A FUNCTIONAL-ANALYTIC METHOD FOR THE STUDY OF DIFFERENCE EQUATIONS

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We will give the generalization of a recently developed functional-analytic method for studying linear and nonlinear, ordinary and partial, difference equations in the ℓ_p^1 and ℓ_p^2 spaces, $p \in \mathbb{N}$, $p \geq 1$. The method will be illustrated by use of two examples concerning a nonlinear ordinary difference equation known as the Putnam equation, and a linear partial difference equation of three variables describing the discrete Newton law of cooling in three dimensions.

1. Introduction

The aim of this paper is to present the generalization of a functional-analytic method, which was recently developed for the study of linear and nonlinear difference equations of one, two, three, and four variables in the Hilbert space

$$\ell_p^2 = \left\{ f(i_1,\dots,i_p) : \mathbb{N}^p \longrightarrow \mathbb{C} : \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} \left| f(i_1,\dots,i_p) \right|^2 < +\infty \right\} \tag{1.1}$$

and the Banach space

$$\ell_p^1 = \left\{ f(i_1,\dots,i_p) : \mathbb{N}^p \longrightarrow \mathbb{C} : \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} \left| f(i_1,\dots,i_p) \right| < +\infty \right\}, \tag{1.2}$$

where $\mathbb{N}^p = \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{p\text{-times}}$ and $p = 1,2,3,4$.

More precisely, this method was introduced for the first time by Ifantis in [5] for the study of linear and nonlinear ordinary difference equations. Later, this method was extended by the authors in [7, 9, 10] in order to study a class of nonlinear ordinary difference equations more general than the one studied in [5]. For the study of linear and

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nonlinear partial difference equations of two variables, we developed a similar functional-analytic method in [11, 12], which was extended in [8] in order to study partial difference equations of three and four variables.

The aim of this paper is to present the generalization of this functional-analytic method for the study of linear and nonlinear partial difference equations of p variables in the Hilbert space ℓ_p^2, defined by (1.1), and the Banach space ℓ_p^1, defined by (1.2), respectively, with $p \in \mathbb{N}$, $p \geq 1$. The motivation for seeking solutions of partial difference equations in the spaces ℓ_p^2 and ℓ_p^1 arises from various problems of mathematics, physics, and biology, such as probability problems, problems concerning integral equations, generating analytic functions, Laurent or z-transforms, numerical schemes, boundary value problems of partial differential equations, problems of quantum mechanics, and problems of population dynamics and epidemiology (for more details, see [11] and the references therein). Also, by assuring the existence of a solution of a difference equation in the space ℓ_p^2 or ℓ_p^1, we obtain information regarding the asymptotic behavior of the unknown sequence for *initial conditions which are in general complex numbers*.

We would like, at this point, to give an outline of the functional-analytic method that we will present in details in Section 2. (For a sketch of the main ideas used in the proofs of our main results, see the beginning of Section 3.) By use of this method, the linear or nonlinear difference equation under consideration is transformed equivalently into a linear or nonlinear operator equation defined in an abstract Hilbert space H or Banach space H_1, respectively. In this way, we can use various results (e.g., fixed point theorems) from the wealth of operator theory, in order to assure the existence of a unique solution of the operator equation in H or H_1. In the case of linear equations, we use the following classical result of operator theory [4, pages 70–71].

Theorem 1.1. *Let T be a linear, bounded operator of the Hilbert space H with $\|T\| < 1$. Then the inverse of $I - T$ exists on H and is uniquely determined and bounded by $\|(I - T)^{-1}\| \leq 1/(1 - \|T\|)$.*

In the case of nonlinear equations, we use the following fixed point theorem of Earle and Hamilton [3].

Theorem 1.2. *Let X be a bounded, connected, and open subset of a Banach space B. Further, let $g : X \to g(X)$ be holomorphic, that is, its Fréchet derivative exists and $g(X)$ lies strictly inside X. Then g has a unique fixed point in X. (By saying that a subset X' of X lies strictly inside X, we mean that there exists $\epsilon > 0$ such that $\|x' - y\| > \epsilon$ for all $x' \in X'$ and $y \in B - X$.)*

For both linear and nonlinear difference equations, we obtain, by use of our method, a bound of the solution of the difference equation under consideration. Moreover, in the case of nonlinear difference equations, we use a constructive technique, which allows us to obtain a region, depending on the initial conditions and the parameters of the equations, where the solution of the difference equation under consideration holds.

We illustrate our method in Section 3 by applying it to two difference equations which arise from a mathematical problem (the Putnam equation) and a physical problem concerning the discrete Newton law of cooling in three dimensions.

2. The functional-analytic method

We denote by H an abstract separable Hilbert space with orthonormal base $\{e_{i_1,\dots,i_p}\}$, $i_1,\dots,i_p = 1,2,\dots$, and elements $u \in H$ which have the form

$$u = \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} (u, e_{i_1,\dots,i_p}) e_{i_1,\dots,i_p}, \tag{2.1}$$

with norm $\|u\|^2 = \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} |(u, e_{i_1,\dots,i_p})|^2$. Also, by H_1 we mean the Banach space consisting of those elements $u \in H$ which satisfy the condition

$$\sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} \left| (u, e_{i_1,\dots,i_p}) \right| < +\infty. \tag{2.2}$$

The norm in H_1 is denoted by $\|u\|_1 = \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} |(u, e_{i_1,\dots,i_p})|$. By $u(i_1,\dots,i_p)$ we mean an element of l_p^2 or l_p^1, and by $u = \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} (u, e_{i_1,\dots,i_p}) e_{i_1,\dots,i_p}$ we mean that element of H or H_1 generated by $u(i_1,\dots,i_p)$.

Finally, we define in H the shift operators V_j, $j = 1,\dots,p$, as follows:

$$V_j e_{i_1,\dots,i_j,\dots,i_p} = e_{i_1,\dots,i_j+1,\dots,i_p}. \tag{2.3}$$

It can be easily seen that their adjoint operators are

$$V_j^* e_{i_1,\dots,i_j,\dots,i_p} = e_{i_1,\dots,i_j-1,\dots,i_p}, \quad i_j = 2,3,\dots, \qquad V_j^* e_{i_1,\dots,1,\dots,i_p} = 0, \tag{2.4}$$

and that

$$||V_j^*|| = ||V_j|| = ||V_j^*||_1 = ||V_j||_1 = 1, \quad j = 1,\dots,p. \tag{2.5}$$

The following proposition is of fundamental importance in our approach.

Proposition 2.1. *The function*

$$\phi : H \longrightarrow l_p^2, \qquad \phi(u) = (u, e_{i_1,\dots,i_p}) = u(i_1,\dots,i_p), \tag{2.6}$$

is an isomorphism from H onto l_p^2.

Proof. We begin by showing that the mapping defined by (2.6) is well defined. Indeed, since $u \in H$, we have

$$\begin{aligned} ||u(i_1,\dots,i_p)||_{l_p^2}^2 &= \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} \left| u(i_1,\dots,i_p) \right|^2 \\ &= \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} \left| (u, e_{i_1,\dots,i_p}) \right|^2 \\ &= \|u\|^2 < +\infty. \end{aligned} \tag{2.7}$$

By use of the properties of an inner product, it is obvious that ϕ is linear. Also, ϕ is a one-to-one mapping onto l_p^2. Indeed, if $u \in H$, $v \in H$ with $\phi(u) = \phi(v)$, then

$$(u - v, e_{i_1,\dots,i_p}) = 0 \Longleftrightarrow u = v, \tag{2.8}$$

because $e_{i_1,\dots,i_p}$ is an orthonormal base of H.

Furthermore, if $\alpha(i_1,\dots,i_p) \in l_p^2$, then there exists $u \in H$ such that $\phi(u) = \alpha(i_1,\dots,i_p)$. This u is given by

$$u = \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} \alpha(i_1,\dots,i_p) e_{i_1,\dots,i_p}, \tag{2.9}$$

and it belongs to H since

$$\|u\|^2 = \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} |\alpha(i_1,\dots,i_p)|^2 = ||\alpha(i_1,\dots,i_p)||_{l_p^2}^2 < +\infty. \tag{2.10}$$

Finally, the mapping ϕ preserves the norm since

$$||\phi(u)||^2 = \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} |u(i_1,\dots,i_p)|^2 = \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} |(u, e_{i_1,\dots,i_p})|^2 = \|u\|^2. \tag{2.11}$$

Thus, the mapping ϕ defined by (2.6) is an isomorphism from H onto l_p^2. □

In a similar way, the following proposition can also be proved.

Proposition 2.2. *The function*

$$\phi : H \longrightarrow l_p^1, \qquad \phi(u) = (u, e_{i_1,\dots,i_p}) = u(i_1,\dots,i_p), \tag{2.12}$$

is an isomorphism from H onto l_p^1.

We call the element u, defined by (2.6) or (2.12), the *abstract form* of $u(i_1,\dots,i_p)$ in H or H_1, respectively. In general, if G is a mapping in $l_p^2(l_p^1)$ and N is a mapping in $H(H_1)$, we call $N(u)$ the *abstract form* of $G(u(i_1,\dots,i_p))$ if

$$G(u(i_1,\dots,i_p)) = (N(u), e_{i_1,\dots,i_p}). \tag{2.13}$$

3. Illustrative examples

In this section, we will illustrate our method using two characteristic examples of difference equations arising in a problem of mathematics and a problem of physics. More precisely, we will establish conditions so that the difference equations under consideration have a unique bounded solution in l_p^1 or l_p^2. Such kind of solutions is extremely useful not only from a mathematical point of view, but also from an applied point of view (see Remarks 3.2 and 3.4).

We would like now to give the main ideas used in the proofs of our results. First, using (2.6) or (2.12), we transform the linear or nonlinear difference equation under consideration into an equivalent linear or nonlinear operator equation in an abstract separable Hilbert H or Banach H_1 space. Then, after some manipulations, we bring the linear operator equation into the form

$$(I - T)u = f, \tag{3.1}$$

where $u \in H$ is the unknown variable, f a known element of H, and $T : H \to H$ a known linear operator. At this point, we impose conditions so that $\|T\| < 1$, in order to apply Theorem 1.1 to the preceding operator equation and obtain information for the initial linear difference equation under consideration.

In the case of nonlinear equations, we do some manipulation in order to write the operator equation in the form

$$u = g(u), \tag{3.2}$$

where $u \in H$ is the unknown variable and $g : X \subset H_1 \to g(X)$ a known nonlinear mapping. Usually, $g(u)$ has the form

$$g(u) = h + \phi(u), \tag{3.3}$$

where h is a known element of H_1 depending on the initial conditions and the nonhomogeneous term (if any) of the initial nonlinear difference equation, and $\phi : H_1 \to H_1$ is a known nonlinear mapping. At this point, we impose conditions on $\|h\|_1$ in order to apply the fixed point Theorem 1.2 to equation $u = g(u)$ and obtain information for the initial nonlinear difference equation under consideration.

3.1. The Putnam equation. Consider the nonlinear, homogeneous, ordinary difference equation

$$\begin{aligned} f(i+3)+f(i+2) &= f(i+4)f(i+3)f(i+2)+f(i+4)f(i+1) \\ &\quad + f(i+4)f(i) - f(i+1)f(i), \quad i = 1,2,\ldots. \end{aligned} \tag{3.4}$$

Equation (3.4) appeared in a problem given in the 25th William Lowell Putnam Mathematical Competition, held on December 5, 1964 (see [1]). This problem is as follows [1]:

"Let p_n, $n = 1,2,\ldots$, be a bounded sequence of integers, which satisfies the recursion

$$p_n = \frac{p_{n-1} + p_{n-2} + p_{n-3}p_{n-4}}{p_{n-1}p_{n-2} + p_{n-3} + p_{n-4}}. \tag{3.5}$$

Show that the sequence eventually becomes periodic."

As mentioned in [1], the solution of this problem is independent of the recurrence relation that the sequence p_n satisfies, as long as p_n is bounded. In the years that passed, it turned out that (3.5) is quite attractive from a mathematical point of view. In this paper, we will prove the following result.

Result 3.1. *The Putnam equation (3.4) has a unique bounded solution in* $\ell_1^1 + \{1\}$ *if*

$$|f(1)-1| + |f(2)-1| + |f(3)-1| + |f(4)-1| < 0.120227, \tag{3.6}$$

which satisfies

$$|f(i)| < 1.236068, \tag{3.7}$$

where the initial conditions $f(1)$, $f(2)$, $f(3)$, *and* $f(4)$ *are in general complex numbers.*

Remark 3.2. (a) It is obvious from the preceding result that the solution of the Putnam equation (3.4) tends to 1 if (3.6) holds. Thus, 1 is a locally asymptotically stable equilibrium point of (3.4) if (3.6) holds.

(b) In [6], it was proved, among other things, that the equilibrium point 1 of (3.4) is globally asymptotically stable for *positive initial conditions*.

Proof of Result 3.1. Equation (3.4) is a nonlinear ordinary difference equation, that is, a difference equation of $p = 1$ variable. As a consequence, we will work in the Banach space ℓ_1^1 and the isomorphic abstract Banach space H_1 with orthonormal base $\{e_i\}$, $i = 1,2,\ldots$. (For reasons of simplicity, we will use the symbol i instead of the symbol i_1.)

First of all, we mention that $\varrho = 1$ is an equilibrium point of (3.4) and we set $f(i) = u(i) + \varrho$. Then (3.4) becomes

$$\begin{aligned}(\varrho^2+2\varrho)u(i+4) &+ (\varrho^2-1)u(i+3) + (\varrho^2-1)u(i+2)\\ &= -u(i+4)u(i+1) - u(i+4)u(i+3)u(i+2) - u(i+4)u(i)\\ &\quad + u(i+1)u(i) - \varrho u(i+4)u(i+3) - \varrho u(i+4)u(i+2) - \varrho u(i+3)u(i+2).\end{aligned} \tag{3.8}$$

Using (2.12), we find the abstract forms of all the terms involved in (3.8). More precisely, we have

$$\begin{gathered}u(i+k) = (u, e_{i+k}) = (u, V_1^k e_i) = \left((V_1^*)^k u, e_i\right), \quad k = 2,3,4,\\ u(i+m)u(i+n) = (u, e_{i+m})(u, e_{i+n})e_i = N_{mn}(u), \quad m,n = 0,1,2,3,4,\\ u(i+4)u(i+3)u(i+2) = (u, e_{i+4})(u, e_{i+3})(u, e_{i+2})e_i = N_2(u).\end{gathered} \tag{3.9}$$

Moreover, we can prove that the nonlinear operators $N_{mn}(u)$, $N_2(u)$ are Frechét-differentiable in H_1. Thus, the abstract form of (3.8) in H_1 is

$$\begin{aligned}(\varrho^2+2\varrho)&(V_1^*)^4 u + (\varrho^2-1)(V_1^*)^3 u + (\varrho^2-1)(V_1^*)^2 u\\ &= -N_{41}(u) - N_2(u) - N_{40}(u) + N_{10}(u) - \varrho N_{43}(u) - \varrho N_{42}(u) - \varrho N_{32}(u) \Longrightarrow (V_1^*)^4 u\\ &= -\frac{1}{3}N_{41}(u) - \frac{1}{3}N_2(u) - \frac{1}{3}N_{40}(u) + \frac{1}{3}N_{10}(u) - \frac{1}{3}N_{43}(u) - \frac{1}{3}N_{42}(u) - \frac{1}{3}N_{32}(u)\end{aligned} \tag{3.10}$$

or, due to the fact that $V^* e_1 = 0$,

$$\begin{aligned} u &= g(u) \\ &= u(1)e_1 + u(2)e_2 + u(3)e_3 + u(4)e_4 \\ &\quad - \frac{1}{3}V^4[N_{41}(u) + N_2(u) + N_{40}(u) - N_{10}(u) + N_{43}(u) + N_{42}(u) + N_{32}(u)]. \end{aligned} \tag{3.11}$$

From the preceding equation we obtain, taking the norm of both parts in H_1,

$$\begin{aligned} \|u\|_1 &= \|g(u)\|_1 \\ &\leq |u(1)| + |u(2)| + |u(3)| + |u(4)| \\ &\quad + \frac{1}{3}[\|N_{41}(u)\|_1 + \|N_2(u)\|_1 + \|N_{40}(u)\|_1 + \|N_{10}(u)\|_1 \\ &\qquad + \|N_{43}(u)\|_1 + \|N_{42}(u)\|_1 + \|N_{32}(u)\|_1] \Longrightarrow \|u\|_1 \\ &\leq |u(1)| + |u(2)| + |u(3)| + |u(4)| + \frac{1}{3}(\|u\|_1^3 + 6\|u\|_1^2). \end{aligned} \tag{3.12}$$

Let $\|u\|_1 \leq R$, R sufficiently large but finite. Then, from (3.12), we have

$$\|u\|_1 \leq |u(1)| + |u(2)| + |u(3)| + |u(4)| + \frac{1}{3}R^3 + 2R^2. \tag{3.13}$$

Let $P(R) = R - 2R^2 - (1/3)R^3$. This function has a maximum at $R_0 = \sqrt{5} - 2 \cong 0.236068$, which is $P_0 \cong 0.120227$. Thus, for $R = R_0$, we find that if

$$|u(1)| + |u(2)| + |u(3)| + |u(4)| \leq P_0 - \epsilon, \quad \epsilon > 0, \tag{3.14}$$

then

$$\|g(u)\|_1 \leq R_0 - \epsilon < R_0, \tag{3.15}$$

for $\|u\|_1 < R_0$. This means that for

$$|u(1)| + |u(2)| + |u(3)| + |u(4)| < P_0, \tag{3.16}$$

g is a holomorphic mapping from $X = B(0,R_0) = \{u \in H_1 : \|u\|_1 < R_0\}$ strictly inside $X = B(0,R_0)$. Indeed, it is obvious that $g(X) \subseteq X$. Moreover, $g(X)$ lies strictly inside X, since if $w \in H_1 - X \Rightarrow \|w\|_1 \geq R_0$ and $w' \in g(X)$, that is, there exists an $f \in X \Rightarrow \|f\|_1 < R_0$ such that $g(f) = w'$, then we find easily that $\|w - w'\| \geq \epsilon > \epsilon/2 = \epsilon_1$. As a consequence, the fixed point theorem of Earle and Hamilton can be applied to (3.11). Thus, for

$$|u(1)| + |u(2)| + |u(3)| + |u(4)| < P_0, \tag{3.17}$$

(3.11) has a unique solution in H_1 bounded by R_0. Equivalently, this means that if (3.17) holds, then the difference equation (3.8) has a unique solution in ℓ_1^1 bounded by R_0. As a consequence, if (3.6) holds, (3.4) has a unique solution in $\ell_1^1 + \{1\}$ bounded by $1 + R_0$. □

3.2. A linear difference equation of three variables describing the discrete Newton law of cooling. Consider the linear, homogeneous, partial difference equation

$$\begin{aligned} &u(i,j,n+1) + [4r(i,j,n) - 1]u(i,j,n) - r(i,j,n)u(i-1,j,n) \\ &\quad - r(i,j,n)u(i+1,j,n) - r(i,j,n)u(i,j-1,n) - r(i,j,n)u(i,j+1,n) = 0, \end{aligned} \tag{3.18}$$

where $i,j,n = 1,2,\ldots$, and $r(i,j,n)$ is a known sequence. Equation (3.18) describes the discrete Newton law of cooling in three dimensions. More precisely, the physical problem that (3.18) describes is the following.

Consider the distribution of heat through a "very long" (so long that it can be labelled by the set of integers) nonuniform thin plate. Let $u(i,j,n)$ be the temperature of the plate at the position (i,j) and time n. At time n, if the temperature $u(i-1,j,n)$ is higher than $u(i,j,n)$, heat will flow from the point $(i-1,j)$ to (i,j) at a rate $r(i,j,n)$. Similarly, heat will flow from the point $(i+1,j)$ to (i,j) at the same rate, $r(i,j,n)$. Thus, the total effect will be

$$\begin{aligned} u(i,j,n+1) - u(i,j,n) &= r(i,j,n)[u(i-1,j,n) - 2u(i,j,n) + u(i+1,j,n)] \\ &\quad + r(i,j,n)[u(i,j-1,n) - 2u(i,j,n) + u(i,j+1,n)], \end{aligned} \tag{3.19}$$

which is essentially (3.18). For (3.18), bounded and/or positive solutions of (3.18) are of interest (see [2]). In this paper, we will prove the following result.

Result 3.3. (a) *Let*

$$\sup_{i,j,n} \left| \frac{1}{4r(i,j,n) - 1} \right| < +\infty, \tag{3.20}$$

$$\sup_{i,j,n} \left| \frac{1}{4r(i,j,n) - 1} \right| \left[1 + 4\sup_{i,j,n} |r(i,j,n)| \right] < 1. \tag{3.21}$$

Then the unique solution of (3.18) in ℓ_3^2 is the zero solution.

(b) *Let*

$$\sup_{i,j,n} |4r(i,j,n) - 1| + 4\sup_{i,j,n} |r(i,j,n)| < 1. \tag{3.22}$$

Then (3.18) has a unique bounded solution in ℓ_3^2, which satisfies

$$|u(i,j,n)| \le \frac{\|u(i,j,1)\|_{\ell^2_{\mathbb{N}^2}}}{1-\sup_{i,j,n}|4r(i,j,n)-1| - 4\sup_{i,j,n}|r(i,j,n)|}, \tag{3.23}$$

provided that the initial conditions $u(i,j,1)$ (which are in general complex) belong to ℓ_2^2.

Proof of Result 3.3. Equation (3.18) is a linear partial difference equation of $p=3$ variables. As a consequence, we will work in the Hilbert space ℓ_3^2 and the isomorphic abstract Hilbert space H with orthonormal base $\{e_{i,j,n}\}$, $i,j,n=1,2,\dots$. (For reasons of simplicity, we will use the symbols i, j, and n instead of the symbols i_1, i_2, and i_3, respectively.)

Using (2.6), we find the abstract forms of all the terms involved in (3.18). More precisely, we have

$$\begin{aligned}
u(i+1,j,n) &= (u,e_{i+1,j,n}) = (u,V_1e_{i,j,n}) = (V_1^*u,e_{i,j,n}),\\
u(i,j+1,n) &= (u,e_{i,j+1,n}) = (u,V_2e_{i,j,n}) = (V_2^*u,e_{i,j,n}),\\
u(i,j,n+1) &= (u,e_{i,j,n+1}) = (u,V_3e_{i,j,n}) = (V_3^*u,e_{i,j,n}),\\
u(i-1,j,n) &= (u,e_{i-1,j,n}) = (u,V_1^*e_{i,j,n}) = (V_1u,e_{i,j,n}),\\
u(i,j-1,n) &= (u,e_{i,j-1,n}) = (u,V_2^*e_{i,j,n}) = (V_2u,e_{i,j,n}),\\
b(i,j,n)u(i,j,n) &= (Bu,e_{i,j,n}),
\end{aligned} \tag{3.24}$$

where B is the diagonal operator $Be_{i,j,n}=b(i,j,n)e_{i,j,n}$ for a sequence $b(i,j,n)$. Thus, the abstract form of (3.18) in H is

$$V_3^*u + R_1u - RV_1u - RV_1^*u - RV_2u - RV_2^*u = 0, \tag{3.25}$$

where R, R_1 are the diagonal operators

$$Re_{i,j,n} = r(i,j,n)e_{i,j,n}, \quad R_1e_{i,j,n} = [4r(i,j,n)-1]e_{i,j,n}, \quad i,j,n\ge 1. \tag{3.26}$$

(a) Due to (3.20), (3.25) is rewritten as follows:

$$(I-T)u = 0, \tag{3.27}$$

where $T = -R_1^{-1}V_3^* + R_1^{-1}RV_1 + R_1^{-1}RV_1^* + R_1^{-1}RV_2 + R_1^{-1}RV_2^*$. But $\|T\| \le \|R_1^{-1}\|(1+4\|R\|) < 1$ due to (3.21). Thus, according to Theorem 1.1, the inverse of $I-T$ exists and is a linear bounded operator in H. Thus, the unique solution of (3.27) in H is the zero solution. Equivalently, this means that the unique solution of (3.18) in ℓ_3^2 is the zero solution.

(b) Since $V_3^*e_{i,j,1}=0$, (3.25) is written as follows:

$$(I-T)u = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} u(i,j,1)e_{i,j,1}, \tag{3.28}$$

where $T = -V_3R_1 + V_3RV_1 + V_3RV_1^* + V_3RV_2 + V_3RV_2^*$. But $\|T\| \leq \|R_1\| + 4\|R\| < 1$ due to (3.22). Thus, the inverse of $I - T$ exists and is a linear operator of H bounded by

$$||(I-T)^{-1}|| \leq \frac{1}{1 - \sup_{i,j,n} |4r(i,j,n) - 1| - 4\sup_{i,j,n} |r(i,j,n)|}. \tag{3.29}$$

Thus, (3.28) has a unique solution in H bounded by

$$\|u\| \leq \frac{|| \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} u(i,j,1)e_{i,j,1} ||}{1 - \sup_{i,j,n} |4r(i,j,n) - 1| - 4\sup_{i,j,n} |r(i,j,n)|}. \tag{3.30}$$

Equivalently, this means that (3.18) has a unique solution in ℓ_3^2, which satisfies (3.23). □

Remark 3.4. (a) Since $u(i,j,n) \in \ell_3^2$, we have $\lim_{i,j,n\to\infty} u(i,j,n) = 0$. The *physical importance* of this fact is that after a long period of time (theoretically infinite), at the end of the plate (which is assumed to be of infinite length), the temperature will tend to zero, which is in agreement with the physical laws.

(b) In [2], (3.18) is mentioned but not studied. More precisely, it is stated there that if the plate has an initial temperature at $n = 0$, then after a quite large time interval, the temperature of the plate will not depend on time, but only on the position (i,j). When this happens, the temperature $u(i,j)$ of the plate will satisfy the linear, homogeneous partial difference equation of two variables, which is characterized as the steady state equation

$$u(i-1,j) + u(i+1,j) + u(i,j-1) + u(i,j+1) - 4u(i,j) = 0. \tag{3.31}$$

This equation has a positive, bounded solution which is $u(i,j) \equiv 1$. (Note that this solution does not belong to ℓ_2^2.) Then an important question is the following [2].

"Do equations of the form

$$\begin{aligned} &\alpha(i,j)u(i-1,j) + \beta(i,j)u(i+1,j) + \gamma(i,j)u(i,j-1) \\ &\quad + \delta(i,j)u(i,j+1) - \sigma(i,j)u(i,j) = 0, \end{aligned} \tag{3.32}$$

where $\alpha(i,j)$, $\beta(i,j)$, $\gamma(i,j)$, $\delta(i,j)$, and $\sigma(i,j)$ are real sequences, have bounded and/or positive solutions?"

The following was proved in [2]: if $\alpha(i,j)$, $\beta(i,j)$, $\gamma(i,j)$, $\delta(i,j)$, and $\sigma(i,j)$ are positive sequences with

$$\sup_{i,j} \left\{ \left| \frac{\alpha(i,j)}{\sigma(i,j)} \right| + \left| \frac{\beta(i,j)}{\sigma(i,j)} \right| + \left| \frac{\gamma(i,j)}{\sigma(i,j)} \right| + \left| \frac{\delta(i,j)}{\sigma(i,j)} \right| \right\} < 1, \tag{3.33}$$

then the unique bounded solution of (3.32) with $i,j = 0, \pm 1, \pm 2, \ldots$ is the zero solution.

In a way similar to the proof of Result 3.3, we can prove the following.

(i) If

$$\sup_{i,j}\left|\frac{\alpha(i,j)}{\sigma(i,j)}\right|+\sup_{i,j}\left|\frac{\beta(i,j)}{\sigma(i,j)}\right|+\sup_{i,j}\left|\frac{\gamma(i,j)}{\sigma(i,j)}\right|+\sup_{i,j}\left|\frac{\delta(i,j)}{\sigma(i,j)}\right|<1, \tag{3.34}$$

then the unique bounded solution of (3.32) in ℓ_2^2 is the zero solution.

Note that (3.34) implies (3.33).

(ii) If $u(i,1)\in\ell_1^2$ and

$$\sup_{i,j}\left|\frac{\alpha(i,j)}{\delta(i,j)}\right|+\sup_{i,j}\left|\frac{\beta(i,j)}{\delta(i,j)}\right|+\sup_{i,j}\left|\frac{\gamma(i,j)}{\delta(i,j)}\right|+\sup_{i,j}\left|\frac{\sigma(i,j)}{\delta(i,j)}\right|<1, \tag{3.35}$$

then (3.32) has a unique bounded solution in ℓ_2^2, which satisfies

$$|u(i,j)|\le\frac{||u(i,1)||_{\ell_{\mathbb{N}}^2}}{1-\sup_{i,j}\left|\frac{\alpha(i,j)}{\delta(i,j)}\right|-\sup_{i,j}\left|\frac{\beta(i,j)}{\delta(i,j)}\right|-\sup_{i,j}\left|\frac{\gamma(i,j)}{\delta(i,j)}\right|-\sup_{i,j}\left|\frac{\sigma(i,j)}{\delta(i,j)}\right|}. \tag{3.36}$$

(iii) If $u(1,j)\in\ell_1^2$ and

$$\sup_{i,j}\left|\frac{\alpha(i,j)}{\beta(i,j)}\right|+\sup_{i,j}\left|\frac{\gamma(i,j)}{\beta(i,j)}\right|+\sup_{i,j}\left|\frac{\delta(i,j)}{\beta(i,j)}\right|+\sup_{i,j}\left|\frac{\sigma(i,j)}{\beta(i,j)}\right|<1, \tag{3.37}$$

then (3.32) has a unique bounded solution in ℓ_2^2, which satisfies

$$|u(i,j)|\le\frac{||u(1,j)||_{\ell_{\mathbb{N}}^2}}{1-\sup_{i,j}\left|\frac{\alpha(i,j)}{\beta(i,j)}\right|-\sup_{i,j}\left|\frac{\gamma(i,j)}{\beta(i,j)}\right|-\sup_{i,j}\left|\frac{\delta(i,j)}{\beta(i,j)}\right|-\sup_{i,j}\left|\frac{\sigma(i,j)}{\beta(i,j)}\right|}. \tag{3.38}$$

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