frac{(-R)(x_n + \delta(-x_n)) - (-R)(x_n)}{\delta} + \varepsilon_n \|x_n\|. \tag{3.10}$$

Note that

$$\lim_{\delta \to 0} \frac{(-R)(x_n + \delta(-x_n)) - (-R)(x_n)}{\delta} \leq (-R)^o(x_n; -x_n) = R^o(x_n; x_n). \tag{3.11}$$

So finally we obtain again (3.6).

Also,

$$\frac{1}{p}\|D(x_n + \delta x_n)\|_p^p - \frac{1}{p}\|Dx_n\| = -\frac{1}{p}\|Dx_n\|_p^p(1 - (1+\delta)^p). \tag{3.12}$$

So if we divide this by δ and let $\delta \to 0$, we have that it is equal to $\|Dx_n\|_p^p$. Finally, there exists $v_n(z) \in \partial\Phi(x_n)$ such that $\langle v_n, x_n \rangle = \Phi^o(x_n; x_n)$ and $w_n \in \partial j_2(z, \tau(x_n(z)))$ such that

$$\langle w_n, x_n \rangle_\Gamma = \psi_1^o(x_n; x_n) \quad \text{with } \psi_1(x) = \int_\Gamma j_2(z, \tau(x(z)))\, d\sigma. \tag{3.13}$$

Note that

$$v_n \in \partial\left(-\int_Z j_1(z, x_n(z))\, dz\right) = -\partial \int_Z j_1(z, x_n(z))\, dz. \tag{3.14}$$

So, from (3.6), it follows that

$$\int_Z v_n x_n(z)\, dz - \|Dx_n\|_p^p - \int_\Gamma w_n x_n\, d\sigma \leq \varepsilon_n \|x_n\|, \tag{3.15}$$

for some $v_n \in \partial(\int_Z j_1(z, x_n(z))\, dz)$.

Suppose that $\{x_n\} \subseteq W^{1,p}(Z)$ was unbounded. Then (at least for a subsequence), we may assume that $\|x_n\| \to \infty$. Let $y_n = x_n/(\|x_n\|)$, $n \geq 1$, and it is easy to see that $\|y_n\| = 1$. By passing to a subsequence if necessary, we may assume that

$$\begin{aligned} y_n &\xrightarrow{w} y && \text{in } W^{1,p}(Z), \\ y_n &\longrightarrow y && \text{in } L^p(Z), \\ y_n(z) &\longrightarrow y(z) && \text{a.e. on } Z \text{ as } n \longrightarrow \infty, \\ |y_n(z)| &\leq k(z) && \text{a.e. on } Z \text{ with } k \in L^p(Z). \end{aligned} \tag{3.16}$$

Recall that from the choice of the sequence $\{x_n\}$, we have $|R(x_n)| \leq M_1$ for some $M_1 > 0$ and all $n \geq 1$,

$$\frac{\theta}{p}\|Dx_n\|_p^p + \theta \int_\Gamma j_2(z, \tau(x_n(z)))\, d\sigma - \theta \int_Z j_1(z, x_n(z))\, dz \leq \theta M_1. \tag{3.17}$$

On the other hand, we have

$$\int_Z v_n x_n(z)\,dz - \|Dx_n\|_p^p - \int_\Gamma w_n x_n\,d\sigma \le \varepsilon_n \|x_n\|. \tag{3.18}$$

Adding inequalities (3.17) and (3.18) we obtain

$$\begin{aligned}\left(\frac{\theta}{p}-1\right)\|Dx_n\|_p^p &+ \int_Z \left(v_n(z)x_n(z) - \theta j_1\left(z, x_n(z)\right)\right) dz \\ &+ \int_\Gamma \left(j_2\left(z, \tau\left(x_n(z)\right)\right) - w_n(z)x_n(z)\right) d\sigma \\ &\le \theta M_1 + \varepsilon_n \|x_n\|.\end{aligned} \tag{3.19}$$

From $H(j_1)$(ii), we have that for all $|x| > M$ and $v \in \partial j_1(z,x)$, $0 < \theta j_1(z,x) \le vx$. From $H(j_1)(i)$, we know that $|v| \le c_1|x|^{p-1} + c_2|x|^{p^*-1}$. Using Lebourg mean value theorem (see Clarke [3, Theorem 2.3.7, page 41]) we have that for all $x \in \mathbb{R}$

$$j_1(z,x) - j_1(z,0) = wx \tag{3.20}$$

with $w \in \partial j_1(z,s)$ where $s \in (0,x)$. Recall that $j_1(z,0) \in L^\infty(Z)$. So

$$\left|j_1(z,x)\right| \le c_1 + c_2|x|^p + c_3|x|^{p^*}. \tag{3.21}$$

So for $|x| \le M$, we have that $|v| \le C$ and $|j_1(z,x)| \le C$ for all $v \in \partial j_1(z,x)$ for some $C > 0$. Thus, there exists some $M > 0$ such that $vx - \theta j_1(z,x) + M \ge 0$ for all $x \in \mathbb{R}$.

Therefore, (3.19) becomes

$$\left(\frac{\theta}{p}-1\right)\|Dx_n\|_p^p + \int_\Gamma \left(j_2\left(z, \tau\left(x_n(z)\right)\right) - w_n(z)x_n(z)\right) d\sigma \le \theta M_1 + \varepsilon_n \|x_n\| + M. \tag{3.22}$$

Dividing by $\|x_n\|^p$, we get

$$\frac{\int_\Gamma \left(j_2\left(z, \tau\left(x_n(z)\right)\right) - w_n(z)x_n(z)\right) d\sigma}{\|x_n\|^p} \longrightarrow 0. \tag{3.23}$$

Indeed, from the Lebourg mean value theorem, we have that for any $x \in \mathbb{R}$,

$$j_2(z,x) - j_2(z,0) = wx, \tag{3.24}$$

with $w \in \partial j_2(z,s)$ where $s \in (0,x)$. From $H(j_2)$, we have that for every $w \in \partial j_2(z,s)$, $|w| \le c_1 + c_2|s|^\mu$. Moreover, note that $j_2(z,0) \in L^\infty$. So,

$$\left|j_2(z,x)\right| \le c_1|x| + c_2|x|^{\mu+1} + c_3. \tag{3.25}$$

Thus,

$$
\begin{aligned}
&\frac{\int_\Gamma \left(j_2(z,\tau(x_n(z))) - w_n(z)x_n(z)\right) d\sigma}{\|x_n\|^p} \\
&\qquad \le \int_\Gamma \frac{c_1 |x_n(z)|}{\|x_n\|^p} d\sigma + \int_\Gamma \frac{c_2 |x_n(z)|^{\mu+1}}{\|x_n\|^p} d\sigma + \frac{c_4}{\|x_n\|^p} \\
&\qquad \le c_1 \frac{\|x_n\|_{L^1(\Gamma)}}{\|x_n\|^p} + c_2 \frac{\|x_n\|^{\mu+1}_{L^{\mu+1}(\Gamma)}}{\|x_n\|^p} + \frac{c_4}{\|x_n\|^p}.
\end{aligned}
\tag{3.26}
$$

Note that

$$
\begin{aligned}
\|x_n\|_{L^1(\Gamma)} &\le K\|x_n\|_{1/q,1,\Gamma} \le C\|x_n\|_{1,p,Z}, \\
\|x_n\|^{\mu+1}_{L^{\mu+1}(\Gamma)} &\le K\|x_n\|^{\mu+1}_{1/q,p,\Gamma} \le C\|x_n\|^{\mu+1}_{1,p,Z},
\end{aligned}
\tag{3.27}
$$

(see Adams [1, page 217]), recall that $\mu+1 < p$. Now we have finished the claim.

Going back to (3.22) we have that $\|Dy_n\| \to 0$. From the weak lower semicontinuity of the norm functional, we have that $\|Dy\| \le \liminf \|Dy_n\| \le \limsup \|Dy_n\| \to 0$. Therefore, we infer that $y_n \to y$ in $W^{1,p}(Z)$ (recall that $y_n \to y$ weakly in $W^{1,p}(Z)$ and $\|Dy_n\| \to \|Dy\| = 0$). So, $y = \xi \in \mathbb{R}$. But, $\|y_n\| = 1$, so $\|y\| = 1$, thus $y = \xi \neq 0$. Suppose that $\xi > 0$, then $x_n(z) \to \infty$. From H(j_1)(ii) we have that, for all $x \in \mathbb{R}$, $j_1(z,x) \ge c_1|x|^\theta - a(z)$. So it is clear that

$$
\begin{aligned}
\frac{R(x_n)}{\|x_n\|^\theta} \le{}& \frac{1}{p}\|Dy_n\|_p^p \frac{1}{\|x_n\|^{\theta-p}} - c_1 \int_Z |y_n(z)|^\theta dz \\
&+ \int_Z \frac{a(z)}{\|x_n\|^\theta} dz + \frac{\int_\Gamma j_2(z,\tau(x_n(z))) d\sigma}{\|x_n\|^\theta}.
\end{aligned}
\tag{3.28}
$$

Recall that from the choice of the sequence we have that

$$
\frac{R(x_n)}{\|x_n\|^\theta} \ge -\frac{M_1}{\|x_n\|^\theta}.
\tag{3.29}
$$

As before it is easy to see that

$$
\frac{\int_\Gamma j_2(z,\tau(x_n(z))) d\sigma}{\|x_n\|^\theta} \longrightarrow 0, \qquad \frac{1}{p}\|Dy_n\|_p^p \frac{1}{\|x_n\|^{\theta-p}} \longrightarrow 0.
\tag{3.30}
$$

So from (3.28) we have that $0 \le -c_1|\xi|^\theta|Z|$. But this is a contradiction. So $\{x_n\} \subseteq W^{1,p}(Z)$ is bounded.

From the properties of the subdifferential of Clarke, we have

$$
\begin{aligned}
\partial R(x_n) &\subseteq \partial\Phi(x_n) + \partial\psi(x_n) \\
&\subseteq \partial\Phi(x_n) + \partial\left(\frac{1}{p}\|Dx_n\|_p^p\right) + \int_\Gamma \partial j_2(z,\tau(x_n(z))) d\sigma
\end{aligned}
\tag{3.31}
$$

(see Clarke [3, page 83]). So we have

$$\langle w_n, y\rangle = \langle Ax_n, y\rangle + \langle \tau(r_n), y\rangle_\Gamma - \int_Z v_n(z)y(z)\,dz \tag{3.32}$$

with $r_n(z) \in \partial j_2(z,\tau(x_n(z)))$, $v_n(z) \in \partial j_1(z,x_n(z))$, and w_n the element with minimal norm of the subdifferential of $\mathbb{R}$ and $A : W^{1,p}(Z) \to W^{1,p}(Z)^*$ such that $\langle Ax, y\rangle = \int_Z (\|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{\mathbb{R}^N})\,dz$. But $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$, so $x_n \to x$ in $L^p(Z)$ and $x_n(z) \to x(z)$ a.e. on Z by virtue of the compact embedding $W^{1,p}(Z) \subseteq L^p(Z)$.

Note that the trace of x_n belongs to $W^{1/q,p}(\Gamma)$, thus, from H(j_2), the trace of $r_n \in L^q(\Gamma)$. Recall that there exists some $K > 0$ such that $\|x_n\|_{1/q,p,\Gamma} \le K\|x_n\|_{1,p,Z}$. Therefore, r_n is bounded in $L^q(\Gamma)$ and moreover, in $(W^{1/q,p}(\Gamma))^*$ (the dual space of $W^{1/q,p}(\Gamma)$). Choose $y = x_n - x$, then we obtain

$$\left|\langle \tau(r_n), x_n - x\rangle_\Gamma\right| \longrightarrow 0. \tag{3.33}$$

With $\langle\cdot,\cdot\rangle_\Gamma$ we denote the natural pairing of $(W^{1/q,p}(\Gamma), (W^{1/q,p}(\Gamma))^*)$.

Then in the limit we have that $\limsup\langle Ax_n, x_n - x\rangle = 0$ (note that v_n is bounded in $L^{p^*}(Z)$). By virtue of inequality (2.11), we have that $Dx_n \to Dx$ in $L^p(Z)$. So we have $x_n \to x$ in $W^{1,p}(Z)$. The claim is proved. □

For every $\xi \in \mathbb{R}$, $\xi \neq 0$, we have

$$\begin{aligned} R(\xi) = \int_\Gamma j_2(z,\xi)\,d\sigma - \int_Z j_1(z,\xi)\,dz &\Longrightarrow \frac{1}{|\xi|^\theta}R(\xi) \\ &\le \frac{1}{|\xi|^\theta}\int_\Gamma j_2(z,\xi)\,d\sigma - \frac{1}{|\xi|^\theta}\int_Z j_1(z,\xi)\,dz. \end{aligned} \tag{3.34}$$

As before we show that

$$-\frac{1}{|\xi|^\theta}\int_Z j_1(z,\xi)\,dz \le -c_1\frac{|\xi|^\theta}{|\xi|^\theta}, \qquad \frac{1}{|\xi|^\theta}\int_\Gamma j_2(z,\xi)\,d\sigma \longrightarrow 0. \tag{3.35}$$

Thus $R(\xi) \to -\infty$ as $|\xi| \to \infty$.

In order to use the Mountain-Pass theorem, it remains to show that there exists $\rho > 0$ such that for $\|x\| = \rho$, $R(x) \ge a > 0$. In fact, we will show that for every sequence $\{x_n\} \subseteq W^{1,p}(Z)$ with $\|x_n\| = \rho_n \downarrow 0$, $R(x_n) > 0$. Indeed, suppose not. Then there exists some sequence $\{x_n\}$ such that $R(x_n) \le 0$. Thus,

$$\frac{1}{p}\|Dx_n\|_p^p \le \int_Z j_1(z,x_n(z))\,dz, \tag{3.36}$$

recall that $j_2 \ge 0$. Dividing this inequality by $\|x_n\|^p$ and letting $y_n(z) = x_n(z)/\|x_n\|$, then

$$\|Dy_n\|_p^p \le \int_Z p\frac{j_1(z,x_n(z))}{\|x_n\|^p}\,dz. \tag{3.37}$$

From H(j_1)(iii) we have that for almost all $z \in Z$, for any $\varepsilon > 0$, we can find $\delta > 0$ such that for $|x| \le \delta$,

$$p j_1(z,x) \le (\theta(z)+\varepsilon)|x|^p. \tag{3.38}$$

On the other hand, as before for almost all $z \in Z$ and all $|x| \ge \delta$, we have

$$p\left|j_1(z,x)\right| \le c_1|x|^p + c_2|x|^{p^*}. \tag{3.39}$$

Thus we can always find $\gamma > 0$ such that $p|j_1(z,x)| \le (\theta(z)+\varepsilon)|x|^p + \gamma|x|^{p^*}$ for all $x \in \mathbb{R}$. Indeed, choose $\gamma \ge c_2 + |\theta(z)+\varepsilon - c_1||\delta|^{p-p^*}$, we obtain

$$\begin{aligned} \|Dy_n\|_p^p &\le \int_Z (\theta(z)+\varepsilon)\left|y_n(z)\right|^p dz + \gamma\int_Z \frac{\left|x_n(z)\right|^{p^*}}{\|x_n\|^p}\,dz \\ &\le \int_Z (\theta(z)+\varepsilon)\left|y_n(z)\right|^p dz + \gamma_1\|x_n\|^{p^*-p}. \end{aligned} \tag{3.40}$$

Here we have used the fact that $W^{1,p}(Z)$ embeds continuously in $L^{p^*}(Z)$. So

$$0 \le \|Dy_n\|_p^p \le \varepsilon\|y_n\|_p^p + \gamma_1\|x_n\|^{p^*-p}. \tag{3.41}$$

Therefore, in the limit we have that $\|Dy_n\|_p \to 0$. Recall that $y_n \to y$ weakly in $W^{1,p}(Z)$. So $\|Dy\|_p \le \liminf\|Dy_n\|_p \le \limsup\|Dy_n\|_p \to 0$. So $\|Dy\|_p = 0$, thus $y = \xi \in \mathbb{R}$. Note that $Dy_n \to Dy$ weakly in $L^p(Z)$ and $\|Dy_n\|_p \to \|Dy\|_p$ so $y_n \to y$ in $W^{1,p}(Z)$. Since $\|y_n\| = 1$, $\|y\| = 1$ so $\xi \ne 0$. Suppose that $\xi > 0$. Going back to (3.40), we have

$$0 \le \int_Z (\theta(z)+\varepsilon)\,y_n^p(z)\,dz + \gamma_1\|x_n\|^{p^*-p}. \tag{3.42}$$

In the limit we have

$$0 \le \int_Z (\theta(z)+\varepsilon)\xi^p\,dz \le \varepsilon\xi^p|Z|, \tag{3.43}$$

recall that $\theta(z) \le 0$. Thus $\int_Z \theta(z)\xi^p\,dz = 0$. But this is a contradiction. So the claim is proved.

By Theorem 2.2, there exists $x \in W^{1,p}(Z)$ such that $0 \in \partial R(x)$. That is, $0 \in \partial\Phi(x) + \partial\psi(x)$. So, we can say that

$$\int_Z w(z)y(z) = \int_Z \|Dx(z)\|^{p-2}(Dx(z),Dy(z))\,dz + \int_\Gamma v(z)x(z)\,d\sigma \tag{3.44}$$

for some $w \in L^{q^*}(Z)$ such that $w(z) \in \partial j_1(z,x(z))$ for some $v \in \partial j_2(z,\tau(x(z)))$ and for every $y \in W^{1,p}(Z)$. Choose now $y = s \in C_o^\infty(Z)$, we obtain

$$\int_Z w(z)s(z) = \int_Z \|Dx(z)\|^{p-2}(Dx(z),Ds(z))\,dz. \tag{3.45}$$

But $\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in L^{q^*}(Z)$ because $w \in L^{q^*}(Z)$ (see Kenmochi [7, Proposition 3.1, page 132]).

Going back to (3.44) and letting $y = C^\infty(Z)$ and finally using [7, the Green formula 1.6], we have that $-\partial x/\partial n_p \in \partial j_2(z, \tau(x(z)))$.

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