STABILITY OF PERIODIC SOLUTIONS OF FIRST-ORDER DIFFERENCE EQUATIONS LYING BETWEEN LOWER AND UPPER SOLUTIONS

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We prove that if there exists $\alpha \le \beta$, a pair of lower and upper solutions of the first-order discrete periodic problem $\Delta u(n) = f(n,u(n))$; $n \in I_N \equiv \{0,\dots,N-1\}$, $u(0) = u(N)$, with f a continuous N-periodic function in its first variable and such that $x + f(n,x)$ is strictly increasing in x, for every $n \in I_N$, then, this problem has at least one solution such that its N-periodic extension to $\mathbb{N}$ is stable. In several particular situations, we may claim that this solution is asymptotically stable.

1. Introduction

It is well known that one of the most important concepts in the qualitative theory of differential and difference equations is the stability of the solutions of the treated problems. Classical tools, as approximation by linear equations or Lyapunov functions, have been developed for both type of equations, see [7] for ordinary differential equations and [8] for difference ones.

More recently, some authors as, among others, de Coster and Habets [6], Nieto [9], or Ortega [10], have proved the stability of solutions of adequate ordinary differential equations that lie between a pair of lower and upper solutions. In this case, fixed points theorems and degree and index theory are the fundamental arguments to deduce the mentioned stability results. Stability for order-preserving operators defined on Banach spaces have been obtained by Dancer in [4] and Dancer and Hess in [5]. On these papers, the authors describe the assymptotic behavior of the iterates that lie between a lower and an upper solution of suitable operators.

Our purpose is to ensure the stability of at least one periodic solution of a first-order difference equation. We will prove such result by using a monotone nondecreasing operator. In this case, the defined operator does not verify the conditions imposed in [5]. The so-obtained results are in the same direction as the ones proved by the authors in [3] for the first-order implicit difference equation $\Delta u(i) = f(i,u(i+1))$ coupled with periodic boundary conditions. In that situation, we give some optimal conditions on function f and on the number of the possible periodic solutions of the considered problem, that warrant the existence of at least one stable solution. The arguments there are different

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from the ones used in this paper because in that situation, due to the lack of uniqueness of solutions of the initial problem, the discrete operator considered here cannot be defined.

The paper is organized as follows. In Section 2, we present some fundamental properties of the set of solutions of initial and periodic problems. In Section 3, we prove the existence of at least one N-periodic stable solution. Finally, Section 4 is devoted to give some examples that point out the, in some sense, optimality of the obtained results.

2. Preliminaries

This paper is devoted to study the stability, by using the method of lower and upper solutions, of the N-periodic solutions of the following first-order difference equation:
(P)

$$\Delta u(n) = f(n, u(n)); \quad n \in \mathbb{N}, \tag{2.1}$$

where, for every $n \in \mathbb{N}$, $\Delta u(n) = u(n+1) - u(n)$, $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, N-periodic in its first variable for some $N \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ given.

Throughout the paper, having $x = (x(0), \dots, x(N))$ and $y = (y(0), \dots, y(N))$, we will say that $x \le y$ in $J_N \equiv \{0, \dots, N\}$ if $x(j) \le y(j)$ for all $j \in J_N$, we will say that $x < y$ in J_N when $x \le y$ in J_N and there is at least one $j_0 \in J_N$ such that $x(j_0) < y(j_0)$, moreover $x \ll y$ in J_N when $x(j) < y(j)$ for all $j \in J_N$. We will denote

$$[x, y] = \{z = (z(0), \dots, z(N));\ x \le z \le y \text{ in } J_N\}. \tag{2.2}$$

We say that $u : \mathbb{N} \to \mathbb{R}$ is an N-periodic solution of problem (P) if it satisfies equation (P) in $\mathbb{N}$ and $u(n) = u(N+n)$ for all $n \in \mathbb{N}$. From the periodicity of f in its first variable, it is obvious that to look for an N-periodic solution of (P) is equivalent to solve equation
(P_N)

$$\Delta u(i) = f(i, u(i)); \quad i \in I_N \equiv \{0, \dots, N-1\}, \qquad u(0) = u(N). \tag{2.3}$$

Below, we will denote $u : J_N \to \mathbb{R}$ and $\bar{u} : \mathbb{N} \to \mathbb{R}$ as a solution of (P_N) and its N-periodic extension to $\mathbb{N}$, respectively.

We define the concept of lower and upper solutions for problem (P_N) as follows.

Definition 2.1. Let $N \in \mathbb{N}^*$ be given. Say that $\alpha = (\alpha(0), \dots, \alpha(N))$ is a lower solution of problem (P_N) if it satisfies

$$\Delta\alpha(i) \le f(i, \alpha(i)); \quad i \in I_N, \qquad \alpha(0) \le \alpha(N). \tag{2.4}$$

The concept of upper solution is given by reversing the previous inequalities.

It is important to note, see [1, 2], that the existence of α and β, a pair of lower and upper solutions of problem (P_N), such that $\alpha \le \beta$ in J_N, does not imply the existence of a solution of this problem.

Now, by defining for each $n \in \mathbb{N}$, $h_n : \mathbb{R} \to \mathbb{R}$ as $h_n(x) := x + f(n,x)$ for all $x \in \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ as $g := h_{N-1} \circ \cdots \circ h_1 \circ h_0$, we have that problem (P_N) has a solution if and only if g has a fixed point.

If for every $n \in I_N$, h_n is a strictly increasing function on $[\alpha(n),\beta(n)]$, with $\alpha \le \beta$ a pair of lower and upper solutions of problem (P_N), then it turns out that for every $n \in I_N$,

$$\alpha(n+1) \le h_n(\alpha(n)) \le h_n(\beta(n)) \le \beta(n+1), \tag{2.5}$$

which implies that $g([\alpha(0),\beta(0)]) \subset [\alpha(0),\beta(0)]$.

Now, we assume the following properties.

(H) There exist α and β a pair of lower and upper solutions that are no solutions of problem (P_N), such that $\alpha \le \beta$ in J_N.

(Hf) $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is N-periodic in its first variable and for all $n \in I_N$, the function $f(n,\cdot)$ is continuous on $[\alpha(n),\beta(n)]$, with α and β given in (H). Moreover, $h_n(x) := x + f(n,x)$ is strictly increasing on $[\alpha(n),\beta(n)]$.

As a consequence of the ideas exposed above, we deduce the following existence result for problem (P).

Lemma 2.2. *Assume that conditions (H) and (Hf) are fulfilled, then problem (P) has at least one N-periodic solution $\bar{u}$ such that $u(\equiv \bar{u}|_{J_N}) \in [\alpha,\beta]$.*

Remark 2.3. One can see in [1] that the previous property is optimal in the sense that if h_n is not monotone increasing on $[\alpha(n),\beta(n)]$ for some $n \in I_N$, then the existence result is not guaranteed.

On the other hand, plainly for every $\xi \in \mathbb{R}$, the initial problem

(P_ξ)

$$\Delta u(n) = f(n,u(n)); \quad n \in \mathbb{N}, \qquad u(0) = \xi, \tag{2.6}$$

has a unique solution which will be denoted throughout the paper as u_ξ.

If we denote its restriction to the interval I_N as

(P_ξ^N)

$$\Delta u(n) = f(n,u(n)); \quad n \in I_N, \qquad u(0) = \xi, \tag{2.7}$$

then, one can see in [2, Example 2.2] that the existence of $\alpha \le \beta$, a pair of lower and upper solutions of this problem, that is,

$$\Delta\alpha(i) - f(i,\alpha(i)) \le 0 \le \Delta\beta(i) - f(i,\beta(i)); \quad i \in I_N, \qquad \alpha(0) \le \xi \le \beta(0), \tag{2.8}$$

is not sufficient to ensure that the unique solution of problem (P_ξ^N) lies in $[\alpha,\beta]$. However, with an analogous argument to the periodic case, whenever h_n is a strictly increasing function for all $n \in I_N$, we derive that this solution belongs to the sector formed by α and β. Moreover, the strict monotony of h_n allows to ensure that the solutions starting at different initial conditions are not equal at any point.

From this arguments, one can prove the following result.

LEMMA 2.4. *If conditions* (H) *and* (Hf) *hold, then for all* $\xi \in [\alpha(0),\beta(0)]$, *the unique solution of the initial problem* (P_ξ^N) *belongs to the sector* $[\alpha,\beta]$. *Moreover, the unique solution* u_ξ *of problem* (P_ξ) *is such that* $u_\xi|_{J_N} \in [\alpha,\beta]$.

If $\xi,\eta \in [\alpha(0),\beta(0)]$ *are such that* $\xi < \eta$, *then,* $\alpha \leq u_\xi|_{J_N} \ll u_\eta|_{J_N} \leq \beta$ *in* J_N.

Below, we give a more precise description of the set of solutions of problem (P_N).

LEMMA 2.5. *Assume conditions* (H) *and* (Hf). *Let* u, $v \in [\alpha,\beta]$ *be two solutions of problem* (P_N). *Then, one of the following statements is true:*

(i) $\alpha \ll u \equiv v \ll \beta$ *in* J_N;

(ii) $\alpha \ll u \ll v \ll \beta$ *in* J_N;

(iii) $\alpha \ll v \ll u \ll \beta$ *in* J_N.

Proof. First we show that $\alpha \ll u$ in J_N.

Suppose that there exists $n_0 \in J_N$ such that $\alpha(n_0) = u(n_0)$, by using inequalities (2.5), we obtain that $\alpha(n) = u(n)$ for all $n \in \{0,\dots,n_0\}$. Therefore, $\alpha(N) = u(N)$, and so $\alpha(n) = u(n)$ for all $n \in J_N$, which implies that α is a solution of (P_N) and it contradicts hypothesis (H).

Hence, $\alpha \ll u$ in J_N. Inequality $v \ll \beta$ in J_N can be proven in a similar way.

Now, if $u(0) = v(0)$, then, from the uniqueness of solutions of the initial problem, we conclude that $u \equiv v$ in J_N, and so assertion (i) holds.

However, if $u(0) < v(0)$, then, from Lemma 2.4 we may assert that $u(n) < v(n)$ for all $n \in J_N$, so that claim (ii) is proved.

Claim (iii) is fulfilled whenever $u(0) > v(0)$. □

Previous result establishes that the set of solutions in $[\alpha,\beta]$ of problem (P_N) (and their N-periodic extensions of problem (P)) is totally ordered and bounded. From the continuity of function f, we know that it is closed. Thus, we conclude that there exist ψ and ϕ the minimum and the maximum of the aforementioned set and, clearly, they match up the minimal and the maximal solutions, respectively, in $[\alpha,\beta]$ of problem (P_N).

3. Stability

In this section, we prove the stability of at least one N-periodic solution $\bar{u}$ of problem (P) such that u belongs to the sector $[\alpha,\beta]$.

Here, we say that $u_{\bar{\xi}} : \mathbb{N} \to \mathbb{R}$, the unique solution of the initial problem $(P_{\bar{\xi}})$, is stable if and only if for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) \in (0,\varepsilon)$ such that $|u_\xi(n) - u_{\bar{\xi}}(n)| < \varepsilon$ for all $n \in \mathbb{N}^*$, whenever $\xi \in (\bar{\xi} - \delta, \bar{\xi} + \delta)$.

It is asymptotically stable if and only if it is stable and there exists $\mu > 0$ such that $\lim_{n\to\infty}(u_\xi - u_{\bar{\xi}})(n) = 0$ for all $\xi \in (\bar{\xi} - \mu, \bar{\xi} + \mu)$.

We will say that it is stable from above if the interval $(\bar{\xi} - \delta, \bar{\xi} + \delta)$ is replaced by $(\bar{\xi}, \bar{\xi} + \delta)$. Similar comment is valid for stable from below and from asymptotically stable from above and from below.

We will call attractivity set of $u_{\bar{\xi}}$ to the biggest interval $V_{\bar{\xi}}$ such that $\bar{\xi} \in V_{\bar{\xi}}$ and $\lim_{n\to\infty}(u_\xi - u_{\bar{\xi}})(n) = 0$ for all $\xi \in V_{\bar{\xi}}$.

Now, for every $u : J_N \to \mathbb{R}$, let $T(u)$ denote the unique solution of problem $(P^N_{u(N)})$.

Note that, from Lemma 2.4, we know that $T([\alpha,\beta]) \subset [\alpha,\beta]$.

Following properties for operator T, carry over.

Proposition 3.1. *Let $\xi \in \mathbb{R}$ be given, then $u_\xi : \mathbb{N} \to \mathbb{R}$ is the solution of the initial problem (P_ξ) if and only if $u_\xi(n) = T^{k_n} v(i_n)$, with $v : J_N \to \mathbb{R}$ the unique solution of the initial problem (P^N_ξ), $k_n \in \mathbb{N}$, $i_n \in I_N$, and $n = k_n N + i_n$.*

Proof. Let u_ξ and v be the unique solutions of (P_ξ) and (P^N_ξ), respectively. By definition, $u_\xi = v$ in J_N. Since f is N-periodic in its first variable, for every $k \in \mathbb{N}$, the values of u_ξ on the intervals $\{kN, \dots, kN + N - 1\}$ match up the values of the unique solution of the initial problem $(P^N_{u_\xi(kN)})$ on I_N, that is, with the values of $T^k v$ on I_N. □

As a straightforward consequence of the previous result, we obtain the following characterization of the set of solutions of problem (P_N).

Corollary 3.2. *Let $u : J_N \to \mathbb{R}$, then u is a solution of problem (P_N) if and only if u is a fixed point of operator T.*

Proposition 3.3. *Assume that conditions (H) and (Hf) are satisfied, then operator T is monotone nondecreasing on $[\alpha,\beta]$.*

Moreover, if $u, v \in [\alpha,\beta]$ are such that $u(N) < v(N)$, then $Tu \ll Tv$ in J_N.

Proof. Let $u, v \in [\alpha,\beta]$ be such that $u \le v$ in J_N. Thus, $(Tu)(0) = u(N) \le v(N) = (Tv)(0)$. The monotonicity of operator T is a direct consequence of condition (Hf).

Last part can be easily obtained from Lemma 2.4. □

Remark 3.4. Note that, despite Proposition 3.3 is valid, operator T is not strictly increasing on $[\alpha,\beta]$, that is, if $u, v \in [\alpha,\beta]$ are such that $u < v$ in J_N, then $Tu < Tv$ in J_N. Indeed, it is enough to consider a pair of functions $u, v \in [\alpha,\beta]$ such that $u(N) = v(N)$. It is obvious that $Tu \equiv Tv$.

This property guarantees that the results given in [5] for strictly increasing operators, defined on Banach spaces, cannot be applied to the operator T defined above.

Proposition 3.5. *Assume that conditions (H) and (Hf) are fulfilled, then $\alpha < T\alpha \ll T\beta < \beta$ in J_N.*

Proof. Let $u = T\alpha$, by definition

$$u(n+1) = u(n) + f(n, u(n)), \quad n \in I_N; \qquad u(0) = \alpha(N) \ge \alpha(0). \tag{3.1}$$

Condition (Hf) ensures that $\alpha \le T\alpha$ in J_N. If the equality holds, then we have that α is a solution of the periodic problem (P_N) which contradicts hypothesis (H). Hence, $\alpha < T\alpha$ in J_N and the proof of the first inequality is complete.

One can prove in a similar way the fact that $T\beta < \beta$ in J_N.

As we have proved Lemma 2.5, $\alpha \ll \beta$ in J_N, so that the second inequality follows from Proposition 3.3. □

Proposition 3.6. *Assume that conditions (H) and (Hf) are fulfilled. Let $\alpha_0 = \alpha$, $\beta_0 = \beta$ and for all $m \geq 1$, $\alpha_m = T\alpha_{m-1}$ and $\beta_m = T\beta_{m-1}$. Then, the two following properties hold.*

(1) *$\{\alpha_m\}_{m\in\mathbb{N}}$ is a strictly increasing sequence which converges uniformly in J_N to ψ, the minimal solution in $[\alpha,\beta]$ of problem (P_N).*

(2) *$\{\beta_m\}_{m\in\mathbb{N}}$ is a strictly decreasing sequence which converges uniformly in J_N to ϕ, the maximal solution in $[\alpha,\beta]$ of problem (P_N).*

Proof. We only prove the first assertion; the second one holds similarly.

If $\alpha_1(0) = \alpha_1(N)$, then α_1 is a solution of problem (P_N). Thus, equality $\alpha_1(0) = \alpha(N)$ establishes that this case is not possible because it contradicts Lemma 2.5.

Moreover, inequality $\alpha_1(0) > \alpha_1(N)$ does not hold, so that $\alpha_1(0) < \alpha_1(N)$. Inequality $\alpha_1(0) = \alpha_0(N) < \psi(N) = \psi(0)$, Proposition 3.3, and Lemma 2.4 yield $\alpha_1 = T\alpha_0 \ll T\alpha_1 \ll \psi = T\psi$ in J_N. Inductively by using Proposition 3.3, we deduce that $\alpha_m \ll T\alpha_m \ll T\psi = \psi$ in J_N.

The conclusion follows from the boundness from above by ψ of the sequence $\{\alpha_m\}_{m\in\mathbb{N}}$ and the definition of ψ. □

This property allows us to deduce the one-sided asymptotic stability of the N-periodic extensions of the extremal solutions of the periodic problem (P_N). The obtained result is the following theorem.

Theorem 3.7. *If assumptions (H) and (Hf) hold, then $\bar{\phi}$ is asymptotically stable from above and $\bar{\psi}$ is asymptotically stable from below. Moreover, sets $[\phi(0),\beta(0)]$ and $[\alpha(0),\psi(0)]$ are contained in the attractivity sets of $\bar{\phi}$ and $\bar{\psi}$, respectively.*

Proof. We only prove the claim for $\bar{\phi}$. The other one can be proven by similar arguments.

Let $\xi \in (\phi(0),\beta(0)]$; we know, from Lemma 2.5, that $(\phi(0),\beta(0)]$ is not empty. There exists, by Proposition 3.6, $i_0 \geq 1$ such that $\xi \in (\beta_{i_0}(0),\beta_{i_0-1}(0)]$.

Let $v : J_N \to \mathbb{R}$ be the unique solution of the initial problem (P_ξ^N); monotony properties of operator T ensure that the sequence $\{T^m v\}_{m\in\mathbb{N}}$ is strictly decreasing in J_N and it converges uniformly in J_N to ϕ.

Proposition 3.1 leads to the desired result. □

Plainly from this result, we may establish the asymptotic stability by assuming uniqueness of solutions in $[\alpha,\beta]$ of problem (P_N).

Corollary 3.8. *If assumptions (H) and (Hf) hold and there exists a unique solution u of problem (P_N) in $[\alpha,\beta]$, then $\bar{u}$ is asymptotically stable. Moreover, set $[\alpha(0),\beta(0)]$ is contained in the attractivity set of $\bar{u}$.*

Whenever f is a strictly decreasing function in its second variable, we achieve the following result.

Corollary 3.9. *Suppose that conditions (H) and (Hf) hold and that for every $n \in I_N$, $f(n,\cdot)$ is strictly decreasing on $[\alpha(n),\beta(n)]$. Then, problem (P_N) has a unique solution u in $[\alpha,\beta]$ and $\bar{u}$ is asymptotically stable. Moreover, set $[\alpha(0),\beta(0)]$ is contained in the attractivity set of $\bar{u}$.*

Proof. Let u and v be two different solutions of problem (P_N) in $[\alpha,\beta]$. From Lemma 2.5, we may assume, without loss of generality, that $u \ll v$ in J_N. As a consequence, we obtain the following contradiction:

$$0 = u(N) - u(0) = \sum_{n=0}^{N-1} f(n,u(n)) > \sum_{n=0}^{N-1} f(n,v(n)) = v(N) - v(0) = 0. \tag{3.2}$$

□

Remark 3.10. One could intend to prove the previous result by replacing decreasing with increasing. However, under this assumption, we have that

$$\alpha(N) - \alpha(0) \le \sum_{n=0}^{N-1} f(n,\alpha(n)) < \sum_{n=0}^{N-1} f(n,\beta(n)) \le \beta(N) - \beta(0), \tag{3.3}$$

which contradicts hypothesis (H).

Below, for every $\xi \in (\psi(0),\phi(0))$, we analyse the behavior of the solution of the initial problem (P_ξ). In order to do this, we study the orbits of the operator T in $[\psi,\phi]$. We achieve similar properties to the ones proved by Dancer and Hess in [5] for strictly increasing operators defined on an arbitrary Banach space. However, as we have noted in the previous section, we cannot deduce the stability results as a consequence of the proved results in that reference, because we are not in the presence of a strictly increasing operator.

Moreover, Theorem 3.7 allows us to prove the following property of one-sided asymptotic stability.

Proposition 3.11. *Assume that conditions (H) and (Hf) hold. Let u, v be two solutions of the periodic problem (P_N) such that $u \ll v$ in J_N and there is no solution of this problem lying between both functions. Then one of the two following assertions holds.*

(1) *The N-periodic solution $\bar{v}$ is asymptotically stable from below. Moreover, set $(u(0), v(0)]$ is contained in the attractivity set of $\bar{v}$.*
(2) *The N-periodic solution $\bar{u}$ is asymptotically stable from above. Moreover, set $[u(0), v(0))$ is contained in the attractivity set of $\bar{u}$.*

Proof. Suppose that there exists $\xi \in (u(0),v(0))$ such that $u_\xi(0) < u_\xi(N)$. By defining of $\alpha_0 : J_N \to \mathbb{R}$ as the unique solution of the initial problem (P_ξ^N), it turns out that α_0 and β verify condition (H) and v is the minimal solution in $[\alpha_0,\beta]$ of problem (P_N).

Hence, the first part of assertion (1) follows from Theorem 3.7.

Now, for all $m \ge 1$, let α_m be the unique solution of the final problem

$$\Delta u(n) = f(n,u(n)); \quad n \in I_N, \qquad u(N) = \alpha_{m-1}(0); \tag{3.4}$$

it follows from condition (Hf) that $\{\alpha_m\}_{m\in\mathbb{N}}$ is a strictly decreasing sequence of lower solutions of problem (P_N) and it converges uniformly in J_N to u.

Now, given $\eta \in (u(0),\xi]$, there exists $m_0 \ge 0$ such that $\eta \in (\alpha_{m_0}(0),\alpha_{m_0-1}(0)]$ and so $u_\eta|_{J_N} \in (\alpha_{m_0},\alpha_{m_0-1}]$. The second part of claim (1) follows from Theorem 3.7.

Statement (2) is true provided that there exists $\xi \in (u(0),v(0))$ such that $u_\xi(0) > u_\xi(N)$.

□

As a consequence of this result, we can ensure asymptotic stability by assuming a finite number of solutions of problem (P_N) in $[\alpha,\beta]$.

Theorem 3.12. *If assumptions* (H) *and* (Hf) *are fulfilled and problem* (P_N) *has a finite number of solutions in* $[\alpha,\beta]$*, then at least one* N*-periodic solution of* (P) *is asymptotically stable.*

Proof. Define

$$C := \{u \in [\alpha,\beta] \ : \bar{u} \text{ is an } N\text{-periodic solution of } (P) \text{ and a.s.b.}\}, \tag{3.5}$$

where a.s.b. means asymptotically stable from below.

From Theorem 3.7, we know that this set is not empty ($\psi \in C$), moreover it is bounded from above by ϕ finite, and by Lemma 2.5 well ordered. Proposition 3.11 establishes that function $\max C$ is asymptotically stable. □

Lastly, we consider the opposite case to the previous one, that is, there are not finite number of solutions of problem (P_N) in $[\alpha,\beta]$. In this situation, we only guarantee stability, not asymptotic stability.

Theorem 3.13. *If assumptions* (H) *and* (Hf) *hold and problem* (P_N) *does not have a finite number of solutions in* $[\alpha,\beta]$*, then at least one* N*-periodic solution of* (P) *is stable.*

Proof. Consider set C defined in (3.5); given that it is not empty, bounded from above, there exists function $u_S := \sup C$. Due to the fact that the set of solutions of (P_N) is closed and well ordered, we conclude that u_S is a solution of problem (P_N).

If u_S is isolated from below, it is clear that $u_S \in C$, and so it is asymptotically stable from below.

Suppose that u_S is not isolated from below, by supremum's definition and Lemma 2.5, there exists a strictly increasing monotone sequence $\{u_m\}_{m\in\mathbb{N}} \subset C$ which converges uniformly in J_N to u_S.

Therefore, given $\epsilon > 0$, we know that there exists $m_0 \in \mathbb{N}$ such that $0 < u_S(n) - u_m(n) < \epsilon$ for all $n \in J_N$ and $m \geq m_0$. Hence, since $\bar{u}_m$ is asymptotically stable from below for every $m \in \mathbb{N}$, Proposition 3.1 together with the nondecreasing properties of operator T guarantees that $\bar{u}_S$ is stable from below.

If u_S is the limit of a decreasing sequence of solutions of (P_N), then it is stable from above. Otherwise, if it is isolated from above, then it is asymptotic stable from above and we conclude the proof. □

4. Examples and counterexamples

In this section, we present two examples which illustrate the results obtained in the previous section. In the first one, we consider a problem with a unique N-periodic solution in the whole space; we show that this solution is asymptotically stable.

In the second example, we show that Theorem 3.13 cannot be improved, in the sense that it is possible to find a nontrivial function f such that the set of N-periodic solutions of problem (P) is not finite and none of these solutions is asymptotically stable.

Example 4.1. Let $N \in \mathbb{N}^*$ be fixed, consider the following problem

$$\Delta u(n) = -\arctan\left(u(n) - i_n\right), \quad n \in [kN, kN+N-1],\ k \in \mathbb{N}, \tag{4.1}$$

where for every $k \in \mathbb{N}$, $i_n \in I_N$ is given as $n = kN + i_n$.

Obviously $f(n,x) = -\arctan(x - i_n)$ is a continuous and strictly decreasing function in its second variable and N-periodic in the first one.

Let $D > 0$, it is clear that, for D large enough, α_D and β_D, defined for every $n \in J_N$ as $\alpha_D(n) = n - D$ and $\beta_D(n) = -n + D$, is a pair of lower and upper solutions that are no solutions of problem (P_N) (with f defined above) and they are such that $\alpha \le \beta$ in J_N, that is, condition (H) is fulfilled. Since f satisfies condition (Hf), we deduce, from Corollary 3.9, that this problem has a unique solution such that its restriction to J_N belongs to $[\alpha_D, \beta_D]$ and it is asymptotically stable.

Due to the fact that $\alpha_D \to -\infty$ and $\beta_D \to +\infty$ whenever $D \to +\infty$, we claim that the restriction to J_N of problem (4.1) has only a solution in $\mathbb{R}^{N+1}$, and so problem (4.1) has a unique N-periodic solution defined in $\mathbb{N}$. Moreover, the attractive set of this solution is the whole space $\mathbb{R}$.

Remark 4.2. Note that in the previous example, we may ensure the character not only local as in the obtained results above, but global of the asymptotic stability of the considered solution.

Example 4.3. Let $f : [-1,2] \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} -\dfrac{x}{2}, & \text{if } x \in [-1,0], \\ \lim_{i\to+\infty} f_i(x), & \text{if } x \in [0,1], \\ -\dfrac{x-1}{2}, & \text{if } x \in [1,2]. \end{cases} \tag{4.2}$$

Here, the functional sequence $f_i : [0,1] \to \mathbb{R}$ is defined in the following way: $f_0(x) = 0$, and for $i \ge 1$, consider $D_i = \bigcup_{j=1}^{2^{i-1}} (a_j^i, b_j^i)$ the union of the $2^{(i-1)}$ open intervals dropped from $[0,1]$ in the ith step of the construction of the classical ternary Cantor set.

We define $f_i(x) = f_{i-1}(x)$ for all $x \notin D_i$ and, for any $j \in \{1,\dots,2^{i-1}\}$,

$$f_i(x) = \begin{cases} \dfrac{a_j - x}{2}, & \text{if } x \in \left[a_j, \dfrac{a_j + b_j}{2}\right], \\ \dfrac{x - b_j}{2}, & \text{if } x \in \left[\dfrac{a_j + b_j}{2}, b_j\right]. \end{cases} \tag{4.3}$$

It is clear that this function is continuous and nonpositive on $[0,1]$, moreover the set of zeros of f is the ternary Cantor set.

If we look for the constant solutions of problem

$$\Delta u(n) = f(u(n)), \quad n \in \mathbb{N}, \tag{4.4}$$

we know that they are the zeros of f, that is, the ternary Cantor set.

On the other hand, since $x + f(x) \leq x$ on $[0,1]$, we have that all the constant solutions are stables from above and solution 0 is asymptotically stable from below. Given that 0 is not isolated in the set of constant solutions of this problem, it is not asymptotically stable from above.

Note that $\alpha \equiv -1$ and $\beta \equiv 2$ is a pair of lower and upper solutions that are no solutions of this problem and conditions (H) and (Hf) hold in $[\alpha,\beta]$ for $N = 1$.

Remark 4.4. It is important to note that in spite of the fact that in the previous example there is not any asymptotic stable solutions of that problem, if we consider solution 0 as a fixed point of operator T, then it is the limit of a strictly increasing sequence of strict lower solutions ($y_n < Ty_n$ in J_N) and a strictly decreasing sequence of strict upper solutions ($Tz_n < z_n$ in J_N); that is, it is a strongly order-stable fixed point of T (see [5]).

Thus we may assert that the concepts of asymptotic stable solution and strongly order-stable fixed point are not equivalent.

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