FIRST- AND SECOND-ORDER DYNAMIC EQUATIONS WITH IMPULSE

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We present existence results for discontinuous first- and continuous second-order dynamic equations on a time scale subject to fixed-time impulses and nonlinear boundary conditions.

1. Introduction

We first briefly survey the recent results for existence of solutions to first-order problems with fixed-time impulses. Periodic boundary conditions using upper and lower solutions were considered in [19], using degree theory. A nonlinear alternative of Leray-Schauder type was used in [15] for initial conditions or periodic boundary conditions. The monotone iterative technique was employed in [14] for antiperiodic and nonlinear boundary conditions. Lower and upper solutions and periodic boundary conditions were studied in [20]. Semilinear damped initial value problems in a Banach space using fixed point theory were investigated in [6]. In [9], existence of solutions for the differential equation $u'(t) = q(u(t))g(t,u(t))$ subject to a general boundary condition is proven, in which g is Carathéodory and $q \in L^\infty$, and existence of lower and upper solutions is assumed. Schauder's fixed point theorem was used there. This generalized an earlier result found in [18]. It appears that little has been done concerning dynamic equations with impulses on time scales (see [4, 5, 16] for earlier results). In Section 2, the present paper uses ideas from [9] to prove an existence result for discontinuous dynamic equations on a time scale subject to fixed-time impulses and nonlinear boundary conditions.

The study of boundary value problems for nonlinear second-order differential equations with impulses has appeared in many papers (see [10, 11, 13] and the references therein). In Section 3, we use ideas from [12, 16] to prove an existence result for second-order dynamic equations on a time scale subject to fixed-time impulses and nonlinear boundary conditions. Nonlinear boundary conditions cover, among others, the periodic and the Dirichlet conditions, and have been introduced for ordinary differential equations by Adje in [1]. Assuming the existence of a lower and an upper solution, we prove that the solution of the boundary value problem stays between them.

In [2], it was shown that the upper and lower solution method will not work for first-order dynamic equations involving Δ-derivatives, unless restrictive assumptions are

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made. Hence, in Section 2, we work with the ∇-derivative. In Section 3, we can use the more conventional Δ-derivative.

The monographs [17, 21] are good general references on impulsive differential equations—discussion of applications may be found in these books. Applications of the results given in this paper could involve those typically modelled on time scales which are subjected to sudden major influences, for example, an insect population sprayed with an insecticide or a financial market affected by a major terrorist attack.

For our purposes, we let $\mathbb{T}$ be a time scale (a closed subset of $\mathbb{R}$), let $[a,b]$ be the closed and bounded interval in $\mathbb{T}$, that is, $[a,b] := \{t \in \mathbb{T} : a \le t \le b\}$ and $a,b \in \mathbb{T}$. For the readers' convenience, we state a few basic definitions on a time scale $\mathbb{T}$ [7, 8].

Obviously a time scale $\mathbb{T}$ may or may not be connected. Therefore, we have the concept of *forward* and *backward jump operators* as follows: define $\sigma,\rho : \mathbb{T} \mapsto \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}. \tag{1.1}$$

If $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, then $t \in \mathbb{T}$ is called *right dense* (rd), *right scattered*, *left dense*, *left scattered*, respectively. We also define the *graininess function* $\mu : \mathbb{T} \mapsto [0,\infty)$ as $\mu(t) = \sigma(t) - t$. The sets $\mathbb{T}^\kappa$, $\mathbb{T}_\kappa$ which are derived from $\mathbb{T}$ are as follows: if $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^\kappa = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}_\kappa = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$. If $f : \mathbb{T} \mapsto \mathbb{R}$ is a function, we define the functions $f^\sigma : \mathbb{T}^\kappa \mapsto \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}^\kappa$, $f^\rho : \mathbb{T}_\kappa \mapsto \mathbb{R}$ by $f^\rho(t) = f(\rho(t))$ for all $t \in \mathbb{T}_\kappa$ and $\sigma^0(t) = \rho^0(t) = t$.

If $f : \mathbb{T} \mapsto \mathbb{R}$ is a function and $t \in \mathbb{T}^\kappa$, then the *delta derivative* of f at a point t is defined to be the number $f^\Delta(t)$ (provided it exists) with the property that, for each $\varepsilon > 0$, there is a neighborhood of U_1 of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \le \varepsilon|\sigma(t) - s|, \tag{1.2}$$

for all $s \in U_1$. If $t \in \mathbb{T}_\kappa$, then we define the *nabla derivative* of f at a point t to be the number $f^\nabla(t)$ (provided it exists) with the property that, for each $\varepsilon > 0$, there is a neighborhood of U_2 of t such that

$$\left|[f(\rho(t)) - f(s)] - f^\nabla(t)[\rho(t) - s]\right| \le \varepsilon|\rho(t) - s|, \tag{1.3}$$

for all $s \in U_2$.

Remark 1.1. If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f^\nabla(t) = f'(t)$, and if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$ and $f^\nabla(t) = \nabla f(t) = f(t) - f(t-1)$.

A function $F : \mathbb{T} \to \mathbb{R}$ is called a Δ-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the Cauchy Δ-integral from a to t of f is defined by

$$\int_a^t f(s)\triangle s = F(t) - F(a) \quad \forall t \in \mathrm{T}. \tag{1.4}$$

A function $\Phi : \mathbb{T} \to \mathbb{R}$ is called a ∇-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $\Phi^{\nabla}(t) = f(t)$ for all $t \in \mathbb{T}_k$. We then define the Cauchy ∇-integral from a to t of f by

$$\int_a^t f(s)\nabla s = \Phi(t) - \Phi(a) \quad \forall t \in \mathbb{T}. \tag{1.5}$$

Note that, in the case $\mathbb{T} = \mathbb{R}$, we have

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\nabla t = \int_a^b f(t)dt, \tag{1.6}$$

and, in the case $\mathbb{T} = \mathbb{Z}$, we have

$$\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k), \qquad \int_a^b f(t)\nabla t = \sum_{k=a+1}^{b} f(k), \tag{1.7}$$

where $a, b \in \mathbb{T}$ with $a \le b$.

There are two types of impulse effects that are studied in the literature. The first is the "fixed-time impulse": a set of times $0 < t_1 < t_2 < \cdots < t_n < T$ is specified, and the solution is required to satisfy

$$u(t_k^+) = I_k(u(t_k)) \tag{1.8}$$

for $k = 1, 2, \ldots, n$, where the functions I_k provide the "impulse." Also studied are "variable-time impulses," in which a set of curves $t = \tau_1(x), t = \tau_2(x), \ldots, t = \tau_n(x)$ is given, and the solution satisfies $u(t^+) = I_k(u(t))$ for $t = \tau_k(u(t))$, $k = 1, 2, \ldots, n$. Impulses of both types introduce discontinuities in the solution. As mentioned in [17] and other works in the reference list, applications involving impulse effects can be found in biology, medicine, physics, economics, pharmacokinetics, and engineering. In this paper, we consider fixed-time impulses. Without loss of generality, we investigate systems with a single impulse.

2. First order

Let $0, t_1, T \in \mathbb{T}$ with $0 < t_1 < T$ and t_1 right dense. Let $J = [0, T] \cap \mathbb{T}$,

$$u^{\nabla}(t) = g(t, u(t)), \quad t \in J \setminus \{t_1\}, \tag{2.1}$$

$$u(t_1^+) = I(u(t_1)), \tag{2.2}$$

$$B(u(0), u(T)) = 0. \tag{2.3}$$

Note that (2.3) covers as special cases many initial and boundary conditions found in the literature. Let $J_1 = [0, t_1] \cap J$, $J_2 = (t_1, T] \cap J$. Define $\int_a^b y(s)\nabla s = \int_{(a,b]} y(s)\nabla s$, where the integrals in Section 2 are with respect to the Lebesgue ∇-measure as defined by Atici and Guseinov in [3].

Let u_i be the restriction of $u : J \to \mathbb{R}$ to J_i, $i = 1,2$, then

$$\begin{aligned}
\mathcal{C}(J_1) &= \{u : J_1 \longrightarrow \mathbb{R} : u \text{ is continuous on } J_1\}, \\
\mathcal{C}(J_2) &= \{u : J_2 \longrightarrow \mathbb{R} : u \text{ is continuous on } J_2 \text{ and } u(t_1^+) \text{ exists}\}, \\
A &= \{u : J \longrightarrow \mathbb{R} : u_1 \in \mathcal{C}(J_1) \text{ and } u_2 \in \mathcal{C}(J_2)\}.
\end{aligned} \tag{2.4}$$

For $u \in A$, let $\|u\| = \sup\{|u(t)| : t \in J\}$. $(A, \|\cdot\|)$ is a Banach space. For $u, v \in A$,

$$[u,v] \equiv \{w \in A : u(t) \le w(t) \le v(t)\ \forall t \in J\}. \tag{2.5}$$

Definition 2.1. $u : \mathbb{T} \to \mathbb{R}$ is a solution of (2.1)–(2.3) if

(i) $u \in A$,

(ii)

$$\begin{aligned}
u(t) &= u(0) + \int_0^t g(s,u(s))\nabla s, \quad t \in J_1, \\
u(t) &= I(u(t_1)) + \int_{t_1}^t g(s,u(s))\nabla s, \quad t \in J_2,
\end{aligned} \tag{2.6}$$

(iii) $B(u(0),u(T)) = 0$.

We call $\alpha : J \to \mathbb{R}$ a lower solution of (2.1)–(2.3) if

(i) $\alpha \in A$,

(ii) $\alpha(b) - \alpha(a) \le \int_a^b g(s,\alpha(s))\nabla s$ for $a \le b$ and $a, b \in J_1$, or $a, b \in J_2$,

(iii) $\alpha(t_1^+) \le I(\alpha(t_1))$,

(iv) $B(\alpha(0),\alpha(T)) \le 0$.

We call $\beta : J \to \mathbb{R}$ an upper solution of (2.1)–(2.3) if it satisfies the same assumptions, but replace $\le$ with $\ge$.

Let $p(t,x) = \max\{\alpha(t), \min\{x, \beta(t)\}\}$. We have the following assumptions throughout this section:

(1) for each $x \in \mathbb{R}$, $g(\cdot,x)$ is Lebesgue ∇-measurable on J,

(2) for a.e. (∇) $t \in J$, $g(t,\cdot)$ is continuous,

(3) there is an $h : J \to [0,\infty)$, $\int_0^T h(s)\nabla s < \infty$ such that $|g(t,p(t,x))| \le h(t)$ a.e. (∇) on J, for all $x \in \mathbb{R}$,

(4) $I : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing,

(5) $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and for each $x \in [\alpha(0), \beta(0)]$, $B(x,\cdot)$ is nonincreasing.

Note. a.e. (∇) denotes the Lebesgue ∇-measure.

Theorem 2.2. *Assume that conditions (1)–(5) are satisfied and α, β are lower and upper solutions of (2.1)–(2.3) with $\alpha(t) \le \beta(t)$ for all $t \in J$. Then, there exists a solution u to (2.1)–(2.3) such that $u \in [\alpha, \beta]$.*

Proof. Our proof follows that of Cabada and Liz [9]. Define an operator $G : A \to A$ by

$$Gu(t) = p(0,\overline{u}(0)) + \int_0^t g(s,p(s,u(s)))\nabla s, \quad t \in J_1, \tag{2.7}$$

$$Gu(t) = I(p(t_1,u(t_1))) + \int_{t_1}^t g(s,p(s,u(s)))\nabla s, \quad t \in J_2, \tag{2.8}$$

where $\overline{u}(0) \equiv u(0) - B(u(0),u(T))$.

Claim 2.3. If u is a fixed point of the operator G, then u is a solution of (2.1)–(2.3) such that $u \in [\alpha,\beta]$.

Proof of Claim 2.3. We assume that $u \in A$ satisfies

$$u(t) = p(0,\overline{u}(0)) + \int_0^t g(s,p(s,u(s)))\nabla s, \quad t \in J_1, \tag{2.9}$$

$$u(t) = I(p(t_1,u(t_1))) + \int_{t_1}^t g(s,p(s,u(s)))\nabla s, \quad t \in J_2. \tag{2.10}$$

Subclaim 1. $u(t) \in [\alpha(t),\beta(t)]$, for all $t \in J$.

Note that, by letting $t = 0$ in the right-hand side of (2.9), we have $\int_{\varnothing} g(s,p(s,u(s)))\nabla s = 0$ and hence $u(0) = p(0,\overline{u}(0))$ which is in $[\alpha(0),\beta(0)]$ by the definition of p. Suppose there exists a $t_1' \in (0,t_1] \cap J$ such that $\alpha(t_1') > u(t_1')$. Since $\alpha(0) \le u(0)$, there exists a $t_2' \in [0,t_1') \cap J$ such that $\alpha(t_2') \le u(t_2')$ and $\alpha(t) > u(t)$ on $(t_2',t_1'] \cap J$. Then, $g(t,p(t,u(t))) = g(t,\alpha(t))$ for all $t \in (t_2',t_1'] \cap J$. We then have, for any $t \in (t_2',t_1'] \cap J$,

$$\begin{aligned}
\int_0^t g(s,p(s,u(s)))\nabla s &= u(t) - p(0,\overline{u}(0)) = u(t_2') - p(0,\overline{u}(0)) + u(t) - u(t_2'), \\
\int_0^{t_2'} g(s,p(s,u(s)))\nabla s &+ u(t) - u(t_2') \\
\Longrightarrow u(t) - u(t_2') &= \int_0^t g(s,p(s,u(s)))\nabla s - \int_0^{t_2'} g(s,p(s,u(s)))\nabla s \\
&= \int_{t_2'}^t g(s,p(s,u(s)))\nabla s = \int_0^{t_2'} g(s,\alpha(s))\nabla s.
\end{aligned} \tag{2.11}$$

From assumption (ii) of the definition of lower solution, we have

$$\alpha(t) - \alpha(t_2') \le \int_{t_2'}^t g(s,\alpha(s))\nabla s. \tag{2.12}$$

We then have

$$u(t) - u(t_2') = \int_{t_2'}^t g(s,\alpha(s))\nabla s \ge \alpha(t) - \alpha(t_2') \tag{2.13}$$

and recalling that $\alpha(t_2') \le u(t_2')$ and $u(t) < \alpha(t)$, this is a contradiction. Hence, $\alpha \le u$ on J_1. Similarly, $u \le \beta$ on J_1.

We then have

$$\alpha(t_1) \le u(t_1) \le \beta(t_1), \tag{2.14}$$

and using the fact that I is nondecreasing, we have

$$\alpha(t_1^+) \le I(\alpha(t_1)) \le I(u(t_1)) \le I(\beta(t_1)) \le \beta(t_1^+). \tag{2.15}$$

We also have $I(u(t_1)) = I(p(t_1,u(t_1))) = u(t_1^+)$ and hence from (2.15) we conclude

$$\alpha(t_1^+) \le u(t_1^+) \le \beta(t_1^+). \tag{2.16}$$

We may now proceed as before to get $\alpha \le u \le \beta$ on J_2, establishing Subclaim 1.

We may apply Subclaim 1 to (2.9) to verify that u satisfies the first equation in property (ii) of a solution to (2.1)–(2.3), and apply Subclaim 1 to (2.10) to verify that u satisfies the second equation in property (ii).

Subclaim 2. $\overline{u}(0) \in [\alpha(0),\beta(0)]$.

Suppose that $\alpha(0) > \overline{u}(0) = u(0) - B(u(0),u(T))$. Thus, $u(0) = p(0,\overline{u}(0)) = \alpha(0)$ and hence $B(u(0),u(T)) > 0$. Using assumption (5), we have $B(\alpha(0),\alpha(T)) \ge B(\alpha(0),u(T)) > 0$, which contradicts α being a lower solution of (2.1)–(2.3). We then have $\alpha(0) \le \overline{u}(0)$ and, similarly, $\overline{u}(0) \le \beta(0)$, establishing Subclaim 2.

As a result of Subclaim 2, we have $u(0) = p(0,\overline{u}(0)) = \overline{u}(0) = u(0) - B(u(0),u(T))$ and hence $B(u(0),u(T)) = 0$, establishing Claim 2.3. □

Claim 2.4. $G: A \to A$ has a fixed point.

Proof of Claim 2.4. We will apply Schauder's fixed point theorem.

Let $K = \|\alpha\| + \|\beta\|$. Define $w: J \to \mathbb{R}$ by $w(t) = K + \int_0^t h(s)\nabla s$.

Let

$$\begin{aligned} S = \{u \in A : |u(0)| \le K,\ |u(t_1^+)| \le K,\ |u(b) - u(a)| \le w(b) - w(a) \\ \text{on } 0 \le a \le b \le t_1 \text{ or } t_1 < a \le b \le T, \text{ where } a,b \in J\}. \end{aligned} \tag{2.17}$$

It can be shown that S is a convex and compact subset of $(A, \|\cdot\|)$.

Subclaim 3. $G(S) \subseteq S$.

Let $u \in S$ and consider Gu. Let $t = 0$ in (2.7) to obtain

$$|Gu(0)| = |p(0,\overline{u}(0))| \le \max\{|\alpha(0)|, |\beta(0)|\} \le K. \tag{2.18}$$

Note that $\alpha(t_1) \le p(t_1,u(t_1)) \le \beta(t_1)$, hence

$$\begin{gathered} \alpha(t_1^+) \le I(\alpha(t_1)) \le I(p(t_1,u(t_1))) \le I(\beta(t_1)) \le \beta(t_1^+), \\ |I(p(t_1,u(t_1)))| \le \max\{|\alpha(t_1^+)|, |\beta(t_1^+)|\} \le K. \end{gathered} \tag{2.19}$$

Let $t \downarrow t_1$ in (2.8) to obtain

$$|Gu(t_1^+)| = |I(p(t_1,u(t_1)))| \le K. \tag{2.20}$$

Let $a, b \in J$ with $0 \le a \le b \le t_1$,

$$|Gu(b) - Gu(a)| = \left| \int_a^b g(s, p(s, u(s))) \nabla s \right| \le \int_a^b h(s) \nabla s = w(b) - w(a). \tag{2.21}$$

Similar results hold for $t_1 < a \le b \le T$.

Subclaim 4. $G : S \to S$ is continuous.

Let $\{u_n\}_{n=1}^{\infty} \subseteq S$ which converges to $u \in S$ in the space $(A, \|\cdot\|)$. Note that $u_n \to u$ uniformly on compact subsets of J. Let $n \in \mathbb{N}$ and $t \in J_1$, then

$$\begin{aligned} & Gu(t) - Gu_n(t) \\ &\quad = Gu(t) - p(0, \overline{u}(0)) - [Gu_n(t) - p(0, \overline{u}_n(0))] + p(0, \overline{u}(0)) - p(0, \overline{u}_n(0)) \\ &\quad = \int_0^t g(s, p(s, u(s))) \nabla s - \int_0^t g(s, p(s, u_n(s))) \nabla s + p(0, \overline{u}(0)) - p(0, \overline{u}_n(0)) \end{aligned} \tag{2.22}$$

and hence

$$\begin{aligned} & |Gu(t) - Gu_n(t)| \\ &\quad \le \int_0^{t_1} |g(s, p(s, u(s))) - g(s, p(s, u_n(s)))| \nabla s + p(0, \overline{u}(0)) - p(0, \overline{u}_n(0)). \end{aligned} \tag{2.23}$$

Now take $\lim_{n\to\infty}$ and apply the Lebesgue dominated convergence theorem and the continuity of g in its second variable and of p to conclude

$$\lim_{n\to\infty} |Gu_n(t) - Gu(t)| = 0. \tag{2.24}$$

Note that (2.23) does not involve t in its right-hand side, so we can conclude that the convergence is uniform on J_1.

A similar argument shows that $Gu_n \to G$ uniformly on compact subsets of J_2.

Hence, by Subclaims 3 and 4, Schauder's fixed point theorem applies to G, finishing the proof of Claim 2.4. □

Claims 2.3 and 2.4 yield the desired result. □

Example 2.5. Let $\mathbb{T} = [0,1] \cup [2,3]$, $t_1 = 2$, $g(t,x) = t^2 + x^2$, $I(x) = x + 1$, $u(0) = 0$. (Note that I is not bounded, as required in [4].)

$\alpha(t) = 0$ is a lower solution .

To construct β on $[0,1]$, we solve $\beta' = 1 + \beta^2$ $(\ge t^2 + \beta^2)$, $\beta(0) = 0$. Then this implies that $\beta(t) = \tan t$. By considering boundary conditions, we have

$$\begin{gathered} \beta(2) = \beta(0) + \int_0^2 \beta^{\nabla}(s) \nabla s = \beta(0) + \int_0^1 \beta'(s) ds + \beta^{\nabla}(2), \\ \beta(2) = \tan 1 + \frac{\beta(2) - \beta(1)}{2 - 1} \Longrightarrow \beta(2) = \beta(2) \Longrightarrow \beta(2) \text{ is arbitrary,} \quad \text{let } \beta(2) = 1, \\ \beta(2^+) = I(\beta(2)) = 2. \end{gathered} \tag{2.25}$$

To construct β on $[2,3]$, we solve $\beta' = 9+\beta^2$ $(\geq t^2+\beta^2)$, $\beta(2) = 2$, then we have

$$\beta = 3\tan\left(\tan^{-1}\left(\frac{2}{3}\right)+3t-6\right). \tag{2.26}$$

Applying Theorem 2.2, we know there exists a solution u such that $0 \leq u(t) \leq \beta(t)$ for $t \in \mathbb{T}$.

3. Second order

In this section, we are concerned with second-order dynamic equations with functional boundary conditions and impulse:

$$y^{\Delta\Delta}(t) = f\left(t, y^{\sigma}(t)\right), \quad t \in \mathbb{T}^{\kappa^2} \equiv [a,b] \setminus \{t_1\}, \tag{3.1}$$

$$L_1\left(y(a), y^{\Delta}(a), y\left(\sigma^2(b)\right), y^{\Delta}\left(\sigma(b)\right)\right) = 0, \tag{3.2}$$

$$L_2\left(y(a), y\left(\sigma^2(b)\right)\right) = 0, \tag{3.3}$$

$$y\left(t_1^+\right) - y\left(t_1^-\right) = r_1, \tag{3.4}$$

$$y^{\Delta}\left(t_1^+\right) - y^{\Delta}\left(t_1^-\right) = I\left(y(t_1), y^{\Delta}\left(t_1^-\right)\right), \tag{3.5}$$

where $t_1 \in \mathbb{T}$ with $a < t_1 < b$ and t_1 right-dense, $r_1 \in \mathbb{R}$, I is a real-valued function and $J = [a,b]$. We set $y^{\Delta}(t_1) = y^{\Delta}(t_1^+)$ if t_1 is left-scattered, and $y^{\Delta}(t_1) = y^{\Delta}(t_1^-)$ if t_1 is left-dense. We note that these impulses are different from those studied in [16]. (3.2) and (3.3) cover many conditions found in the literature such as separated and nonseparated boundary conditions, respectively,

$$\begin{aligned} L_1(x,y,z,w) &= x, \qquad & L_2(x,y) &= y, \\ L_1(x,y,z,w) &= y-z, \qquad & L_2(x,y) &= y-x, \end{aligned} \tag{3.6}$$

as in [7, Chapter 4].

Let $J_1 = [a, t_1]$, $J_2 = (t_1, b]$. We define the following spaces of functions.

Let y_i be the restriction of $y : J \to \mathbb{R}$ to J_i, $i = 1,2$, then

$$\begin{gathered} \mathscr{C}'(J_1) = \left\{y : J_1 \longrightarrow \mathbb{R} : y \text{ and } y^{\Delta} \text{ are continuous on } J_1\right\}, \\ \mathscr{C}'(J_2) = \left\{y : J_2 \longrightarrow \mathbb{R} : y \text{ and } y^{\Delta} \text{ are continuous on } J_2 \text{ and } y\left(t_1^+\right) \text{ and } y^{\Delta}\left(t_1^+\right) \text{ exist}\right\}, \\ A = \left\{y : J \longrightarrow \mathbb{R} : y_1 \in \mathscr{C}'(J_1) \text{ and } y_2 \in \mathscr{C}'(J_2)\right\}. \end{gathered} \tag{3.7}$$

For $y \in A$, let $\|y\| = \sup\{|y(t)| : t \in J\}$. $(A, \|\cdot\|)$ is a Banach space. For $x, y \in A$,

$$[x,y] \equiv \left\{z \in A : x(t) \leq z(t) \leq y(t) \ \forall t \in J\right\}. \tag{3.8}$$

Now we introduce the concept of lower and upper solutions of problem (3.1)–(3.5) as follows.

Definition 3.1. The functions α and β are, respectively, a lower and an upper solution of problem (3.1)–(3.5) if the following properties hold:

(i) $\alpha, \beta \in A$;

(ii)

$$\begin{gathered} \alpha^{\Delta\Delta}(t) \geq f(t,\alpha^{\sigma}(t)) \quad \text{on } t \in [a,b] \setminus \{t_1\}, \\ L_1(\alpha(a),\alpha^{\Delta}(a),\alpha(\sigma^2(b)),\alpha^{\Delta}(\sigma(b))) \geq 0, \\ L_2(\alpha(a),\alpha(\sigma^2(b))) = 0, \qquad L_2(\alpha(a),\cdot) \text{ is injective}, \\ \alpha(t_1^+) - \alpha(t_1^-) = r_1, \\ \alpha^{\Delta}(t_1^+) - \alpha^{\Delta}(t_1^-) \geq I(\alpha(t_1),\alpha^{\Delta}(t_1^-)); \end{gathered} \tag{3.9}$$

(iii)

$$\begin{gathered} \beta^{\Delta\Delta}(t) \leq f(t,\beta^{\sigma}(t)) \quad \text{on } t \in [a,b] \setminus \{t_1\}, \\ L_1(\beta(a),\beta^{\Delta}(a),\beta(\sigma^2(b)),\beta^{\Delta}(\sigma(b))) \leq 0, \\ L_2(\beta(a),\beta(\sigma^2(b))) = 0, \qquad L_2(\beta(a),\cdot) \text{ is injective}, \\ \beta(t_1^+) - \beta(t_1^-) = r_1, \\ \beta^{\Delta}(t_1^+) - \beta^{\Delta}(t_1^-) \leq I(\beta(t_1),\beta^{\Delta}(t_1^-)). \end{gathered} \tag{3.10}$$

We assume the following conditions are satisfied for the functions f, L_1 and L_2, and I.

(F) The function $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous.

(L) $L_1 \in C(\mathbb{R}^4, \mathbb{R})$ is nondecreasing in the second variable, nonincreasing in the fourth. Moreover, $L_2 : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and it is nonincreasing with respect to its first variable.

(I) I is continuous and strictly increasing with respect to the first variable and nonincreasing in the second variable.

We consider the following modified truncated problem:

$$y^{\Delta\Delta}(t) - y^{\sigma}(t) = f(t, p(\sigma(t), y^{\sigma}(t))) - p(\sigma(t), y^{\sigma}(t)), \quad t \in \mathbb{T}^{\kappa^2} \equiv [a,b] \setminus \{t_1\}, \tag{3.11}$$

$$y(a) = L_1^*(y(a), y^{\Delta}(a), y(\sigma^2(b)), y^{\Delta}(\sigma(b))), \tag{3.12}$$

$$y(\sigma^2(b)) = L_2^*(y(a), y(\sigma^2(b))), \tag{3.13}$$

$$r_1 = y(t_1^+) - y(t_1^-), \tag{3.14}$$

$$y^{\Delta}(t_1^+) - y^{\Delta}(t_1^-) = I(y(t_1), y^{\Delta}(t_1^-)), \tag{3.15}$$

where $p(t,y) = \min\{\max\{\alpha(t), y\}, \beta(t)\}$,

$$\begin{aligned} L_1^*(x,y,z,w) &= p(a, x + L_1(x,y,z,w)) \quad \forall (x,y,z,w) \in \mathbb{R}^4, \\ L_2^*(x,y) &= p(\sigma^2(b), y - L_2(x,y)) \quad \forall (x,y) \in \mathbb{R}^2. \end{aligned} \tag{3.16}$$

Theorem 3.2. *Assume that conditions (F) and (L) are satisfied. If there exist a lower solution α and an upper solution β of (3.1)–(3.5) such that $\alpha \leq \beta$ on $\mathbb{T}$, then the BVP (3.11)–(3.15) has a solution.*

Proof. It is not difficult to verify that the problem

$$\begin{gathered} y^{\Delta\Delta} - y^{\sigma} = 0, \quad t \in [a,b], \\ y(a) = y(\sigma^2(b)) = 0, \end{gathered} \tag{3.17}$$

has only the trivial solution.

By using [7, Theorem 4.67 and Corollary 4.74], we have that for every $h \in \mathbb{C}_{rd}[a,b]$ and $A, B \in \mathbb{R}$, the problem

$$\begin{aligned} &y^{\Delta\Delta} - y^{\sigma} = h(t), \quad t \in [a,b], \\ &y(a) = A, \qquad y(\sigma^2(b)) = B, \end{aligned} \tag{3.18}$$

has a solution $y(t)$ if and only if the operator

$$Qy(t) = Ay_1(t) + By_2(t) + \int_a^{\sigma(b)} G(t,s)h(s)\Delta s \tag{3.19}$$

has a fixed point. Here $y_1(t)$, $y_2(t)$ are the solutions of the linear homogeneous equation $y^{\Delta\Delta} - y^{\sigma} = 0$, $t \in [a,b]$ and satisfy the boundary conditions $y_1(a) = 1$, $y_1(\sigma^2(b)) = 0$ and $y_2(a) = 0$, $y_2(\sigma^2(b)) = 1$.

G is called the Green's function of the Dirichlet problem. One can verify that (see [7, page 169]) it is continuous in $[a,\sigma^2(b)] \times [a,\sigma^2(b)]$ and $G^{\Delta}(\cdot,s)$ is continuous at $t \neq s = \sigma(s)$ and bounded in $[a,\sigma^2(b)]$.

Define

$$\begin{aligned} Qy(t) = {} & L_1^*(y(a), y^{\Delta}(a), y(\sigma^2(b)), y^{\Delta}(\sigma(b)))y_1(t) + L_2^*(y(a), y(\sigma^2(b)))y_2(t) \\ & + \int_a^{\sigma(b)} G(t,s)\{f(s, p(\sigma(s), y^{\sigma}(s))) - p(\sigma(s), y(\sigma(s)))\}\Delta s + L(t,y(t)), \end{aligned} \tag{3.20}$$

where

$$L(t,y) = \begin{cases} \dfrac{y_2(t)}{W}[y_1(t_1)I(y(t_1), y^{\Delta}(t_1^-)) - r_1 y_1^{\Delta}(t_1)], & a \leq t \leq t_1, \\ \dfrac{y_1(t)}{W}[y_2(t_1)I(y(t_1), y^{\Delta}(t_1^-)) - r_1 y_2^{\Delta}(t_1)], & t_1 \leq t \leq \sigma^2(b), \end{cases} \tag{3.21}$$

where $W = y_2(t_1)y_1{}^{\Delta}(t_1) - y_1(t_1)y_2{}^{\Delta}(t_1)$.

One can easily observe that Q has a fixed point y if and only if y is a solution of (3.11)–(3.15).

Since L_1, L_2, p, and G are bounded and continuous, it can be shown that there exists $R > 0$ such that the compact operator $Q : S \to S$ where $S = \{y \in A : \|y\| \leq R\}$.

Since S is a closed, bounded, and convex set, in view of the Tychonoff-Schauder fixed point theorem, there is at least one fixed point of Q. □

THEOREM 3.3. *Assume that conditions* (F) *and* (I) *are satisfied. Let* α *and* β *be a lower and upper solution, respectively, of the problem (3.1)–(3.5) such that* $\alpha \leq \beta$ *on* $\mathbb{T}$. *Then every solution of the BVP (3.11)–(3.15) belongs to the sector* $[\alpha, \beta]$.

Proof. Let y be a solution of (3.11)–(3.15), by definition of L_1^* and L_2^*, we know that $y(a) \in [\alpha(a),\beta(a)]$ and $y(\sigma^2(b)) \in [\alpha(\sigma^2(b)),\beta(\sigma^2(b))]$. We will prove that $y \in [\alpha,\beta]$ for $t \in (a,\sigma^2(b))$.

Consider $z(t) = y(t) - \beta(t)$. By definition of β, z is continuous on $[a,\sigma^2(b)]$. Suppose, to the contrary, there is a $t^* \in (a,t_1) \cup (t_1,\sigma^2(b))$ such that $(y-\beta)(t^*) = \max_{t\in\mathbb{T}}\{y(t) - \beta(t)\} > 0$.

Suppose that t^* is left scattered. In this case, we have that

$$y^\Delta(t^*) \le \beta^\Delta(t^*), \qquad y^{\Delta\Delta}(\rho(t^*)) \le \beta^{\Delta\Delta}(\rho(t^*)). \tag{3.22}$$

Consequently, by using condition F, we arrive at the following contradiction:

$$\begin{aligned} 0 &> y^{\Delta\Delta}(\rho(t^*)) - \beta^{\Delta\Delta}(\rho(t^*)) - (y(t^*) - \beta(t^*)) \\ &\ge f(\rho(t^*),\beta(t^*)) - f(\rho(t^*),\beta(t^*)) = 0. \end{aligned} \tag{3.23}$$

When t^* is left dense the contradiction holds in a similar way.

Now suppose $t^* = t_1$.

Case 1. t_1 is left scattered.

Then we have $z^\Delta(t_1^+) \le 0$ and $z^\Delta(t_1^+) = z^\Delta(t_1)$. Consequently, by using condition (I), we arrive at the following contradiction:

$$\begin{aligned} 0 &= z^\Delta(t_1^+) - z^\Delta(t_1^-) \\ &= y^\Delta(t_1^+) - y^\Delta(t_1^-) - [\beta^\Delta(t_1^+) - \beta^\Delta(t_1^-)] \\ &\ge I(y(t_1),y^\Delta(t_1)) - I(\beta(t_1),\beta^\Delta(t_1)) \\ &> I(\beta(t_1),y^\Delta(t_1)) - I(\beta(t_1),\beta^\Delta(t_1)) \ge 0. \end{aligned} \tag{3.24}$$

Case 2. t_1 is left dense.

For sufficiently small $\epsilon > 0$, we have

$$z^\Delta(s) \ge 0 \quad \text{for } s \in (t_1 - \epsilon, t_1), \qquad z^\Delta(s^*) \le 0 \quad \text{for } s^* \in (t_1, t_1 + \epsilon). \tag{3.25}$$

Then the contradiction holds in a similar way.

Analogously, the fact that $\alpha(t) \le y(t)$ for all $t \in \mathbb{T}$ can be shown. □

Theorem 3.4. *Assume that* (L) *holds. If* $y \in [\alpha,\beta]$ *is a solution of (3.11)–(3.15), then* y *satisfies equalities (3.1)–(3.5).*

Proof. If $y(\sigma^2(b)) - L_2(y(a),y(\sigma^2(b))) < \alpha(\sigma^2(b))$, the definition of L_2^* gives us that $y(\sigma^2(b)) = \alpha(\sigma^2(b))$.

Now using (L), we obtain a contradiction:

$$\begin{aligned} \alpha(\sigma^2(b)) &> y(\sigma^2(b)) - L_2(y(a),y(\sigma^2(b))) \\ &\ge \alpha(\sigma^2(b)) - L_2(\alpha(a),\alpha(\sigma^2(b))) = \alpha(\sigma^2(b)). \end{aligned} \tag{3.26}$$

Analogously, we arrive at $\alpha(\sigma^2(b)) \le y(\sigma^2(b)) - L_2(y(a),y(\sigma^2(b))) \le \beta(\sigma^2(b))$ and (3.13) implies that $L_2(y(a),y(\sigma^2(b))) = 0$.

To prove that $L_1(y(a), y^\Delta(a), y(\sigma^2(b)), y^\Delta(\sigma(b))) = 0$, it is enough using (3.12) to show

$$\alpha(a) \le y(a) + L_1\big(y(a), y^\Delta(a), y(\sigma^2(b)), y^\Delta(\sigma(b))\big) \le \beta(a). \tag{3.27}$$

If $y(a) + L_1(y(a), y^\Delta(a), y(\sigma^2(b)), y^\Delta(\sigma(b))) < \alpha(a)$, then $y(a) = \alpha(a)$ implies that $0 = L_2(y(a), y(\sigma^2(b))) = L_2(\alpha(a), \alpha(\sigma^2(b)))$.

Since L_2 is injective with respect to the second variable, we have $y(\sigma^2(b)) = \alpha(\sigma^2(b))$.

Using the definition of L_1, we obtain a contradiction:

$$\begin{aligned} \alpha(a) &> y(a) + L_1\big(y(a), y^\Delta(a), y(\sigma^2(b)), y^\Delta(\sigma(b))\big) \\ &\ge \alpha(a) + L_1\big(\alpha(a), \alpha^\Delta(a), \alpha(\sigma^2(b)), \alpha^\Delta(\sigma(b))\big) \ge \alpha(a). \end{aligned} \tag{3.28}$$

Here we used the fact that $y \in [\alpha, \beta]$, and $\alpha(a) = y(a)$, $\alpha(\sigma^2(b)) = y(\sigma^2(b))$, consequently, it follows that $y^\Delta(a) \ge \alpha^\Delta(a)$ and $y^\Delta(\sigma(b)) \le \alpha^\Delta(\sigma(b))$. The other inequality holds similarly. □

Example 3.5. Let $\mathbb{T}$ be any time scale and let the point 1/2 be a right-dense point in $\mathbb{T} \cap [0,1]$. We define f, L_1, and L_2 in the following way:

$$\begin{gathered} f(t,y) = y \sinh\big((y-1)^2\big), \qquad L_1(x,y,z,w) = 1 - x, \\ L_2(x,y) = -y, \qquad I(x,y) = x - 1. \end{gathered} \tag{3.29}$$

Next we consider the following boundary value problem:

$$y^{\Delta\Delta}(t) = f\big(t, y^\sigma(t)\big), \quad t \in [0,1]^{\kappa^2} \setminus \left\{\frac{1}{2}\right\}, \tag{3.30}$$

$$y(0) = 1, \tag{3.31}$$

$$y(1) = 0, \tag{3.32}$$

$$y\left(\frac{1}{2}^+\right) - y\left(\frac{1}{2}^-\right) = -1, \tag{3.33}$$

$$y^\Delta\left(\frac{1}{2}^+\right) - y^\Delta\left(\frac{1}{2}^-\right) = I\left(y\left(\frac{1}{2}\right), y^\Delta\left(\frac{1}{2}^-\right)\right). \tag{3.34}$$

One can easily verify that

$$\alpha(t) = \begin{cases} 0, & \text{if } t \in \left[0, \dfrac{1}{2}\right], \\ 2(t-1), & \text{if } t \in \left(\dfrac{1}{2}, 1\right], \end{cases} \tag{3.35}$$

is a lower solution and

$$\beta(t) = \begin{cases} 2t+1, & \text{if } t \in \left[0, \dfrac{1}{2}\right], \\ -2(t-1), & \text{if } t \in \left(\dfrac{1}{2}, 1\right], \end{cases} \tag{3.36}$$

is an upper solution of the problem (3.30)–(3.34).

Theorem 3.3 assures that there exists a solution $y(t)$ of the problem (3.30)–(3.34) such that $y \in [\alpha,\beta]$. We note that

$$y(t) = \begin{cases} 1, & \text{if } t \in \left[0, \dfrac{1}{2}\right], \\ 0, & \text{if } t \in \left(\dfrac{1}{2}, 1\right], \end{cases} \tag{3.37}$$

is one such solution.

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