PERTURBATIONS NEAR RESONANCE FOR THE p-LAPLACIAN IN $\mathbb{R}^N$

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We study a multiplicity result for the perturbed p-Laplacian equation $-\Delta_p u - \lambda g(x)|u|^{p-2}u = f(x,u) + h(x)$ in $\mathbb{R}^N$, where $1 < p < N$ and λ is near λ_1, the principal eigenvalue of the weighted eigenvalue problem $-\Delta_p u = \lambda g|u|^{p-2}u$ in $\mathbb{R}^N$. Depending on which side λ is from λ_1, we prove the existence of one or three solutions. This kind of results was firstly obtained by Mawhin and Schmitt (1990) for a semilinear two-point boundary value problem.

1. Introduction

In this paper, we study a class of p-Laplacian equations of the form

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u + f(x,u) + h(x) \quad \text{in } D^{1,p}(\mathbb{R}^N), \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, and $g \geq 0$ is a weight function. Here, $D^{1,p}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{1,p}} = \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right)^{1/p}. \tag{1.2}$$

This space, which is motivated by the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, where $p^* = Np/(N-p)$, is in fact a reflexive Banach space characterized by

$$D^{1,p}(\mathbb{R}^N) = \left\{u \in L^{p^*}(\mathbb{R}^N);\ \frac{\partial u}{\partial x_i} \in L^p(\mathbb{R}^N),\ 1 \leq i \leq N\right\}. \tag{1.3}$$

We refer the reader to Ben-Naoum et al. [5] for a quite complete discussion on the space $D^{1,p}(\mathbb{R}^N)$.

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Our study is based on a bifurcation result by Mawhin and Schmitt [15], related to the two-point boundary value problem,

$$-u'' - \lambda u = f(x,u) + h, \quad u(0) = u(\pi) = 0. \tag{1.4}$$

By assuming that f is bounded and satisfying a sign condition, they obtained the following result. If λ is sufficiently near to λ_1 from left, where $\lambda_1 = 1$ is the first eigenvalue of the corresponding linear problem, then (1.4) has at least three solutions. If $1 \leq \lambda < 4$, then problem (1.4) has at least one solution. Some extensions and variations of their result were considered by other authors (cf. Badiale and Lupo [3], Chiappinelli et al. [7], Sanchez [18], and Ma et al. [13]). In [14], the multiplicity part of that result was extended to the p-Laplacian operator in bounded domains, using critical point theory. Our objective is to extend this problem to the p-Laplacian in $\mathbb{R}^N$, with λ approaching to λ_1 from left and from right.

In order to state the Mawhin-Schmitt problem in the context of $D^{1,p}(\mathbb{R}^N)$, we recall some facts about the eigenvalue problem for the weighted p-Laplacian in $\mathbb{R}^N$

$$-\Delta_p u = \lambda g|u|^{p-2}u \quad \text{in } D^{1,p}(\mathbb{R}^N), \tag{1.5}$$

where $g \in L^\infty(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N)$ is a locally Hölder continuous weight function. It is known that for $g \geq 0$, there exists a first eigenvalue $\lambda_1 = \lambda_1(g)$, characterized by

$$\lambda_1 = \inf\left\{\|u\|^p_{D^{1,p}};\ u \in D^{1,p}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} g|u|^p\,dx = 1\right\}, \tag{1.6}$$

which is simple and positive. This implies that

$$\int_{\mathbb{R}^N} |\nabla u|^p\,dx \geq \lambda_1 \int_{\mathbb{R}^N} g|u|^p\,dx \quad \forall u \in D^{1,p}(\mathbb{R}^N). \tag{1.7}$$

Besides, the corresponding eigenfunction φ_1 belongs to $D^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and may be taken positive (see a complete proof in [12]). Putting

$$W = \left\{w \in D^{1,p}(\mathbb{R}^N);\ \int_{\mathbb{R}^N} g|\varphi_1|^{p-2}\varphi_1 w\,dx = 0\right\} \tag{1.8}$$

and $V = \operatorname{span}\{\varphi_1\}$, we have from the simplicity of λ_1,

$$D^{1,p}(\mathbb{R}^N) = V \oplus W. \tag{1.9}$$

Then, since λ_1 is also isolated (see [11]), we have

$$\lambda_2 := \inf\left\{\|w\|^p_{D^{1,p}};\ w \in W,\ \int_{\mathbb{R}^N} g|w|^p\,dx = 1\right\}, \tag{1.10}$$

which satisfies $\lambda_1 < \lambda_2$. In addition,

$$\int_{\mathbb{R}^N} |\nabla w|^p \, dx \geq \lambda_2 \int_{\mathbb{R}^N} g|w|^p \, dx \quad \forall w \in W. \tag{1.11}$$

Next, we make some basic assumptions on the function f. We assume that $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the growth condition

$$|f(x,u)| \leq a(x)|u|^{\sigma-1} + b(x), \tag{1.12}$$

with $1 < \sigma < p$, $a \geq 0$, $a \in L^\infty(\mathbb{R}^N) \cap L^{(p^*/\sigma)'}(\mathbb{R}^N)$, and $b \in L^{p^{*\prime}}(\mathbb{R}^N)$. Some of our hypotheses are given upon the primitive $F(x,u) = \int_0^u f(x,s)\, ds$, namely, there exists $\gamma \in L^1(\mathbb{R}^N)$ such that

$$F(x,u) \geq \gamma(x) \quad \text{a.e. in } \mathbb{R}^N, \ \forall u \in \mathbb{R}. \tag{1.13}$$

We also consider the following: there exist $\alpha \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \cap L^{(p^*/\mu)'}(\mathbb{R}^N)$ and $\beta \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \cap L^{p^{*\prime}}(\mathbb{R}^N)$ satisfying

$$pF(x,u) - f(x,u)u \geq \alpha(x)|u|^\mu + \beta(x) \quad \text{a.e. in } \mathbb{R}^N, \ \forall u \in \mathbb{R}, \tag{1.14}$$

and $1 < \mu \leq \sigma < p$.

Now we are in a position to state our results.

Theorem 1.1. *Assume that (1.12) and (1.13) hold. If in addition*

$$\lim_{|u|\to\infty} F(x,u) = +\infty \quad \textit{a.e. in } \mathbb{R}^N, \tag{1.15}$$

then for any $h \in L^{p^{*\prime}}(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} h(x)\varphi_1(x)\, dx = 0, \tag{1.16}$$

problem (1.1) has at least three solutions when λ is sufficiently close to λ_1 from left.

Theorem 1.2. *Assume that (1.12) and (1.14) hold with $\alpha \geq \max\{a,g\}$. Assume further that $\lambda_1 \leq \lambda < \lambda_2$. Then for any $h \in L^{p^{*\prime}}(\mathbb{R}^N)$ satisfying $|h| \leq \alpha$, problem (1.1) has at least one solution.*

Since (1.14) implies (1.15), under the hypotheses of Theorem 1.2, we get an extension of the original work of Mawhin and Schmitt [15] to the p-Laplacian in $\mathbb{R}^N$. We note that our results do not assume f bounded nor satisfying a sign condition. Theorem 1.2 is related to a class of double resonance problems introduced in [6] for semilinear elliptic equations. It was not considered for the p-Laplacian, even in bounded domains. Condition (1.14) was early used in [1, 8, 10] for example, as an Ambrosetti-Rabinowitz type condition [2]. A simple example of g satisfying all the hypotheses of both theorems is

$$f(x,u) = \sigma a(x)|u|^{\sigma-2}u, \tag{1.17}$$

where $a \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \cap L^{(p^*/\sigma)'}(\mathbb{R}^N)$ and $1 < \mu = \sigma < p-1$.

The proofs of the theorems are given in Section 3. In Section 2, we present some preliminary results on the variational setting of the p-Laplacian equations in $D^{1,p}(\mathbb{R}^N)$ and the related Palais-Smale compactness.

2. Preliminaries

We begin with some standard facts upon the variational formulation of problem (1.1). Let $J_\lambda : D^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ be the functional defined by

$$J_\lambda(u) = \int_{\mathbb{R}^N} \left[\frac{1}{p} |\nabla u(x)|^p - \frac{\lambda}{p} g(x) |u(x)|^p - F(x,u(x)) - h(x)u(x) \right] dx. \tag{2.1}$$

It is proved in do Ó [10], that J_λ is of class $C^1(\mathbb{R}^N)$ and

$$\begin{aligned} \langle J'_\lambda(u), \varphi \rangle = & \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx - \lambda \int_{\mathbb{R}^N} g|u|^{p-2} u\varphi \, dx \\ & - \int_{\mathbb{R}^N} f(x,u)\varphi \, dx - \int_{\mathbb{R}^N} h\varphi \, dx, \end{aligned} \tag{2.2}$$

for all $\varphi \in D^{1,p}(\mathbb{R}^N)$. In addition, the critical points of J_λ are precisely the weak solutions of (1.1).

Next we recall a compactness result which is proved in [5].

Lemma 2.1 (see [5]). *The functional*

$$u \longmapsto \int_{\mathbb{R}^N} m(x) |u(x)|^q \, dx \tag{2.3}$$

is well defined and weakly continuous in $D^{1,p}(\mathbb{R}^N)$*, for* $1 \le q < p^*$ *and* $m \in L^{(p^*/q)'}(\mathbb{R}^N)$.

As a consequence, under the conditions of the lemma, there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} m(x) |u(x)|^q \, dx \le \|m\|_{L^{(p^*/q)'}} \|u\|^q_{L^{p^*}} \le C\|u\|^q_{D^{1,p}}. \tag{2.4}$$

Lemma 2.2. *Assume that (1.12) holds. Then the Nemytskii mapping,*

$$u \longmapsto f(x,u) \tag{2.5}$$

is compact from $D^{1,p}(\mathbb{R}^N)$ *to* $L^{p^{*\prime}}(\mathbb{R}^N)$.

Proof. Put $r = p^{*\prime}$ and $q = (\sigma - 1)r$ so that $q < p^*$ and

$$\left(\frac{p^*}{\sigma}\right)' = r\left(\frac{p^*}{q}\right)'. \tag{2.6}$$

Then we get from (1.12) that $a^r \in L^{(p^*/q)'}(\mathbb{R}^N)$. Now let (u_n) be a sequence such that $u_n \rightharpoonup u$ weakly for some $u \in D^{1,p}(\mathbb{R}^N)$. Then from Lemma 2.1 we have $a^{r/q}u_n \to a^{r/q}u$ strongly in $L^q(\mathbb{R}^N)$. It follows that

$$a^{r/q}u_n \longrightarrow a^{r/q}u, \quad |a^{r/q}u_n| \le k \quad \text{a.e. in } \mathbb{R}^N, \tag{2.7}$$

for some $k \in L^q(\mathbb{R}^N)$. Hence, for all n and a.e. $x \in \mathbb{R}^N$,

$$|f(x,u_n(x))|^r \le 2^r\left(a(x)^r|u_n(x)|^{(\sigma-1)r} + |b(x)|^r\right) \le 2^r\left(|k(x)|^q + |b(x)|^r\right). \tag{2.8}$$

Since the last term is an integrable function, from Lebesgue theorem, we infer that $f(x,u_n) \to f(x,u)$ strongly in $L^r(\mathbb{R}^N)$. □

Next, we do some remarks about the Palais-Smale condition for J_λ. We recall that J_λ is said to satisfy the Palais-Smale condition at level c, $(\mathrm{PS})_c$, if every sequence for which

$$J(u_n) \longrightarrow c, \qquad \|J'(u_n)\|_{(D^{1,p})^*} \longrightarrow 0 \tag{2.9}$$

possesses a convergent subsequence. When J satisfies $(\mathrm{PS})_c$ for all $c \in \mathbb{R}^N$, we simply say that J satisfies the (PS) condition. In Theorem 1.2, we use a weaker version of the (PS) condition due to Cerami (cf. [4]). We say that J satisfies the Palais-Smale-Cerami condition, (PSC), if every sequence, for which

$$J(u_n) \quad \text{is bounded,} \qquad \left(1 + \|u_n\|_{D^{1,p}}\right)\|J'(u_n)\|_{(D^{1,p})^*} \longrightarrow 0, \tag{2.10}$$

possesses a convergent subsequence.

Lemma 2.3. *Assume that condition (1.12) holds. Then any bounded sequence satisfying (2.9) or (2.10) possesses a convergent subsequence.*

Proof. Let (u_n) be a bounded sequence satisfying (2.9). Then, passing to a subsequence if necessary, there exists $u \in D^{1,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $D^{1,p}(\mathbb{R}^N)$ and also in $L^{p^*}(\mathbb{R}^N)$. Consequently,

$$\lim_{n\to\infty} \langle J_\lambda'(u_n), u_n - u\rangle = 0. \tag{2.11}$$

On the other hand, from Lemma 2.2, we know that $f(x,u_n) \to f(x,u)$ strongly in $L^{p^{*\prime}}(\mathbb{R}^N)$ and therefore,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(f(x,u_n)+h)(u_n-u)\,dx = 0. \tag{2.12}$$

Noting that

$$\int_{\mathbb{R}^N} g|u_n|^{p-1}|u_n-u|\,dx \le \left(\int_{\mathbb{R}^N} g|u_n|^p\,dx\right)^{1/p'}\left(\int_{\mathbb{R}^N} g|u_n-u|^p\,dx\right)^{1/p}, \tag{2.13}$$

and since $g \in L^{(p^*/p)'}$, Lemma 2.1 implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} g|u_n|^{p-2}u_n(u_n-u)\,dx = 0. \tag{2.14}$$

Combining (2.11) with (2.12) and (2.14), we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla(u_n-u)\,dx = 0. \tag{2.15}$$

But since we also have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\cdot\nabla(u_n-u)\,dx = 0, \tag{2.16}$$

it follows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot\nabla(u_n-u)\,dx = 0. \tag{2.17}$$

Then from a well-known argument based on the Clarkson inequality (cf. Tolksdorf [19]), we conclude that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n - \nabla u|^p\,dx = 0. \tag{2.18}$$

This completes the proof since (2.10) implies (2.9). □

3. Proofs of Theorems 1.1 and 1.2

The proof of Theorem 1.1 is based on the Ekeland's variational principle and the Ambrosetti-Rabinowitz Mountain-Pass theorem [2]. Theorem 1.2 is proved using the saddle point theorem of Rabinowitz [17].

Proof of Theorem 1.1. We divide the proof in several steps.
Step 1 (the coerciveness of J_λ). Since $\lambda < \lambda_1$ and (1.12) holds, from (1.7) and (2.4) we get

$$J_\lambda(u) \ge \left(\frac{\lambda_1-\lambda}{p\lambda_1}\right)\|u\|^p_{D^{1,p}} - C\|u\|^\sigma_{D^{1,p}} - C\|u\|_{D^{1,p}}, \tag{3.1}$$

where $C > 0$ denotes several constants. Then J_λ is coercive as a consequence of the assumption that $1 < \sigma < p$. This implies that any sequence satisfying (2.9) must be bounded, and therefore Lemma 2.2 implies that J_λ satisfies the $(\mathrm{PS})_c$ for all $c \in \mathbb{R}$. Similarly, from (1.11),

$$J_{\lambda_1}(w) \geq \left(\frac{\lambda_2 - \lambda_1}{p\lambda_2}\right) \|w\|^p_{D^{1,p}} - C\|w\|^\sigma_{D^{1,p}} - C\|w\|_{D^{1,p}}, \tag{3.2}$$

which shows that J_{λ_1} is coercive in W. Noting that $J_{\lambda_1} \leq J_\lambda$ for all $\lambda < \lambda_1$, we have that

$$m = \inf_W J_{\lambda_1} \leq \inf_W J_\lambda. \tag{3.3}$$

Step 2 (estimating J_λ in V). From (1.16) we have for $t \in \mathbb{R}$,

$$J_\lambda(t\varphi_1) = \left(\frac{\lambda_1 - \lambda}{p}\right) \int_{\mathbb{R}^N} |t\varphi_1(x)|^p \, dx - \int_{\mathbb{R}^N} F(x, t\varphi_1(x)) \, dx. \tag{3.4}$$

Now, from (1.13) there exist constants $R, C > 0$ such that

$$\int_{|x|>R} F(x, t\varphi_1(x)) \, dx \geq \int_{|x|>R} \gamma(x) \, dx \geq -C, \quad \forall t \in \mathbb{R}. \tag{3.5}$$

Choosing $t^+ > 0$ sufficiently large, we get from (1.15) that

$$\int_{|x|\leq R} F(x, t^+\varphi_1(x)) \, dx > -m + C. \tag{3.6}$$

Then we have

$$\int_{\mathbb{R}^N} F(x, t^+\varphi_1(x)) \, dx > -m, \tag{3.7}$$

so that

$$J_\lambda(t^+\varphi_1) \leq \left(\frac{\lambda_1 - \lambda}{p}\right) \int_{\mathbb{R}^N} |t^+\varphi_1(x)|^p \, dx + m. \tag{3.8}$$

Then for λ sufficiently near to λ_1, $J_\lambda(t^+\varphi_1) < m$. The same conclusion holds for a $t^- < 0$.

Step 3 (the existence of the first two solutions). Put

$$\mathbb{O}^\pm = \{u \in D^{1,p}(\mathbb{R}^N);\ u = \pm t\varphi_1 + w \text{ with } t > 0,\ w \in W\}. \tag{3.9}$$

Then from Step 2, for λ sufficiently near to λ_1,

$$-\infty < \inf_{\mathbb{O}^\pm} I_\lambda < m. \tag{3.10}$$

Now let $u_n \in \mathbb{O}^+$ be a sequence satisfying (2.9) for $c < m$. Then from coerciveness of J_λ, (u_n) has a convergent subsequence, say, (u_n) itself. Noting that $W = \partial\mathbb{O}^+$ and $\inf_W J_\lambda \geq m$ (Step 1), we conclude that (u_n) converges to an interior point $u \in \mathbb{O}^+$. This means that J_λ satisfies the $(\mathrm{PS})_c$ condition inside $\mathbb{O}^+$ for all $c < m$. Then applying the Ekeland variational principle in $\overline{\mathbb{O}^+}$, we see that J_λ has a critical point u^+ as a local minimum in $\mathbb{O}^+$. (See complete argument in [16].) Similarly, we obtain a critical point u^- of J_λ in $\mathbb{O}^-$. Taking into account that $\mathbb{O}^- \cap \mathbb{O}^+ = \emptyset$, the existence of two weak solutions of (1.1) is proved.

Step 4 (the third solution). To fix ideas, suppose that $J_\lambda(u^+) \leq J_\lambda(u^-)$. If u^- is not an isolated critical point, then J_λ has at least three solutions. Otherwise, putting

$$I(u) = J_\lambda(u+u^-) - J_\lambda(u^-), \qquad e = u^+ - u^-, \tag{3.11}$$

we have that $I(0)=0$, $I(e) \leq 0$, and there exist $r, \rho > 0$ such that $I(u) \geq \rho$ if $\|u\|_{D^{1,p}} = r$. Then, since $I' = J_\lambda'$ and I also satisfies the (PS) condition, from the Mountain-Pass theorem, the number

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)), \tag{3.12}$$

where

$$\Gamma = \{\gamma \in C([0,1], D^{1,p}(\mathbb{R}^N));\ \gamma(0) = u^-,\ \gamma(1) = u^+\} \tag{3.13}$$

is a critical value of J_λ. Noting that all paths joining u^- to u^+ pass through W, we have $c \geq m$. Therefore we have obtained a third critical point of J_λ. The proof is now complete. □

Proof of Theorem 1.2. The proof is based on the arguments from [8, 10].

Step 1 (the growth of F). We prove that for some $C_1, C_2 > 0$,

$$\int_{\mathbb{R}^N} F(x, t\varphi_1)\,dx \geq C_1 \|t\varphi_1\|^{\mu}_{D^{1,p}} - C_2. \tag{3.14}$$

In fact, from (1.14) we have

$$\frac{d}{du}\left(\frac{F(x,u)}{|u|^p}\right) \leq -\alpha(x)|u|^{\mu-p-2}u - \beta(x)|u|^{-p-2}u \quad (u > 0). \tag{3.15}$$

Integrating from $u > 0$ to $+\infty$, and noting that $F(x,\theta)/(\theta^p) \to 0$ as $\theta \to \infty$, we get

$$F(x,u) \geq \frac{\alpha(x)}{p-\mu}|u|^{\mu} + \frac{\beta(x)}{p}. \tag{3.16}$$

Since this inequality holds for $u < 0$, we have

$$\int_{\mathbb{R}^N} F(x, t\varphi_1)\,dx \geq C|t|^{\mu} - C_2, \tag{3.17}$$

and inequality (3.14) follows.

Step 2 (the (PSC) condition). Let (u_n) be a sequence satisfying (2.9). Then from Lemma 2.2, it suffices to prove that (u_n) is bounded. In fact, first we note that (u_n) satisfies

$$\begin{aligned}\langle J_\lambda'(u_n), u_n\rangle - pJ_\lambda(u_n) &= \int_{\mathbb{R}^N} \left[pF(x,u_n) - f(x,u_n)u_n + (p-1)hu_n\right] dx \\ &\geq \int_{\mathbb{R}^N} \alpha|u|^{\mu}\,dx + \int_{\mathbb{R}^N} \beta\,dx + (p-1)\int_{\mathbb{R}^N} hu_n\,dx.\end{aligned} \tag{3.18}$$

Now, since $|h| \leq \alpha$,

$$\left|\int_{\mathbb{R}^N} hu_n\,dx\right| \leq \int_{\mathbb{R}^N} \alpha\,|\,u_n\,|\,dx \leq \|\alpha\|_{L^1}^{1/\mu'}\left(\int_{\mathbb{R}^N} \alpha\,|\,u_n\,|^{\mu}\,dx\right)^{1/\mu}. \tag{3.19}$$

Then from the boundedness of $\langle J_\lambda'(u_n), u_n\rangle - pJ_\lambda(u_n)$, we deduce that

$$\int_{\mathbb{R}^N} \alpha\,|\,u_n\,|^{\mu}\,dx \leq C + C\left(\int_{\mathbb{R}^N} \alpha\,|\,u_n\,|^{\mu}\,dx\right)^{1/\mu}, \tag{3.20}$$

so that

$$\int_{\mathbb{R}^N} \alpha\,|\,u_n\,|^{\mu}\,dx \leq C. \tag{3.21}$$

Now we use an interpolation inequality. Since $0 < \mu < p < p^*$, there exists $t \in (0,1)$ such that

$$1 = \frac{p(1-t)}{\mu} + \frac{pt}{p^*}. \tag{3.22}$$

Then from Hölder inequality,

$$\begin{aligned}\int_{\mathbb{R}^N} \alpha(x)|u|^p\,dx &= \int_{\mathbb{R}^N} \left(\alpha^{1/\mu}|u|\right)^{p(1-t)}\left(\alpha^{1/p^*}|u|\right)^{pt}\,dx \\ &\leq \left(\int_{\mathbb{R}^N} \alpha|u|^{\mu}\,dx\right)^{p(1-t)/\mu}\left(\int_{\mathbb{R}^N} \alpha|u|^{p^*}\,dx\right)^{pt/p^*}.\end{aligned} \tag{3.23}$$

Using (3.21) and (2.4),

$$\int_{\mathbb{R}^N} \alpha(x)\,|\,u_n(x)\,|^{p}\,dx \leq C\|u_n\|_{D^{1,p}}^{tp}. \tag{3.24}$$

Taking into account the boundedness of $J_\lambda(u_n)$,

$$\frac{1}{p}||u_n||^p_{D^{1,p}} \le C + \frac{\lambda}{p}\int_{\mathbb{R}^N} g\,|\,u_n\,|^p\,dx + \int_{\mathbb{R}^N} F(x,u_n)\,dx + \int_{\mathbb{R}^N} hu_n\,dx, \tag{3.25}$$

and since $\alpha \ge \max\{a, g\}$,

$$\begin{aligned}\frac{1}{p}||u_n||^p_{D^{1,p}} \le C + \frac{\lambda}{p}\int_{\mathbb{R}^N} \alpha\,|\,u_n\,|^p\,dx + \frac{1}{\sigma}\int_{\mathbb{R}^N} \alpha\,|\,u_n\,|^\sigma\,dx \\ + \int_{\mathbb{R}^N} b\,|\,u_n\,|\,dx + \int_{\mathbb{R}^N} hu_n\,dx.\end{aligned} \tag{3.26}$$

Consequently, if $\mu = \sigma$ we have, from (3.24),

$$||u_n||^p_{D^{1,p}} \le C\Big(1 + ||u_n||^{tp}_{D^{1,p}} + ||u_n||_{D^{1,p}}\Big). \tag{3.27}$$

Otherwise, we have $\mu < \sigma < p^*$, and as before, we get $s \in (0,1)$ such that

$$\int_{\mathbb{R}^N} \alpha\,|\,u_n\,|^\sigma\,dx \le C||u_n||^{s\sigma}_{D^{1,p}}. \tag{3.28}$$

Then

$$||u_n||^p_{D^{1,p}} \le C\Big(1 + ||u_n||^{tp}_{D^{1,p}} + ||u_n||^{s\sigma}_{D^{1,p}} + ||u_n||_{D^{1,p}}\Big). \tag{3.29}$$

In both cases, we see that $\|u_n\|_{D^{1,p}}$ is uniformly bounded.
Step 3 (the saddle point theorem). It is well known that the (PS) condition can be replaced by the (PSC) condition in the saddle point theorem of Rabinowitz (see [4, 17]). Then to conclude that J_λ has a critical point it suffices to show that

$$\lim_{\|v\|_{D^{1,p}}\to\infty} J_\lambda(v) = -\infty, \qquad \lim_{\|w\|_{D^{1,p}}\to\infty} J_\lambda(w) = +\infty, \tag{3.30}$$

where $v \in V$ and $w \in W$, as defined in (1.9). Now, from (3.14),

$$J_\lambda(t\varphi_1) \le -\left(\frac{\lambda-\lambda_1}{p\lambda_1}\right)||t\varphi_1||^p_{D^{1,p}} - C_1||t\varphi_1||^\mu_{D^{1,p}} + C||t\varphi_1||_{D^{1,p}} + C_2. \tag{3.31}$$

Since $\lambda \ge \lambda_1$, the first part of (3.30) holds. Finally, since $\lambda < \lambda_2$, the argument in Step 1 of the proof of Theorem 1.1 implies the second statement of (3.30). The proof is now complete. □

Note 3.1. Just before the completion of this paper, we noticed that P. De Nápoli and M. C. Mariani [9] studied problem (1.1) in the same framework of our Theorem 1.1. However, they considered only the case $\lambda \to \lambda_1$ from left. Our assumptions on f are slightly more general.

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