BOUNDEDNESS AND VANISHING OF SOLUTIONS FOR A FORCED DELAY DYNAMIC EQUATION

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We give conditions under which all solutions of a time-scale first-order nonlinear variable-delay dynamic equation with forcing term are bounded and vanish at infinity, for arbitrary time scales that are unbounded above. A nontrivial example illustrating an application of the results is provided.

1. Delay dynamic equation with forcing term

Following Hilger's landmark paper [8], a rapidly expanding body of literature has sought to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time-scale calculus, where a time scale is simply any nonempty closed set of real numbers. This paper illustrates this new understanding by extending some continuous results from differential equations to dynamic equations on time scales, thus including as corollaries difference equations and q-difference equations. Throughout this work, we consider the nonlinear forced delay dynamic equation

$$x^{\Delta}(t) = -p(t)f\left(x(\tau(t))\right) + r(t), \quad t \in [t_0, \infty)_{\mathbb{T}},\ t_0 \geq 0, \tag{1.1}$$

where $\mathbb{T}$ is a time scale unbounded above, $f : \mathbb{R} \to \mathbb{R}$ is continuous, and the functions $p : \mathbb{T} \to (0, \infty)$ and $r : \mathbb{T} \to \mathbb{R}$ are both right-dense continuous. Moreover, the variable delay $\tau : \mathbb{T} \to \mathbb{T}$ is increasing with $\tau(t) \leq t$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ such that $\lim_{t\to\infty} \tau(t) = \infty$. The initial function associated with (1.1) takes the form $x(t) = \psi(t)$ for $t \in [\tau(t_0), t_0]$, where ψ is rd-continuous on $[\tau(t_0), t_0]$. Equation (1.1) is studied extensively by Qian and Sun [13] in the case when $\mathbb{T} = \mathbb{R}$. See also related discussions on unforced delay equations by Matsunaga et al. [12] in the continuous case, and by Erbe et al. [6] or Zhang and Yan [14]

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in the discrete case. Other papers on delay dynamic equations include [1–3]. For more on dynamic equations on time scales, skip ahead to the appendix, Section 5, or consult the recent texts by Bohner and Peterson [4, 5]. To clarify some notation, take $\tau^{-1}(t) := \sup\{s : \tau(s) \le t\}$, $\tau^{-(n+1)}(t) = \tau^{-1}(\tau^{-n}(t))$ for $t \in [\tau(t_0), \infty)_{\mathbb{T}}$, and $\tau^{n+1}(t) = \tau(\tau^n(t))$ for $t \in [\tau^{-3}(t_0), \infty)_{\mathbb{T}}$. By our choice of the delay τ, there exists large $T \in \mathbb{T}$ such that $\tau(t) \ge t_0$ and $\tau^2(t) \le \tau(t) \le t \le \tau^{-1}(\sigma(t))$ for all $t \ge T$. In addition, we always suppose that

(H1) the continuous function f satisfies $|f(x)| < |x|$ and $xf(x) > 0$ for $x \ne 0$, with

$$f^{\dagger}(x) := \max\left\{ \sup_{0 \le u \le |x|} f(u), \sup_{0 \le u \le |x|} (-f(-u)) \right\} \quad x \in \mathbb{R}; \tag{1.2}$$

(H2) using the delay τ, the forcing function r satisfies

$$\sum_{n=0}^{\infty} \int_{\tau^{1-n}(t_0)}^{\infty} |r(s)| \Delta s < \infty; \tag{1.3}$$

(H3) the coefficient function p satisfies

$$\int_{\tau(t)}^{\sigma(t)} p(s)\Delta s \le \lambda \quad \forall t \in [t_0, \infty)_{\mathbb{T}}, \qquad \int_{t_0}^{\infty} p(s)\Delta s = \infty, \tag{1.4}$$

where

$$\lambda := \frac{3}{2} + \frac{1}{2} \frac{\inf\{\mu(t) : t \in \mathbb{T}\}}{\sup\{\tau^{-1}(\sigma(t)) - t : t \in \mathbb{T}\}}; \tag{1.5}$$

it is understood that $\lambda = 3/2$ if either $\inf\{\mu(t)\} = 0$ or $\sup\{\tau^{-1}(\sigma(t)) - t\} = \infty$.

2. Background lemmas

We will need Lemma 2.1 in the proof of Lemma 2.2.

Lemma 2.1 [1, Lemma 2.1]. *For a right-dense continuous function $p : \mathbb{T} \to \mathbb{R}$ and points $a, t \in \mathbb{T}$,*

$$\int_a^t \left(p(s) \int_a^{\sigma(s)} p(u)\Delta u \right) \Delta s = \frac{1}{2} \left(\int_a^t p(s)\Delta s \right)^2 + \frac{1}{2} \int_a^t \mu(s) p^2(s) \Delta s. \tag{2.1}$$

Lemma 2.2. *Assume (H1), (H2), (H3) hold. Let x be a solution of (1.1), and assume there exists $t_1 \in (\tau^{-2}(T), \infty)_{\mathbb{T}}$ such that $\tau^2(t_1) \ge t_0$ and $x(t_1)x^{\sigma}(t_1) \le 0$. If for some constant $M > 0$, $|x(t)| \le M$ for $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$, then*

$$|x(t)| \le f^{\dagger}(M) + \lambda \int_{\tau(t_1)}^{t} |r(s)| \Delta s \quad \text{for } t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}. \tag{2.2}$$

Proof. The techniques employed here syncretize and extend ideas from [13, 14]. We concentrate on the case where $x(t) \ge -M$ for $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$; the case where $x(t) \le M$ for $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$ is similar and is omitted. Since $x(t_1)x^{\sigma}(t_1) \le 0$, there exists a real number

$\xi \in [t_1 - 1, t_1]$ such that

$$x(t_1) + [x^\sigma(t_1) - x(t_1)](\xi - t_1 + 1) = 0. \tag{2.3}$$

By (H1), $f^\dagger$ is nonnegative and nondecreasing, thus $f(x(t)) \geq -f^\dagger(x(t)) \geq -f^\dagger(M)$ for $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$. From (1.1), we have

$$x^\Delta(t) \leq p(t)f^\dagger(M) + |r(t)|, \quad t \in [\tau(t_1), \tau^{-1}(t_1)]_{\mathbb{T}}, \tag{2.4}$$

so that integration and the fundamental theorem yield

$$x(t_1) - x(\tau(t)) \leq f^\dagger(M)\int_{\tau(t)}^{t_1} p(s)\Delta s + \int_{\tau(t)}^{t_1} |r(s)|\Delta s, \quad t \in [t_1, \tau^{-1}(t_1)]_{\mathbb{T}}. \tag{2.5}$$

Using the characterization of ξ in (2.3), we obtain that for $t \in [t_1, \tau^{-1}(t_1)]_{\mathbb{T}}$,

$$\begin{aligned}
x(\tau(t)) &\geq x(t_1) - f^\dagger(M)\int_{\tau(t)}^{t_1} p(s)\Delta s - \int_{\tau(t)}^{t_1} |r(s)|\Delta s \\
&= -[x^\sigma(t_1) - x(t_1)](\xi - t_1 + 1) - f^\dagger(M)\int_{\tau(t)}^{t_1} p(s)\Delta s - \int_{\tau(t)}^{t_1} |r(s)|\Delta s \\
&\geq -f^\dagger(M)\left[(\xi - t_1)\int_{t_1}^{\sigma(t_1)} p(s)\Delta s + \int_{\tau(t)}^{\sigma(t_1)} p(s)\Delta s\right] \\
&\quad - (\xi - t_1 + 1)\mu(t_1)|r(t_1)| - \int_{\tau(t)}^{t_1} |r(s)|\Delta s,
\end{aligned} \tag{2.6}$$

where we used (2.4) and Theorem 5.4(4) to arrive at the last line. Continuing in this manner, from (H1) and the fact that $f^\dagger(x) < x$ for positive x, we see that

$$\begin{aligned}
x^\Delta(t) &\leq p(t)f^\dagger\Bigg(f^\dagger(M)\left[(\xi - t_1)\int_{t_1}^{\sigma(t_1)} p(s)\Delta s + \int_{\tau(t)}^{\sigma(t_1)} p(s)\Delta s\right] \\
&\qquad\qquad + (\xi - t_1 + 1)\mu(t_1)|r(t_1)| + \int_{\tau(t)}^{t_1} |r(s)|\Delta s\Bigg) \\
&\leq p(t)\int_{\tau(t)}^{\sigma(t_1)} \left(f^\dagger(M)p(s) + |r(s)|\right)\Delta s \\
&\quad - p(t)(t_1 - \xi)\mu(t_1)\left(f^\dagger(M)p(t_1) + |r(t_1)|\right)
\end{aligned} \tag{2.7}$$

for $t \in [t_1, \tau^{-1}(t_1)]_{\mathbb{T}}$. Now by (H3) and the choice of ξ, we know that

$$0 \leq \zeta := (t_1 - \xi)\int_{t_1}^{\sigma(t_1)} p(s)\Delta s + \int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} p(s)\Delta s \leq \lambda, \tag{2.8}$$

which we consider in the following two cases.

Case 1. Suppose that ζ defined in (2.8) satisfies $\zeta \in (0,1)$. For $t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$, we have

$$\begin{aligned}
x(t) &= x^{\sigma}(t_1) + \int_{\sigma(t_1)}^{t} x^{\Delta}(s)\Delta s \\
&\overset{(2.3)}{=} [x^{\sigma}(t_1) - x(t_1)](t_1 - \xi) + \int_{\sigma(t_1)}^{t} x^{\Delta}(s)\Delta s \\
&\overset{\text{Theorem 5.4}}{=} (t_1 - \xi)\mu(t_1)x^{\Delta}(t_1) + \int_{\sigma(t_1)}^{t} x^{\Delta}(s)\Delta s \\
&\overset{(2.7)}{\leq} (t_1 - \xi)\mu(t_1)p(t_1)\int_{\tau(t_1)}^{\sigma(t_1)} (f^{\dagger}(M)p(s) + |r(s)|)\Delta s \\
&\quad - (t_1 - \xi)^2\mu(t_1)^2 p(t_1)(f^{\dagger}(M)p(t_1) + |r(t_1)|) \\
&\quad - (t_1 - \xi)\mu(t_1)(f^{\dagger}(M)p(t_1) + |r(t_1)|)\int_{\sigma(t_1)}^{t} p(s)\Delta s \\
&\quad + \int_{\sigma(t_1)}^{t} p(s)\left(\int_{\tau(s)}^{\sigma(t_1)} (f^{\dagger}(M)p(u) + |r(u)|)\Delta u\right)\Delta s \\
&\leq f^{\dagger}(M)\bigg\{(t_1 - \xi)\mu(t_1)p(t_1)\left[\int_{\tau(t_1)}^{\sigma(t_1)} p(s)\Delta s - (t_1 - \xi)\mu(t_1)p(t_1)\right] \\
&\qquad + \int_{\sigma(t_1)}^{t} p(s)\left[\int_{\tau(s)}^{\sigma(t_1)} p(u)\Delta u - (t_1 - \xi)\mu(t_1)p(t_1)\right]\Delta s\bigg\} \\
&\quad + (t_1 - \xi)\mu(t_1)p(t_1)\int_{\tau(t_1)}^{\sigma(t_1)} |r(s)|\Delta s + \int_{\sigma(t_1)}^{t} p(s)\int_{\tau(s)}^{\sigma(t_1)} |r(u)|\Delta u\,\Delta s,
\end{aligned} \tag{2.9}$$

where the last inequality follows from simple factoring and the dropping of the negative terms involving $|r(t_1)|$. Continuing,

$$\begin{aligned}
x(t) &\overset{(\mathrm{H3})}{\leq} f^{\dagger}(M)\bigg\{(t_1 - \xi)\mu(t_1)p(t_1)[\lambda - (t_1 - \xi)\mu(t_1)p(t_1)] \\
&\qquad + \int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} p(s)\left[\lambda - \int_{\sigma(t_1)}^{\sigma(s)} p(u)\Delta u - (t_1 - \xi)\mu(t_1)p(t_1)\right]\Delta s\bigg\} \\
&\quad + (t_1 - \xi)\int_{t_1}^{\sigma(t_1)} p(s)\Delta s\int_{\tau(t_1)}^{\sigma(t_1)} |r(u)|\Delta u + \int_{\sigma(t_1)}^{t} p(s)\int_{\tau(s)}^{\sigma(t_1)} |r(u)|\Delta u\,\Delta s \\
&\overset{(2.8)}{\leq} f^{\dagger}(M)\bigg\{-[(t_1 - \xi)\mu(t_1)p(t_1)]^2 - (t_1 - \xi)\mu(t_1)p(t_1)\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} p(s)\Delta s \\
&\qquad + \lambda\zeta - \int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} p(s)\left(\int_{\sigma(t_1)}^{\sigma(s)} p(u)\Delta u\right)\Delta s\bigg\} \\
&\quad + \left(\int_{t_1}^{\tau^{-1}(\sigma(t_1))} p(s)\Delta s\right)\left(\int_{\tau(t_1)}^{t} |r(s)|\Delta s\right).
\end{aligned} \tag{2.10}$$

Using Lemma 2.1 on the last double integral involving p,

$$\begin{aligned} x(t) \le{}& f^{\dagger}(M)\bigg\{-\left[(t_1-\xi)\mu(t_1)p(t_1)\right]^2-(t_1-\xi)\mu(t_1)p(t_1)\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))}p(s)\Delta s \\ &\qquad +\lambda\zeta-\frac{1}{2}\left(\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))}p(s)\Delta s\right)^2-\frac{1}{2}\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))}\mu(s)p(s)^2\Delta s\bigg\} \\ &+\lambda\int_{\tau(t_1)}^{t}|r(s)|\Delta s \\ ={}& f^{\dagger}(M)\left(\lambda\zeta-\left[\frac{\zeta^2}{2}+\frac{\left((t_1-\xi)\mu(t_1)p(t_1)\right)^2}{2}+\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))}\frac{\mu(s)}{2}\left(p(s)\right)^2\Delta s\right]\right) \\ &+\lambda\int_{\tau(t_1)}^{t}|r(s)|\Delta s. \end{aligned} \tag{2.11}$$

Define

$$m(s):=\begin{cases}(t_1-\xi)\sqrt{\mu(s)}\,p(s), & s\le t_1,\\ \sqrt{\mu(s)}\,p(s), & s>t_1,\end{cases} \tag{2.12}$$

so that m is right-dense continuous and

$$x(t)\le f^{\dagger}(M)\left(\lambda\zeta-\frac{\zeta^2}{2}-\frac{1}{2}\int_{t_1}^{\tau^{-1}(\sigma(t_1))}m^2(s)\Delta s\right)+\lambda\int_{\tau(t_1)}^{t}|r(s)|\Delta s. \tag{2.13}$$

By the Cauchy-Schwarz inequality [4, Theorem 6.15],

$$\begin{aligned} &\int_{t_1}^{\tau^{-1}(\sigma(t_1))}m^2(s)\Delta s \\ &\quad\ge\frac{1}{\tau^{-1}(\sigma(t_1))-t_1}\left(\int_{t_1}^{\tau^{-1}(\sigma(t_1))}m(s)\Delta s\right)^2 \\ &\quad=\frac{1}{\tau^{-1}(\sigma(t_1))-t_1}\left((t_1-\xi)\left(\mu(t_1)\right)^{3/2}p(t_1)+\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))}p(s)\sqrt{\mu(s)}\Delta s\right)^2 \\ &\quad\overset{(1.5)}{\ge}2\left(\lambda-\frac{3}{2}\right)\zeta^2. \end{aligned} \tag{2.14}$$

Thus, for $t\in[\sigma(t_1),\tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$,

$$x(t)\le f^{\dagger}(M)\left(\lambda\zeta-\frac{\zeta^2}{2}-\left(\lambda-\frac{3}{2}\right)\zeta^2\right)+\lambda\int_{\tau(t_1)}^{t}|r(s)|\Delta s. \tag{2.15}$$

If $q(x) := \lambda x - x^2/2 - (\lambda - 3/2)x^2$, then $q'(0) > 0$ and $q'(1) = 2 - \lambda \geq 0$ by the choice of λ in (1.5), so that q is increasing on $[0,1]$. Consequently,

$$x(t) \leq f^{\dagger}(M) + \lambda \int_{\tau(t_1)}^{t} |r(s)|\Delta s, \quad t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}. \tag{2.16}$$

Case 2. Suppose $1 \leq \zeta \leq \lambda$ for ζ as in (2.8). Actually, from (H3), we have in this case that $\int_{t_1}^{\tau^{-1}(\sigma(t_1))} p(s)\Delta s \in [1,\lambda]$. Note that

$$g(t) := \int_{t}^{\tau^{-1}(\sigma(t_1))} p(s)\Delta s - 1, \quad t \in [t_1, \tau^{-1}(\sigma(t_1))]_{\mathbb{T}} \tag{2.17}$$

is a delta-differentiable and decreasing function, so that by [4, Theorem 1.16(i)], g is continuous on $t \in [t_1, \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$. Since $g(t_1) \geq 0$ and $g(\tau^{-1}(\sigma(t_1))) = -1 < 0$, by the intermediate value theorem [4, Theorem 1.115], there exists $t_2 \in [t_1, \tau^{-1}(\sigma(t_1)))_{\mathbb{T}}$ such that either $g(t_2) = 0$ or $g(t_2) > 0 > g^{\sigma}(t_2)$. Either way,

$$\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} p(s)\Delta s < 1 \leq \int_{t_2}^{\tau^{-1}(\sigma(t_1))} p(s)\Delta s = \mu(t_2)p(t_2) + \int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} p(s)\Delta s, \tag{2.18}$$

ergo there exists a real number $\phi \in [t_2 - 1, t_2)$ such that

$$\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} p(s)\Delta s + (t_2 - \phi)\mu(t_2)p(t_2) = 1. \tag{2.19}$$

Using (2.3) and (2.4), we have for $t \in [t_1, t_2]_{\mathbb{T}}$ that

$$\begin{aligned}
x(t) &= (t_1 - \xi)\mu(t_1)x^{\Delta}(t_1) + \int_{\sigma(t_1)}^{t} x^{\Delta}(s)\Delta s \\
&\leq (t_1 - \xi)\mu(t_1)\left(p(t_1)f^{\dagger}(M) + |r(t_1)|\right) + \int_{\sigma(t_1)}^{t} \left(p(s)f^{\dagger}(M) + |r(s)|\right)\Delta s \\
&\leq f^{\dagger}(M)\left((t_1 - \xi)\mu(t_1)p(t_1) + \int_{\sigma(t_1)}^{t_2} p(s)\Delta s\right) \\
&\quad + (t_1 - \xi)\mu(t_1)|r(t_1)| + \int_{\sigma(t_1)}^{t} |r(s)|\Delta s \\
&\leq f^{\dagger}(M)\int_{t_1}^{t_2} p(s)\Delta s + \int_{t_1}^{t} |r(s)|\Delta s < f^{\dagger}(M) + \lambda \int_{\tau(t_1)}^{t} |r(s)|\Delta s,
\end{aligned} \tag{2.20}$$

where the last inequality follows from our choice of t_2. For $t \in [\sigma(t_2), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$, with (2.3), we see that

$$\begin{aligned} x(t) &= (t_1-\xi)\mu(t_1)x^{\Delta}(t_1) + \int_{\sigma(t_1)}^{t} x^{\Delta}(s)\Delta s \\ &= \left[(t_1-\xi)\mu(t_1)x^{\Delta}(t_1) + (\phi - t_2+1)\mu(t_2)x^{\Delta}(t_2) + \int_{\sigma(t_1)}^{t_2} x^{\Delta}(s)\Delta s\right] \\ &\quad + \left[(t_2-\phi)\mu(t_2)x^{\Delta}(t_2) + \int_{\sigma(t_2)}^{t} x^{\Delta}(s)\Delta s\right] = S_1+S_2, \end{aligned} \tag{2.21}$$

where S_1 is the first grouping and S_2 is the second. Using (2.4) for S_1 and (2.7) for S_2,

$$\begin{aligned} S_1 &\le f^{\dagger}(M)\left((t_1-\xi)\mu(t_1)p(t_1) + (\phi-t_2)\mu(t_2)p(t_2) + \int_{\sigma(t_1)}^{\sigma(t_2)} p(s)\Delta s\right) \\ &\quad + (t_1-\xi)\mu(t_1)|r(t_1)| + (\phi-t_2)\mu(t_2)|r(t_2)| + \int_{\sigma(t_1)}^{\sigma(t_2)} |r(s)|\Delta s, \\ S_2 &\le f^{\dagger}(M)(t_2-\phi)\mu(t_2)p(t_2)\left[\int_{\tau(t_2)}^{\sigma(t_1)} p(s)\Delta s - (t_1-\xi)\mu(t_1)p(t_1)\right] \\ &\quad + f^{\dagger}(M)\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} p(s)\left[\int_{\tau(s)}^{\sigma(t_1)} p(u)\Delta u - (t_1-\xi)\mu(t_1)p(t_1)\right]\Delta s \\ &\quad + (t_2-\phi)\mu(t_2)p(t_2)\left(\int_{\tau(t_2)}^{\sigma(t_1)} |r(s)|\Delta s - (t_1-\xi)\mu(t_1)|r(t_1)|\right) \\ &\quad + \int_{\sigma(t_2)}^{t} p(s)\left(\int_{\tau(s)}^{\sigma(t_1)} |r(u)|\Delta u - (t_1-\xi)\mu(t_1)|r(t_1)|\right)\Delta s. \end{aligned} \tag{2.22}$$

Then continuing for $t \in [\sigma(t_2), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$ while recalling (2.19), we have

$$\begin{aligned} x(t) \le f^{\dagger}(M)\Bigg(&\left[(t_1-\xi)\mu(t_1)p(t_1) + (\phi-t_2)\mu(t_2)p(t_2) + \int_{\sigma(t_1)}^{\sigma(t_2)} p(s)\Delta s\right] \\ &\times\left[\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} p(s)\Delta s + (t_2-\phi)\mu(t_2)p(t_2)\right] \\ &+ (t_2-\phi)\mu(t_2)p(t_2)\left[\int_{\tau(t_2)}^{\sigma(t_1)} p(s)\Delta s - (t_1-\xi)\mu(t_1)p(t_1)\right] \end{aligned}$$

$$+\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} p(s)\left[\int_{\tau(s)}^{\sigma(t_1)} p(u)\Delta u-(t_1-\xi)\mu(t_1)p(t_1)\right]\Delta s\Bigg)$$

$$+(t_1-\xi)\mu(t_1)\,|\,r(t_1)\,|+(\phi-t_2)\mu(t_2)\,|\,r(t_2)\,|+\int_{\sigma(t_1)}^{\sigma(t_2)}|\,r(s)\,|\,\Delta s$$

$$+(t_2-\phi)\mu(t_2)p(t_2)\left(\int_{\tau(t_2)}^{\sigma(t_1)}|\,r(s)\,|\,\Delta s-(t_1-\xi)\mu(t_1)\,|\,r(t_1)\,|\right)$$

$$+\int_{\sigma(t_2)}^{t} p(s)\left(\int_{\tau(s)}^{\sigma(t_1)}|\,r(u)\,|\,\Delta u-(t_1-\xi)\mu(t_1)\,|\,r(t_1)\,|\right)\Delta s. \tag{2.23}$$

Proceeding by rearranging,

$$\begin{aligned}x(t)\le f^{\dagger}(M)\Bigg(&\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} p(s)\left[(\phi-t_2)\mu(t_2)p(t_2)+\int_{\tau(s)}^{\sigma(t_2)} p(u)\Delta u\right]\Delta s\\&+(t_2-\phi)\mu(t_2)p(t_2)\left[(\phi-t_2)\mu(t_2)p(t_2)+\int_{\tau(t_2)}^{\sigma(t_2)} p(s)\Delta s\right]\Bigg)\\&+(t_1-\xi)\mu(t_1)\,|\,r(t_1)\,|\int_{t}^{\tau^{-1}(\sigma(t_1))} p(s)\Delta s+\int_{\sigma(t_2)}^{t} p(s)\left(\int_{\tau(s)}^{\sigma(t_1)}|\,r(u)\,|\,\Delta u\right)\Delta s\\&+(t_2-\phi)\mu(t_2)\left(p(t_2)\int_{\tau(t_2)}^{\sigma(t_1)}|\,r(s)\,|\,\Delta s-|\,r(t_2)\,|\right)+\int_{\sigma(t_1)}^{\sigma(t_2)}|\,r(s)\,|\,\Delta s.\end{aligned} \tag{2.24}$$

Using (H3) in the first two lines and properties of delta integrals in the last two lines, we arrive at

$$\begin{aligned}x(t)\le{}& f^{\dagger}(M)\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} p(s)\left((\phi-t_2)\mu(t_2)p(t_2)+\lambda-\int_{\sigma(t_2)}^{\sigma(s)} p(u)\Delta u\right)\Delta s\\&+f^{\dagger}(M)(t_2-\phi)\mu(t_2)p(t_2)[(\phi-t_2)\mu(t_2)p(t_2)+\lambda]\\&+\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} p(s)\int_{\tau(s)}^{\sigma(t_1)}|\,r(u)\,|\,\Delta u\,\Delta s+\int_{\sigma(t_1)}^{\sigma(t_2)}|\,r(s)\,|\,\Delta s\\&+\left(\int_{t_2}^{\sigma(t_2)} p(s)\Delta s\right)\left(\int_{\tau(t_2)}^{\sigma(t_1)}|\,r(s)\,|\,\Delta s\right).\end{aligned} \tag{2.25}$$

Applying (2.19) to the terms involving $f^\dagger(M)$ and combining some of the remaining integrals, we see that

$$\begin{aligned} x(t) &\leq f^\dagger(M)\bigg(\lambda - \int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} p(s)\int_{\sigma(t_2)}^{\sigma(s)} p(u)\Delta u\,\Delta s - \left[(t_2-\phi)\mu(t_2)p(t_2)\right]^2 \\ &\qquad - (t_2-\phi)\mu(t_2)p(t_2)\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} p(s)\Delta s\bigg) \\ &\quad + \left(\int_{t_2}^{\tau^{-1}(\sigma(t_1))} p(s)\Delta s\right)\left(\int_{\tau(t_2)}^{\sigma(t_1)} |r(s)|\Delta s\right) + \int_{\sigma(t_1)}^{\sigma(t_2)} |r(s)|\Delta s \\ &\leq f^\dagger(M)\left(\lambda - \frac{1}{2} - \frac{1}{2}\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} \mu(s)\big(p(s)\big)^2\Delta s - \frac{1}{2}\left[(t_2-\phi)\mu(t_2)p(t_2)\right]^2\right) \\ &\quad + \left(\int_{t_2}^{\tau^{-1}(\sigma(t_1))} p(s)\Delta s\right)\left(\int_{\tau(t_2)}^{\sigma(t_2)} |r(s)|\Delta s\right) \end{aligned} \tag{2.26}$$

using Lemma 2.1 and (2.19) again to arrive at the first line, and using the choice of t_2 for the second. Thus, as in (2.15), for $t \in [\sigma(t_2), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$,

$$\begin{aligned} x(t) &\leq f^\dagger(M)\left(\lambda - \frac{1}{2} - \left(\lambda - \frac{3}{2}\right)\right) + \lambda\int_{\tau(t_1)}^{t} |r(s)|\Delta s \\ &= f^\dagger(M) + \lambda\int_{\tau(t_1)}^{t} |r(s)|\Delta s. \end{aligned} \tag{2.27}$$

□

Lemma 2.3. *Suppose that (H1)–(H3) hold. Let x be a solution of (1.1) and let $t_1 \in \mathbb{T}$ be as in Lemma 2.2. Then x is a bounded solution of (1.1).*

Proof. The techniques used here are similar to those on $\mathbb{R}$ found in [13]. Let $M := \max\{|x(t)| : t \in [\tau^2(t_1), t_1]_{\mathbb{T}}\}$. Then by Lemma 2.2,

$$|x(t)| \leq f^\dagger(M) + \lambda\int_{\tau(t_1)}^{t} |r(s)|\Delta s, \quad t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}. \tag{2.28}$$

To prove that x is a bounded solution of (1.1), let

$$t_1^* := \sup\{t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}} : x(t)x^\sigma(t) \leq 0\}; \tag{2.29}$$

for $n \geq 2$, take

$$\begin{aligned} t_n' &:= \min\{t \in [\tau^{1-n}(\sigma(t_1)), \tau^{-n}(\sigma(t_1))]_{\mathbb{T}} : x(t)x^\sigma(t) \leq 0\}, \\ t_n^* &:= \sup\{t \in [\tau^{1-n}(\sigma(t_1)), \tau^{-n}(\sigma(t_1))]_{\mathbb{T}} : x(t)x^\sigma(t) \leq 0\}. \end{aligned} \tag{2.30}$$

If there is no generalized zero in $[\tau^{1-n}(\sigma(t_1)),\tau^{-n}(\sigma(t_1))]_{\mathbb{T}}$, take

$$t'_n := \tau^{1-n}(\sigma(t_1)), \qquad t^*_n := \tau^{-n}(\sigma(t_1)). \tag{2.31}$$

By Lemma 2.2, for $t \in [\sigma(t_1),\sigma(t^*_1)]_{\mathbb{T}}$,

$$|x(t)| \le f^{\dagger}(M)+\lambda\int_{\tau(t_1)}^{t}|r(s)|\Delta s \le M+\lambda\int_{\tau(t_1)}^{\sigma(t^*_1)}|r(s)|\Delta s. \tag{2.32}$$

If $t'_2 \in [\sigma(t^*_1),\tau^{-1}(\sigma(t^*_1))]_{\mathbb{T}}$, then

$$|x(t)| \le \sup_{t\in[\tau^2(t^*_1),t^*_1]_{\mathbb{T}}}\{|x(t)|\}+\lambda\int_{\tau(t^*_1)}^{t}|r(s)|\Delta s, \tag{2.33}$$

so that

$$|x(t)| \le M+\lambda\int_{\tau(t_1)}^{t^*_1}|r(s)|\Delta s+\lambda\int_{\tau(t^*_1)}^{t'_2}|r(s)|\Delta s, \quad t\in[\sigma(t^*_1),t'_2]_{\mathbb{T}}. \tag{2.34}$$

On the other hand, if $t'_2 > \tau^{-1}(\sigma(t^*_1))$, then x has constant sign on $[\sigma(t^*_1),t'_2]_{\mathbb{T}}$. By (1.1) and the fact that $p, xf(x)>0$,

$$|x(t)| \le x(\tau^{-1}(\sigma(t^*_1)))+\int_{\tau^{-1}(\sigma(t^*_1))}^{t'_2}|r(s)|\Delta s, \quad t\in[\tau^{-1}(\sigma(t^*_1)),t'_2]_{\mathbb{T}}. \tag{2.35}$$

Moreover, as above,

$$|x(t)| \le M+\lambda\int_{\tau(t_1)}^{t^*_1}|r(s)|\Delta s+\lambda\int_{\tau(t^*_1)}^{\tau^{-1}(\sigma(t^*_1))}|r(s)|\Delta s, \quad t\in[\sigma(t^*_1),\tau^{-1}(\sigma(t^*_1))]_{\mathbb{T}}, \tag{2.36}$$

so that

$$\begin{aligned}|x(t)| &\le M+\lambda\int_{\tau(t_1)}^{t^*_1}|r(s)|\Delta s+\lambda\int_{\tau(t^*_1)}^{\tau^{-1}(\sigma(t^*_1))}|r(s)|\Delta s+\int_{\tau^{-1}(\sigma(t^*_1))}^{t'_2}|r(s)|\Delta s\\ &\le M+\lambda\int_{\tau(t_1)}^{t^*_1}|r(s)|\Delta s+\lambda\int_{\tau(t^*_1)}^{t'_2}|r(s)|\Delta s, \quad t\in[\sigma(t^*_1),t'_2]_{\mathbb{T}}.\end{aligned} \tag{2.37}$$

Since $t^*_2 - t'_2 \le \tau^{-2}(\sigma(t_1))-\tau^{-1}(\sigma(t_1))$, on $[t'_2,t^*_2]_{\mathbb{T}}$ we have

$$\begin{aligned}|x(t)| &\le \sup_{t\in[\tau^2(t'_2),t'_2]_{\mathbb{T}}}\{|x(t)|\}+\lambda\int_{\tau(t'_2)}^{t}|r(s)|\Delta s\\ &\le M+\lambda\int_{\tau(t_1)}^{t^*_1}|r(s)|\Delta s+\lambda\int_{\tau(t^*_1)}^{t'_2}|r(s)|\Delta s+\lambda\int_{\tau(t'_2)}^{t^*_2}|r(s)|\Delta s.\end{aligned} \tag{2.38}$$

In the same way for $t \in [t_2^*, t_3']_{\mathbb{T}}$ as for the case $t \in [t_1^*, t_2']_{\mathbb{T}}$, we arrive at

$$\begin{aligned}
|x(t)| &\le \sup_{t\in[\tau^2(t_2^*),t_2^*]_{\mathbb{T}}} \{|x(t)|\} + \lambda\int_{\tau(t_2^*)}^{t_3'} |r(s)|\Delta s \\
&\le M + \lambda\left(\int_{\tau(t_1)}^{t_1^*} |r(s)|\Delta s + \int_{\tau(t_1^*)}^{t_2'} |r(s)|\Delta s + \int_{\tau(t_2')}^{t_2^*} |r(s)|\Delta s + \int_{\tau(t_2^*)}^{t_3'} |r(s)|\Delta s\right) \\
&\le M + 2\lambda\int_{\tau(t_1)}^{t_2'} |r(s)|\Delta s + 2\lambda\int_{\tau(t_2')}^{t_3'} |r(s)|\Delta s \\
&\le M + 2\lambda\int_{\tau(t_1)}^{\tau^{-2}(t_1)} |r(s)|\Delta s + 2\lambda\int_{t_1}^{\tau^{-3}(t_1)} |r(s)|\Delta s.
\end{aligned} \tag{2.39}$$

For $t \in [t_3', t_3^*]_{\mathbb{T}}$,

$$\begin{aligned}
|x(t)| &\le \sup_{t\in[\tau^2(t_3'),t_3']_{\mathbb{T}}} \{|x(t)|\} + \lambda\int_{\tau(t_3')}^{t_3^*} |r(s)|\Delta s \\
&\le M + 2\lambda\int_{\tau(t_1)}^{\tau^{-2}(t_1)} |r(s)|\Delta s + 2\lambda\int_{t_1}^{\tau^{-3}(t_1)} |r(s)|\Delta s + \lambda\int_{\tau(t_3')}^{t_3^*} |r(s)|\Delta s.
\end{aligned} \tag{2.40}$$

Consequently, for $t \in [t_3^*, t_4']_{\mathbb{T}}$,

$$\begin{aligned}
|x(t)| &\le \sup_{t\in[\tau^2(t_3^*),t_3^*]_{\mathbb{T}}} \{|x(t)|\} + \lambda\int_{\tau(t_3^*)}^{t_4'} |r(s)|\Delta s \\
&\le M + 2\lambda\int_{\tau(t_1)}^{\tau^{-2}(t_1)} |r(s)|\Delta s + 2\lambda\int_{t_1}^{\tau^{-3}(t_1)} |r(s)|\Delta s \\
&\quad + \lambda\int_{\tau(t_3')}^{t_3^*} |r(s)|\Delta s + \lambda\int_{\tau(t_3^*)}^{t_4'} |r(s)|\Delta s \\
&\le M + 2\lambda\int_{\tau(t_1)}^{\tau^{-2}(t_1)} |r(s)|\Delta s + 2\lambda\int_{t_1}^{\tau^{-3}(t_1)} |r(s)|\Delta s + 2\lambda\int_{\tau^{-1}(t_1)}^{\tau^{-4}(t_1)} |r(s)|\Delta s \\
&\le M + 2\lambda\int_{\tau(t_1)}^{t_1} |r(s)|\Delta s + 4\lambda\int_{t_1}^{\tau^{-1}(t_1)} |r(s)|\Delta s + 6\lambda\int_{\tau^{-1}(t_1)}^{\tau^{-2}(t_1)} |r(s)|\Delta s \\
&\quad + 4\lambda\int_{\tau^{-2}(t_1)}^{\tau^{-3}(t_1)} |r(s)|\Delta s + 2\lambda\int_{\tau^{-3}(t_1)}^{\tau^{-4}(t_1)} |r(s)|\Delta s.
\end{aligned} \tag{2.41}$$

Through recursion, for $t \in [t_n^*, t'_{n+1}]_{\mathbb{T}}$, we obtain

$$\begin{aligned} |x(t)| \le{}& M + 2\lambda \int_{\tau(t_1)}^{t_1} |r(s)|\Delta s + 4\lambda \int_{t_1}^{\tau^{-1}(t_1)} |r(s)|\Delta s + 6\lambda \int_{\tau^{-1}(t_1)}^{\tau^{-2}(t_1)} |r(s)|\Delta s \\ &+ \cdots + 6\lambda \int_{\tau^{2-n}(t_1)}^{\tau^{1-n}(t_1)} |r(s)|\Delta s + 4\lambda \int_{\tau^{1-n}(t_1)}^{\tau^{-n}(t_1)} |r(s)|\Delta s + 2\lambda \int_{\tau^{-n}(t_1)}^{\tau^{-n-1}(t_1)} |r(s)|\Delta s, \end{aligned} \tag{2.42}$$

and for $t \in [t'_{n+1}, t^*_{n+1}]_{\mathbb{T}}$,

$$\begin{aligned} |x(t)| &\le \sup_{t \in [\tau^2(t'_{n+1}), t'_{n+1}]_{\mathbb{T}}} \{|x(t)|\} + \lambda \int_{\tau(t'_{n+1})}^{t^*_{n+1}} |r(s)|\Delta s \\ &\le M + 6\lambda \int_{\tau(t_1)}^{\tau^{-n-1}(t_1)} |r(s)|\Delta s. \end{aligned} \tag{2.43}$$

Now as both t'_n and t^*_n go to infinity as n goes to infinity, by (H2) the solution x must be bounded. □

LEMMA 2.4. *Suppose that (H1)–(H3) hold. Let x be a solution of (1.1) and let $t_1 \in \mathbb{T}$ be as in Lemma 2.2. Then*

$$|x(t)| \le f^{\dagger}(B) + \lambda \int_{\tau(t_1)}^{t} |r(s)|\Delta s, \quad t \in [\sigma(t_1), \infty)_{\mathbb{T}}, \tag{2.44}$$

where $B := \sup_{t \ge t_0} |x(t)|$.

Proof. By Lemma 2.3, x is a bounded solution of (1.1). Set $B := \sup_{t \ge t_0} |x(t)|$, but suppose that (2.44) is false. Then there exists

$$T_1 := \inf\left\{t > \tau^{-1}(\sigma(t_1)) : |x(t)| > f^{\dagger}(B) + \lambda \int_{\tau(t_1)}^{t} |r(s)|\Delta s\right\}. \tag{2.45}$$

Clearly

$$|x(t)| \le f^{\dagger}(B) + \lambda \int_{\tau(t_1)}^{t} |r(s)|\Delta s, \quad t \in [\sigma(t_1), T_1)_{\mathbb{T}}, \tag{2.46}$$

and we have the following cases.

Case 1. (A) Suppose $x(T_1) > f^{\dagger}(B) + \lambda \int_{\tau(t_1)}^{T_1} |r(s)|\Delta s$. By continuity and the choice of T_1, T_1 is a left-scattered point with $|x(\rho(T_1))| \le f^{\dagger}(B) + \lambda \int_{\tau(t_1)}^{\rho(T_1)} |r(s)|\Delta s$ and $x^{\Delta}(\rho(T_1)) > 0$. By (1.1) and (H1), $x(\tau(\rho(T_1))) < 0$. Set

$$T_0 := \max\{t \in [\tau(\rho(T_1)), \rho(T_1))_{\mathbb{T}} : x(t)x^{\sigma}(t) \le 0\}. \tag{2.47}$$

Then $x(T_0)x^{\sigma}(T_0) \le 0$ and $\tau^2(t_1) \le \tau^3(T_1) \le \tau^2(T_0) \le T_0 < T_1$. By (2.46),

$$|x(t)| \le f^{\dagger}(B) + \lambda \int_{\tau(t_1)}^{t} |r(s)|\Delta s, \quad t \in [\tau^2(T_0), T_0]_{\mathbb{T}}. \tag{2.48}$$

Consequently, from Lemma 2.2,

$$|x(t)| \leq f^{\dagger}(B)+\lambda\int_{\tau(t_1)}^{t}|r(s)|\Delta s \quad \text{on } [\sigma(T_0),\tau^{-1}(\sigma(T_0))]_{\mathbb{T}}. \tag{2.49}$$

Since $\tau(\rho(T_1)) \leq T_0 < \rho(T_1)$ and τ is increasing, $\sigma(T_0) \leq T_1$ and

$$f^{\dagger}(B)+\lambda\int_{\tau(t_1)}^{T_1}|r(s)|\Delta s < x(T_1) \leq f^{\dagger}(B)+\lambda\int_{\tau(t_1)}^{T_1}|r(s)|\Delta s, \tag{2.50}$$

a contradiction.

(B) Suppose $x(T_1) = f^{\dagger}(B)+\lambda\int_{\tau(t_1)}^{T_1}|r(s)|\Delta s$. Then T_1 is a right-dense point, $x^{\Delta}(T_1) \geq 0$, and there exists $T_2 \in (T_1,\tau^{-1}(T_1)]_{\mathbb{T}}$ such that $x(t) > f^{\dagger}(B)+\lambda\int_{\tau(t_1)}^{t}|r(s)|\Delta s$ on $(T_1, T_2]_{\mathbb{T}}$. By (1.1) and (H1), $x(\tau(T_1)) \leq 0$. Set

$$T_0 := \max\{t \in [\tau(T_1),T_1)_{\mathbb{T}} : x(t)x^{\sigma}(t) \leq 0\}. \tag{2.51}$$

Then $x(T_0)x^{\sigma}(T_0) \leq 0$ and $\tau^2(t_1) \leq \tau^3(T_2) \leq \tau^2(T_0) \leq T_0 < T_2$. By (2.46),

$$|x(t)| \leq f^{\dagger}(B)+\lambda\int_{\tau(t_1)}^{t}|r(s)|\Delta s, \quad t \in [\tau^2(T_0),T_0]_{\mathbb{T}}. \tag{2.52}$$

As a result, from Lemma 2.2,

$$|x(t)| \leq f^{\dagger}(B)+\lambda\int_{\tau(t_1)}^{t}|r(s)|\Delta s \quad \text{on } [\sigma(T_0),\ \tau^{-1}(\sigma(T_0))]_{\mathbb{T}}. \tag{2.53}$$

Since $\tau(T_1) \leq T_0 < T_2$ and τ is increasing, $\sigma(T_0) \leq T_2 \leq \tau^{-1}(\sigma(T_0))$ and

$$f^{\dagger}(B)+\lambda\int_{\tau(t_1)}^{T_2}|r(s)|\Delta s < x(T_2) \leq f^{\dagger}(B)+\lambda\int_{\tau(t_1)}^{T_2}|r(s)|\Delta s, \tag{2.54}$$

a contradiction.

Case 2. If $x(T_1) \leq -f^{\dagger}(B)-\lambda\int_{\tau(t_1)}^{t}|r(s)|\Delta s$, then (2.46) implies either $x^{\Delta}(T_1) \leq 0$ or $x^{\Delta}(\rho(T_1)) < 0$. Again by (1.1) and (H1), either $x(\tau(T_1)) \geq 0$ or $x(\tau(\rho(T_1))) > 0$. Pick T_0 as above for either case. Just as above, either case leads to a contradiction. □

3. Solutions of (1.1) go to zero

We now state our main result on the global asymptotic behavior of solutions of (1.1).

Theorem 3.1. *If (H1), (H2), (H3) hold, then every solution of (1.1) goes to zero in the limit.*

Proof. If x is a nonoscillatory solution of (1.1), assume without loss of generality that x is eventually positive. Then there exist $M > 0$ and $T_0 \geq t_0$ such that

$$0 < x(t) \leq |x(t_0)| + \int_{t_0}^{t}|r(s)|\Delta s < M, \quad t \in (T_0,\infty)_{\mathbb{T}}. \tag{3.1}$$

Suppose that $\liminf_{t\to\infty} x(t) = 2\varepsilon$ for some $\varepsilon > 0$. Pick $T \in (\tau^{-1}(t_0), \infty)_{\mathbb{T}}$ such that $x(t) \geq \varepsilon$ for $t > \tau(T)$. Since f is continuous and $f(x) > 0$ for $x > 0$, $d := \inf_{\varepsilon \leq x \leq M} f(x) > 0$. By (1.1),

$$x^{\Delta}(t) = -p(t)f(x(\tau(t))) + r(t) \leq -dp(t) + r(t), \quad t \geq T. \tag{3.2}$$

Integrating from T to t, we see that

$$x(t) \leq x(T) - d\int_T^t p(s)\Delta s + \int_T^t |r(s)| \quad \Delta s \longrightarrow -\infty \tag{3.3}$$

as $t \to \infty$ by (H2) and (H3), a contradiction of x eventually positive. Consequently, $\liminf_{t\to\infty} x(t) = 0$, so there exists an increasing unbounded sequence $\{t_n\}_{n=1}^{\infty}$ in $\mathbb{T}$ such that $\lim_{n\to\infty} x(t_n) = 0$. Let $M' := \limsup_{t\to\infty} x(t)$. Again there exists a sequence $\{t'_n\}_{n=1}^{\infty}$ in $\mathbb{T}$ with $t'_n \geq t_n$ such that $\lim_{n\to\infty} x(t'_n) = M'$. Using (H2) and the fact that $x^{\Delta}(t) \leq r(t)$,

$$0 < x(t'_n) \leq x(t_n) + \int_{t_n}^{t'_n} |r(s)| \quad \Delta s \longrightarrow 0,\ n \longrightarrow \infty. \tag{3.4}$$

Hence $M' = 0$ and x goes to zero.

Now let x be an oscillatory solution of (1.1). By Lemma 2.4, (2.44) holds. By the oscillatory nature of x, there exists a sequence $\{t_n^*\}$ in $\mathbb{T}$ such that

$$x(t_n^*)x^{\sigma}(t_n^*) \leq 0, \qquad \tau(t_1^*) \geq \tau^{-1}(t_0), \qquad \tau(t_{n+1}^*) > \tau^{-1}(t_n^*). \tag{3.5}$$

As in [13], we consider the discrete sequence $\{X_n\}$ given by

$$X_1 = B := \sup_{t\geq t_0} |x(t)|, \qquad X_{n+1} = f^{\dagger}(X_n) + \lambda\int_{\tau(t_n^*)}^{\infty} |r(s)|\Delta s. \tag{3.6}$$

Just as in the proof of Lemma 2.3, we arrive at

$$\sup_{t\in[\tau^2(t_n^*),t_n^*]_{\mathbb{T}}} |x(t)| \leq X_n, \qquad \sup_{t\geq\sigma(t_n^*)} |x(t)| \leq X_{n+1}. \tag{3.7}$$

Note that

$$\sum_{n=1}^{\infty} \lambda\int_{\tau(t_n^*)}^{\infty} |r(s)|\Delta s \leq \sum_{n=0}^{\infty} \lambda\int_{\tau^{1-n}(t_1^*)}^{\infty} |r(s)|\Delta s \leq \lambda\sum_{n=0}^{\infty} \int_{\tau^{1-n}(t_1^*)}^{\infty} |r(s)|\Delta s; \tag{3.8}$$

by (H2),

$$\sum_{n=1}^{\infty} \lambda\int_{\tau(t_n^*)}^{\infty} |r(s)|\Delta s =: \sum_{n=1}^{\infty} b_n < \infty. \tag{3.9}$$

Since X_n satisfies the difference equation $X_{n+1} = f^{\dagger}(X_n) + b_n$, using [13, Lemma 2.3] we have that X_n goes to zero as $n \to \infty$. By the choice of X_n, the solution x of (1.1) satisfies $\lim_{t\to\infty} x(t) = 0$. □

Corollary 3.2. *If (H1) and (H3) hold, then every solution of the unforced equation*

$$x^{\Delta}(t)+p(t)f(x(\tau(t)))=0, \quad t\in[t_0,\infty)_{\mathbb{T}}, \tag{3.10}$$

goes to zero in the limit.

Remark 3.3. The results of this paper could easily be modified to show that every solution of

$$x^{\Delta}(t)=\ominus p(t)f(x(\tau(t)))-\ominus r(t), \quad t\in[t_0,\infty)_{\mathbb{T}}, \quad t_0\geq 0, \tag{3.11}$$

goes to zero in the limit, for appropriately adjusted hypotheses (H1), (H2), (H3), where $\ominus p(t):=-p(t)/(1+\mu(t)p(t))$ for $t\in\mathbb{T}$ and $p\in\mathcal{R}$ (see Definition 5.5).

4. Forced delay equation on isolated time scales

Let $\mathbb{T}$ be a time scale unbounded above, with every point both left and right scattered, and consider the food-limited population model [7, 13] given by the delay differential equation

$$y'(t)=py(t)\frac{N-y(t-\tau)}{N+cpy(t-\tau)}, \tag{4.1}$$

where y is the population density, $p>0$ is a constant growth rate, $N>0$ is the carrying capacity of the habitat, $\tau>0$ is the time delay, and $c>0$ is constant. From this we obtain the following modified equation:

$$\frac{1}{y(t)}\frac{dy(t)}{dt}=p\frac{N-y(\lfloor t\rfloor-\lfloor\tau\rfloor)}{N+cpy(\lfloor t\rfloor-\lfloor\tau\rfloor)}, \tag{4.2}$$

where $\lfloor t\rfloor:=\sup\{s\in\mathbb{T}:s\leq t\}$ is the "time-scale" part of the continuous variable t. On any interval of the form $[s,\sigma(s))$, integrate (4.2) from s to t to obtain for $s\leq t<\sigma(s)$ that

$$y(t)=y(s)\exp\left(p\frac{N-y(s-\lfloor\tau\rfloor)}{N+cpy(s-\lfloor\tau\rfloor)}(t-s)\right). \tag{4.3}$$

Replacing t by $\sigma(s)$,

$$y^{\sigma}(s)=y(s)\exp\left(p\mu(s)\frac{N-y(s-\lfloor\tau\rfloor)}{N+cpy(s-\lfloor\tau\rfloor)}\right). \tag{4.4}$$

Note that if $\mathbb{T}=\mathbb{Z}$ and $\lfloor\tau\rfloor=0$, then $\sigma(s)=s+1$ and $\mu(s)\equiv 1$, and this is the simple genotype selection model suggested in [9] and [11, Exercise 1.18(6)]. Thus for any isolated time scale $\mathbb{T}$ that is unbounded above, we consider

$$\frac{y^{\sigma}(t)}{y(t)}=\exp\left(p\mu(t)\frac{N-y(\tau(t))}{N+cpy(\tau(t))}+r(t)\right) \tag{4.5}$$

for some delay $\tau : \mathbb{T} \to \mathbb{T}$ and some function $r : \mathbb{T} \to \mathbb{R}$ satisfying (H2). In (4.5), let $y = Ne^x$, $y^\sigma = Ne^{x\circ\sigma}$, and $y \circ \tau = Ne^{x\circ\tau}$ to obtain

$$x^\Delta(t) = -p\frac{e^{x(\tau(t))} - 1}{1 + cpe^{x(\tau(t))}} + \frac{r(t)}{\mu(t)}. \tag{4.6}$$

If

$$f(x) := \frac{e^x - 1}{1 + cpe^x}, \tag{4.7}$$

then f is continuous with $xf(x) > 0$ for $x \neq 0$ and $f(0) = 0$. As shown in [13], if $cp > 1/3$, then $|f(x)| < |x|$ for $x \neq 0$ as well.

Theorem 4.1. *Suppose $cp > 1/3$ and $\mathbb{T}$ is an isolated time scale with $t_0 \in \mathbb{T}$. If*

$$\sum_{n=0}^{\infty} \sum_{t\in[\tau^{1-n}(t_0),\infty)_{\mathbb{T}}} |r(t)| < \infty, \tag{4.8}$$

$$p(\sigma(t) - \tau(t)) \leq \lambda \quad \forall t \in [t_0, \infty)_{\mathbb{T}}, \tag{4.9}$$

then every positive solution of (4.5) goes to N in the limit.

Proof. Let y be a positive solution of (4.5). As above, the substitution $y = Ne^x$ makes x a solution of (4.6). Since $cp > 1/3$, [13, Theorem 3.1] shows that (H1) is satisfied. To check (H2), note that on isolated time scales,

$$\sum_{n=0}^{\infty} \int_{\tau^{1-n}(t_0)}^{\infty} \frac{|r(s)|\Delta s}{\mu(s)} = \sum_{n=0}^{\infty} \sum_{t\in[\tau^{1-n}(t_0),\infty)_{\mathbb{T}}} |r(t)| < \infty \tag{4.10}$$

by assumption. In the same way, (H3) is satisfied for constant $p > 0$, as

$$\int_{\tau(t)}^{\sigma(t)} p(s)\Delta s = p(\sigma(t) - \tau(t)) \leq \lambda, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.11}$$

Hence by Theorem 3.1, every solution x of (4.6) goes to zero in the limit. But then every positive solution $y = Ne^x$ of (4.5) goes to N. □

Example 4.2. Let $\mathbb{T} = h\mathbb{Z}$ for some $h \in (0, 1)$, $c > 0$, and let $\tau(t) := t - hk$ for $t \in \mathbb{T}$ and $k \in \mathbb{N}$. If

$$\sum_{t=-k}^{\infty} t|r(th)| < \infty, \tag{4.12}$$

$$\frac{1}{3c} < p \leq \frac{3k + 4}{2h(k + 1)^2}, \tag{4.13}$$

then every positive solution of (4.5) goes to N in the limit.

Proof. Observe that $\lambda = (3k+4)/2(k+1)$, and $\sigma(t) - \tau(t) = h(k+1)$. Now we show that (4.12) is equivalent to (4.8) on $h\mathbb{Z}$. In fact, both will be shown to be equivalent to

$$\sum_{t=-k}^{\infty} \sum_{s=t}^{\infty} |r(sh)| < \infty; \tag{4.14}$$

the idea of these three equivalences is adapted from the real case found in [10, Lemma 3.3]. First note that (4.8), (4.12), and (4.14) all imply that $\sum_{t=-k}^{\infty} |r(th)| < \infty$. To see that (4.8) implies (4.14), we switch the order of summing in (4.14) to get that

$$\begin{aligned} \sum_{t=-k}^{\infty} \sum_{s=t}^{\infty} |r(sh)| &= \sum_{n=0}^{\infty} \sum_{t=(n-1)k}^{nk-1} \sum_{s=t}^{\infty} |r(sh)| \\ &= \sum_{n=0}^{\infty} \left(\sum_{t=(n-1)k}^{nk-1} (t-(n-1)k+1)|r(th)| + k \sum_{t=nk}^{\infty} |r(th)| \right). \end{aligned} \tag{4.15}$$

As a result,

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{t=nk}^{\infty} |r(th)| &\le \frac{1}{k} \sum_{t=-k}^{\infty} \sum_{s=t}^{\infty} |r(sh)| = \frac{1}{k} \sum_{n=0}^{\infty} \sum_{t=(n-1)k}^{nk-1} \sum_{s=t}^{\infty} |r(sh)| \\ &\le \frac{1}{k} \sum_{n=0}^{\infty} \sum_{t=(n-1)k}^{nk-1} \sum_{s=(n-1)k}^{\infty} |r(sh)| = \sum_{n=0}^{\infty} \sum_{t=(n-1)k}^{\infty} |r(th)|. \end{aligned} \tag{4.16}$$

Therefore (4.8) implies (4.14). And since

$$\sum_{t=-k}^{\infty} \sum_{s=t}^{\infty} |r(sh)| = \sum_{s=-k}^{\infty} \sum_{t=-k}^{s} |r(sh)| = \sum_{t=-k}^{\infty} (t+1-k)|r(th)|, \tag{4.17}$$

(4.12) implies (4.14). Therefore (4.8) implies (4.12). Thus all the hypotheses of Theorem 4.1 are met. □

5. Appendix on time scales

The definitions below merely serve as a preliminary introduction to the time-scale calculus; they can be found in the context of a much more robust treatment than is allowed here in the textbooks [4, 5] and the references therein.

Definition 5.1. Define the forward (backward) jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}$ (resp., $\rho(t)$ at t for $t > \inf \mathbb{T}$) by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\}), \quad \forall t \in \mathbb{T}. \tag{5.1}$$

Also define $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ if $\sup \mathbb{T} < \infty$, and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ if $\inf \mathbb{T} > -\infty$. Define the graininess function $\mu : \mathbb{T} \to \mathbb{R}$ by $\mu(t) = \sigma(t) - t$.

Throughout this work, the assumption is made that $\mathbb{T}$ is unbounded above and has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Also assume throughout that $a < b$ are points in $\mathbb{T}$ and define the time-scale interval $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$. The jump operators σ and ρ allow the classification of points in a time scale in the following way: if $\sigma(t) > t$, the point t is right-scattered, while if $\rho(t) < t$, then t is left-scattered. If $\sigma(t) = t$, the point t is right-dense; if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is left-dense.

Definition 5.2. Fix $t \in \mathbb{T}$ and let $y : \mathbb{T} \to \mathbb{R}$. Define $y^{\Delta}(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$, there is a neighborhood U of t such that for all $s \in U$,

$$\left| [y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s] \right| \le \epsilon \left| \sigma(t) - s \right|. \tag{5.2}$$

Call $y^{\Delta}(t)$ the (delta) derivative of y at t.

Definition 5.3. If $F^{\Delta}(t) = f(t)$, then define the (Cauchy) delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a). \tag{5.3}$$

The following theorem is due to Hilger [8].

THEOREM 5.4. *Assume that $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$.*

(1) *If f is differentiable at t, then f is continuous at t.*

(2) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}. \tag{5.4}$$

(3) *If f is differentiable and t is right-dense, then*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{5.5}$$

(4) *If f is differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$.*

Next we define the important concept of right-dense continuity. An important fact concerning right-dense continuity is that every right-dense continuous function has a delta antiderivative [4, Theorem 1.74]. This implies that the delta definite integral of any right-dense continuous function exists.

Definition 5.5. A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous (denoted by $f \in C_{\mathrm{rd}}(\mathbb{T};\mathbb{R})$) provided that f is continuous at every right-dense point $t \in \mathbb{T}$, and $\lim_{s \to t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$. A function p is regressive provided that $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$, and

$$\mathcal{R} := \{ p \in C_{\mathrm{rd}}(\mathbb{T};\mathbb{R}) : 1 + \mu(t)p(t) \neq 0,\ t \in \mathbb{T} \}. \tag{5.6}$$

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