PERIODIC SOLUTIONS OF NONLINEAR VECTOR DIFFERENCE EQUATIONS

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Essentially nonlinear difference equations in a Euclidean space are considered. Conditions for the existence of periodic solutions and solution estimates are derived. Our main tool is a combined usage of the recent estimates for matrix-valued functions with the method of majorants.

1. Introduction and notation

Periodic solutions of difference equations in Euclidean and Banach spaces have been considered by many authors, see, for example, [1–3, 5–10, 12] and the references therein. Mainly equations with separated linear parts and scalar equations were investigated. In this paper, we consider essentially nonlinear systems in a Euclidean space. We prove the existence of periodic solutions and derive the estimates for their norms.

Let $\mathbb{C}^n$ be the set of all complex n-vectors with an arbitrary norm $\|\cdot\|$, I is the unit matrix, $R_s(A)$ denotes the spectral radius of a matrix A, and

$$\Omega(r) = \{z \in \mathbb{C}^n : \|z\| \le r\}. \tag{1.1}$$

Consider in $\mathbb{C}^n$ the equation

$$x(t+1) = B(x(t),t)x(t) + F(x(t),t) \quad (t = 0,1,2,\dots), \tag{1.2}$$

where $F(\cdot,t)$ continuously maps $\Omega(r)$ into $\mathbb{C}^n$, and $B(z,t)$ are $n \times n$-matrices continuous in $z \in \Omega(r)$ and dependent on $t = 0,1,\dots$. In addition, $F(v,t)$ and $B(v,t)$ are periodic in t:

$$\begin{aligned} F(z,t) &= F(z,t+T) \quad (z \in \Omega(r);\ t = 0,1,\dots), \\ B(z,t) &= B(z,t+T) \quad (z \in \Omega(r);\ t = 0,1,\dots) \end{aligned} \tag{1.3}$$

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for some positive integer T. It is also assumed that there are nonnegative constants ν and μ, such that

$$\|F(z,t)\| \leq \nu\|z\| + \mu \quad (z \in \Omega(r),\ t = 0,1,2,\dots,T-1). \tag{1.4}$$

Denote by $\omega(r,T)$ *the set of the finite sequences* $h = \{v(k)\}_{k=0}^{T-1}$ *whose elements* $v(k)$ *belong to* $\Omega(r)$.

For an $h = \{v(k)\}_{k=0}^{T} \in \omega(r,T)$, put

$$\begin{gathered} U_h(t,s) = B(v(t-1),t-1)B(v(t-2),t-2)\cdots B(v(s),s), \\ U_h(t,t) = I \quad (0 \leq s < t \leq T) \end{gathered} \tag{1.5}$$

and assume that

$$I - U_h(T,0) \text{ is invertible} \quad \forall h \in \omega(r,T). \tag{1.6}$$

2. Statement of the main result

Theorem 2.1. *Under conditions (1.3)–(1.6), with the notation*

$$\begin{aligned} M(r,T) := \sup_{h\in\omega(r,T);\ k=0,\dots,T-1} & \sum_{j=0}^{T-1} \|U_h(k,0)(I-U_h(T,0))^{-1}U_h(T,j+1)\| \\ & + \sum_{j=0}^{k-1} \|U_h(k,j+1)\| \end{aligned} \tag{2.1}$$

suppose that

$$M(r,T)(\nu r + \mu) < r. \tag{2.2}$$

Then system (1.2) has a T-periodic solution. Moreover, that periodic solution satisfies the estimates

$$\max_{j=0,1,\dots,T-1} \|x(j)\| \leq \frac{\mu M(r,T)}{1-\nu M(r,T)} < r. \tag{2.3}$$

We remark that if $F(0,t) \neq 0$ for some t in $\{0,1,\dots,T-1\}$, then the solution found in the above theorem cannot be trivial.

For instance, let

$$\|B(z,t)\| \leq q < 1 \quad (z \in \Omega(r),\ t = 0,\dots,T-1). \tag{2.4}$$

Then $\|U_h(k,j)\| \leq q^{k-j}$ and

$$\|(I-U_h(T,0))^{-1}\| \leq \frac{1}{1-q^T}. \tag{2.5}$$

Therefore

$$M(r,T) \le \sum_{j=0}^{T-1} \frac{1}{1-q^T} q^{T-j-1} + \max_k \sum_{j=0}^{k-1} q^{k-j-1} \le \sum_{j=0}^{T-1} q^j \left(\frac{1}{1-q^T} + 1 \right) = \frac{2-q^T}{1-q^T} \sum_{j=0}^{T-1} q^j. \tag{2.6}$$

But

$$\sum_{j=0}^{T-1} q^j = \frac{1-q^T}{1-q}. \tag{2.7}$$

Thus

$$M(r,T) \le \frac{2-q^T}{1-q}. \tag{2.8}$$

Now Theorem 2.1 implies the following corollary.

Corollary 2.2. *Under conditions (1.3)–(1.4) and (2.4), suppose that*

$$(r\nu + \mu)\frac{2-q^T}{1-q} < r. \tag{2.9}$$

Then system (1.2) has a T-periodic solution. Moreover that periodic solution satisfies the estimates

$$\max_{j=0,1,\dots,T-1} ||x(j)|| \le \frac{\mu(2-q^T)}{1-q-\nu(2-q^T)} \le r. \tag{2.10}$$

3. Proof of Theorem 2.1

To achieve our goal, let us first consider the nonhomogeneous periodic problem

$$y(t+1) = B(\nu(t),t)y(t) + f(t), \quad t = 0,1,\dots,T-1 \tag{3.1}$$

$$y(0) = y(T), \tag{3.2}$$

where $\{f(t)\}_{k=0}^{T-1}$ is a given sequence in $\mathbb{C}^n$ and $h = \{\nu(t)\} \in \omega(r,T)$. Thanks to the Variation of constants formula, solution of (3.1) is given by

$$y(k) = U_h(k,0)y(0) + \sum_{j=0}^{k-1} U_h(k-1,j+1)f(j), \quad k = 1,\dots,T. \tag{3.3}$$

Thus, the periodic boundary value problem (3.1), (3.2) has a solution provided

$$y(0) = y(T) = U_h(T,0)y(0) + \sum_{j=0}^{T-1} U_h(T,j+1)f(j), \tag{3.4}$$

or

$$y(0) = \left(I - U_h(T,0)\right)^{-1} \sum_{j=0}^{T-1} U_h(T, j+1) f(j), \tag{3.5}$$

and in such a case, this solution is given by

$$y(k) = U_h(k,0)\left(I - U_h(T,0)\right)^{-1} \sum_{j=0}^{T-1} U_h(T, j+1) f(j) + \sum_{j=0}^{k-1} U_h(k, j+1) f(j), \quad k = 1,\ldots,T, \tag{3.6}$$

and thus its maximum norm satisfies the inequality

$$\max_{j=0,1,\ldots,T-1} \|y(j)\| \le M(r,T) \max_{j=0,1,\ldots,T-1} \|f(j)\|. \tag{3.7}$$

Let us consider the nonlinear periodic problem (1.2), (3.2).

Lemma 3.1. *Under conditions (1.4), (1.6), and (2.2), the periodic problem (1.2), (3.2) has at least one solution* $\{x(t)\}_{t=0}^{T} \in \omega(r,T)$. *Moreover, that solution satisfies estimates (2.3).*

Proof. For an arbitrary $h = \{v(t)\} \in \omega(r,T)$, define a mapping Z by

$$\begin{aligned} (Zh)(k) = {} & U_h(k,0)\left(I - U_h(T,0)\right)^{-1} \sum_{j=0}^{T-1} U_h(T, j+1) F(v(j), j) \\ & + \sum_{j=0}^{k-1} U_h(k, j+1) F(v(j), j), \quad k = 0,\ldots,T-1. \end{aligned} \tag{3.8}$$

Due to (2.2),

$$\begin{aligned} \max_{j=0,1,\ldots,T-1} \|(Zh)(j)\| &\le \max_{t=0,\ldots,T-1} \|F(v(t),t)\| M(r,T) \\ &\le \left(\nu \max_{j=0,\ldots,T-1} \|v(j)\| + \mu\right) M(r,T) \le \nu r + \mu. \end{aligned} \tag{3.9}$$

So Z continuously maps $\omega(r,T)$ into itself. By Browder's fixed point theorem, Z has a fixed point $x \in \omega(r,T)$, cf. [11]. It is easily checked that the point is the desired solution of problem (1.2), (3.2).

Furthermore, if $\{x(t)\}_{t=0}^{T} \in \omega(r,T)$ is a solution of (1.2), (3.2), then in view of (3.7) and (1.4), we will have the relations

$$\max_{j=0,1,\ldots,T-1} \|x(j)\| \le \max_{t=0,1,\ldots,T-1} \|F(x(t),t)\| M(r,T) \le \left(\nu \max_{j=0,\ldots,T} \|x(j)\| + \mu\right) M(r,T), \tag{3.10}$$

which implies (2.3), since under (2.2) $\nu M(r,T) < 1$. The proof is complete. □

Assertion of Theorem 2.1 follows from the previous lemma and the periodicity of $F(\cdot,t)$ and $B(\cdot,t)$ in t.

4. Systems with linear majorants

In this section and the next one it is assumed that the norm *is ideal*. That is the vectors $z = (z_k)_{k=1}^n$ and $|z| = (|z_k|)_{k=1}^n$ have the same norm. For example,

$$\|z\| = \|z\|_p = \left[\sum_{k=1}^{n} |z_k|^p\right]^{1/p} \quad (1 \le p < \infty). \tag{4.1}$$

Let there be a variable matrix $W(t) = (w_{jk}(t))_{j,k=1}^n$ $t = 0,\dots,T$ independent of z with nonnegative entries, such that the relation

$$|B(z,t)| \le W(t) \quad (z \in \Omega(r),\ t = 0,\dots,T-1) \tag{4.2}$$

is valid with a positive $r < \infty$. Then we will say that $B(\cdot,t) = (b_{\{jk\}}(\cdot,t))_{j,k=1}^n$ has in $\Omega(r)$ *the linear majorant* $W(t)$.

Inequality (4.2) means that

$$|b_{jk}(z,t)| \le w_{jk}(t) \quad (j,k = 1,\dots,n;\ z \in \Omega(r),\ t = 1,2,\dots,T). \tag{4.3}$$

Let us introduce the equation

$$y(t+1) = W(t)y(t) \quad (t = 1,2,\dots). \tag{4.4}$$

Lemma 4.1. *Let $B(\cdot,t)$ have a linear majorant $W(t)$ in the ball $\Omega(r)$. Then*

$$\|U_h(t,s)\| \le \|V(t,s)\| \quad (h \in \omega(r,T),\ 0 \le s < t \le T-1), \tag{4.5}$$

where $V(t,s) = W(t-1)W(t-2)\cdots W(s)$.

Proof. Clearly,

$$\|U_h(t,s)\| = \|B(v(t-1),t-1)\cdots B(v(s),s)\| \le \|W(t-1)\cdots W(s)\|. \tag{4.6}$$

This proves the result. □

Furthermore, assume that the spectral radius of $V(T,0)$ is less than one. Then the matrix $I - V(T,0)$ is positively invertible. Put

$$m(W,T) := \sup_{k=0,\dots,T-1} \sum_{j=0}^{T-1} \|V(k,0)(I - V(T,0))^{-1}V(T,j+1)\| + \sum_{j=0}^{k-1} \|V(k,j+1)\|. \tag{4.7}$$

Now Theorem 2.1 implies the following theorem.

Theorem 4.2. *Under conditions (1.3)–(1.4) and (4.2) assume that the evolution operator of (4.4) satisfy the inequality $R_s(V(T,0)) < 1$. In addition, suppose that*

$$(r\nu + \mu)m(W,T) < r. \tag{4.8}$$

Then system (1.2) has a T-periodic solution. Moreover, that periodic solution satisfies the estimates

$$\max_{j=0,1,\ldots,T-1} ||x(j)|| \le \frac{\mu m(W,T)}{1-\nu m(W,T)} \le r. \tag{4.9}$$

5. Systems with constant majorants

Assume that in (4.2) $W(t) \equiv W_0$ is a constant matrix. Then we will say that $B(h,t)$ has in set $\Omega(r)$ *the constant majorant* $W(t)$. In this case $V(t,s) = W_0^{t-s}$. Set

$$m(W_0,T) = \max_{k=0,\ldots,T-1} \{||W_0^k (I-W_0^T)^{-1}|| + 1\} \sum_{j=0}^{T-1} ||W_0^j||. \tag{5.1}$$

Now Theorem 4.2 yields the following theorem.

Theorem 5.1. *Under conditions (1.3)–(1.4) assume that $B(\cdot,s)$ has in $\Omega(r)$ a constant majorant W_0, and $R_s(W_0) < 1$. In addition, suppose that*

$$(\mu + r\nu) m(W_0,T) < r. \tag{5.2}$$

Then system (1.2) has a T-periodic solution. Moreover, that periodic solution satisfies the estimates

$$\max_{j=0,1,\ldots,T-1} ||x(j)|| \le \frac{\mu m(W_0,T)}{1-\nu m(W_0,T)} < r. \tag{5.3}$$

Let us derive an estimate for $m(W_0;T)$ in terms of the eigenvalues and the Frobenius norm of W_0 as follows. Let $\|\cdot\|_2$ be the Euclidean norm in $\mathbb{C}^n$, and A be an $n\times n$-matrix. Let $\lambda_1(A),\ldots,\lambda_n(A)$ be the eigenvalues of A including their multiplicities. We will make use of the following quantity

$$g(A) = \left\{ N^2(A) - \sum_{i=1}^{n} |\lambda_i(A)|^2 \right\}^{1/2}, \tag{5.4}$$

where $N(A)$ is the Frobenius (Hilbert-Schmidt) norm of A, that is, $N^2(A) = \text{Trace}(AA^*)$. Below we give simple estimates for $g(A)$.

Next, we recall that the following estimates are valid:

$$||A^m||_2 \le \sum_{k=0}^{n-1} R_s^{m-k}(A) g^k(A) \frac{C_m^k}{\sqrt{k!}} \quad (m = 0,1,\ldots), \tag{5.5}$$

$$||(A-\lambda I)^{-1}||_2 \le \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}\rho^{k+1}(A,\lambda)}, \tag{5.6}$$

where

$$C_m^k = \frac{m!}{(m-k)!k!} \tag{5.7}$$

and $\rho(A,\lambda)$ is the distance between $\lambda \in \mathbb{C}$ and the spectrum of A. Estimates (5.5) and (5.6) are proved in [4, pages 12 and 21]. Thus,

$$||W_0^m||_2 \le \theta_m(W_0), \quad m = 0,1,2,\ldots, \tag{5.8}$$

where

$$\theta_m(W_0) = \sum_{k=0}^{n-1} R_s^{m-k}(W_0) g^k(W_0) \frac{C_m^k}{\sqrt{k!}}. \tag{5.9}$$

Furthermore, due to (5.6)

$$||(W_0^T - I)^{-1}||_2 \le \nu(T, W_0), \tag{5.10}$$

where

$$\nu(T,W_0) = \sum_{k=0}^{n-1} \frac{g^k(W_0^T)}{\sqrt{k!}(1 - R_s^T(W_0))^{k+1}}. \tag{5.11}$$

Then

$$m(W_0;T) \le \widetilde{M}(W_0;T), \tag{5.12}$$

where

$$\widetilde{M}(W_0;T) := \left\{\nu(T,W_0) \max_{k=0,\ldots,T-1} \theta_k(W_0) + 1\right\} \sum_{j=0}^{T-1} \theta_j(W_0). \tag{5.13}$$

Under the condition, $R_s(W_0) < 1$ we have

$$\max_{k=0,\ldots,T-1} \theta_k(W_0) \le 2^{T-1} \sum_{k=0}^{n-1} \frac{g^k(W_0)}{\sqrt{k!}}. \tag{5.14}$$

Note also that $g(W_0^T) \le N^T(W_0)$. Moreover, if A is a normal matrix: $AA^* = A^*A$, then $g(A) = 0$. The following inequalities are also true

$$\begin{gathered} g^2(A) \le N^2(A) - |\operatorname{Trace} A^2|, \\ g^2(A) \le \frac{1}{2} N^2(A^* - A), \end{gathered} \tag{5.15}$$

cf. [4, Section 2.1].

Now Theorem 5.1 implies the following theorem.

Theorem 5.2. *Under conditions (1.3)–(1.4), assume that $B(\cdot,t)$ has in $\Omega(r)$ a constant majorant W_0 and $R_s(W_0) < 1$. In addition, let*

$$(\mu + r\nu)\widetilde{M}(W_0;T) < r. \tag{5.16}$$

Then system (1.2) has a T-periodic solution. Moreover, that periodic solution satisfies the estimates

$$\max_{j=0,1,\dots,T-1} \|x(j)\| \le \frac{\mu \widetilde{M}(W_0,T)}{1-\nu \widetilde{M}(W_0,T)} \le r. \tag{5.17}$$

As an example, let W_0 be a normal matrix, then $g(W_0)=0$, $\theta_m(W_0)=R_s^m(W_0)\le 1$ and

$$\widetilde{M}(W_0,T)=\frac{1}{1-R_s^T(W_0)}. \tag{5.18}$$

Now we can directly apply the previous theorem.

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