ON THE SYSTEM OF RATIONAL DIFFERENCE EQUATIONS $x_{n+1} = f(y_{n-q}, x_{n-s}), y_{n+1} = g(x_{n-t}, y_{n-p})$

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We study the global behavior of positive solutions of the system of rational difference equations $x_{n+1} = f(y_{n-q}, x_{n-s})$, $y_{n+1} = g(x_{n-t}, y_{n-p})$, $n = 0,1,2,\ldots$, where $p,q,s,t \in \{0,1,2,\ldots\}$ with $s \geq t$ and $p \geq q$, the initial values $x_{-s}, x_{-s+1},\ldots,x_0, y_{-p}, y_{-p+1},\ldots,y_0 \in (0,+\infty)$. We give sufficient conditions under which every positive solution of this system converges to the unique positive equilibrium.

1. Introduction

In this paper, we study the convergence of positive solutions of a system of rational difference equations. Recently there has been published quite a lot of works concerning the behavior of positive solutions of systems of rational difference equations [1–7, 9, 11]. Not only these results are valuable in their own right, but also they can provide insight into their differential counterparts.

Papaschinopoulos and Schinas [10] studied the oscillatory behavior, the periodicity, and the asymptotic behavior of the positive solutions of systems of rational difference equations

$$x_{n+1} = A + \frac{x_{n-1}}{y_n}, \quad y_{n+1} = A + \frac{y_{n-1}}{x_n}, \quad n = 0,1,\ldots, \tag{1.1}$$

where $A \in (0,+\infty)$ and the initial values $x_{-1}, x_0, y_{-1}, y_0 \in (0,+\infty)$.

Recently, Kulenović and Nurkanović [8] investigated the global asymptotic behavior of solutions of systems of rational difference equations

$$x_{n+1} = \frac{a + x_n}{b + y_n}, \quad y_{n+1} = \frac{d + y_n}{e + x_n}, \quad n = 0,1,\ldots, \tag{1.2}$$

where $a,b,d,e \in (0,+\infty)$ and the initial values $x_0, y_0 \in (0,+\infty)$.

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In this paper, we consider the more general equation

$$x_{n+1} = f(y_{n-q}, x_{n-s}), \qquad y_{n+1} = g(x_{n-t}, y_{n-p}), \tag{1.3}$$

where $p, q, s, t \in \{0, 1, 2, \ldots\}$ with $s \geq t$ and $p \geq q$, the initial values $x_{-s}, x_{-s+1}, \ldots, x_0, y_{-p}, y_{-p+1}, \ldots, y_0 \in (0, +\infty)$ and f satisfies the following hypotheses.

(H_1) $f(u,v), g(u,v) \in C(E \times E, (0,+\infty))$ with $a = \inf_{(u,v)\in E\times E} f(u,v) \in E$ and $b = \inf_{(u,v)\in E\times E} g(u,v) \in E$, where $E \in \{(0,+\infty), [0,+\infty)\}$.

(H_2) $f(u,v)$ and $g(u,v)$ are decreasing in u and increasing in v.

(H_3) Equation

$$x = f(y,x), \qquad y = g(x,y) \tag{1.4}$$

has a unique positive solution $x = \overline{x}$, $y = \overline{y}$.

(H_4) $f(b,x)$ has only one fixed point in the interval $(a,+\infty)$, denoted by A, and $g(a,y)$ has only one fixed point in the interval $(b,+\infty)$, denoted by B.

(H_5) For every $w \in E$, $f(w,x)/x$ and $g(w,x)/x$ are nonincreasing in x in $(0,+\infty)$.

2. Main results

Theorem 2.1. *Assume that (H_1)–(H_5) hold and $\{(x_n, y_n)\}$ is a positive solution of (1.3), then there exists a positive integer N such that*

$$f(B,a) \leq x_n \leq A, \quad g(A,b) \leq y_n \leq B, \quad \text{for } n \geq N. \tag{2.1}$$

Proof. Since $a = \inf_{(u,v)\in E\times E} f(u,v) \in E$ and $b = \inf_{(u,v)\in E\times E} g(u,v) \in E$, we have

$$\begin{aligned} \overline{x} &= f(\overline{y}, \overline{x}) > f(\overline{y}+1, \overline{x}) \geq a, \\ \overline{y} &= g(\overline{x}, \overline{y}) > g(\overline{x}+1, \overline{y}) \geq b. \end{aligned} \tag{2.2}$$

Claim 1. $g(A,b) < \overline{y} < B$ and $f(B,a) < \overline{x} < A$.

Proof of Claim 1. If $B \leq \overline{y}$, then it follows from (H_2), (H_4), and (H_5) that

$$B = g(a,B) > g(\overline{x},B) = B\frac{g(\overline{x},B)}{B} \geq B\frac{g(\overline{x},\overline{y})}{\overline{y}} = B, \tag{2.3}$$

which is a contradiction. Therefore $\overline{y} < B$. In a similar fashion it is true that $\overline{x} < A$.

Since $\overline{y} < B$ and $\overline{x} < A$, we have that

$$f(B,a) < f(\overline{y},\overline{x}) = \overline{x}, \qquad g(A,b) < g(\overline{x},\overline{y}) = \overline{y}, \tag{2.4}$$

Claim 1 is proven. □

Claim 2. (i) For all $n \geq q+1$, $x_{n+1} \leq x_{n-s}$ if $x_{n-s} > A$ and $x_{n+1} \leq A$ if $x_{n-s} \leq A$.

(ii) For all $n \geq t+1$, $y_{n+1} \leq y_{n-p}$ if $y_{n-p} > B$ and $y_{n+1} \leq B$ if $y_{n-p} \leq B$.

Proof of Claim 2. We only prove (i) (the proof of (ii) is similar). Obviously

$$x_{n+1} = f(y_{n-q}, x_{n-s}) \le f(b, x_{n-s}). \tag{2.5}$$

If $x_{n-s} \le A$, then $x_{n+1} \le f(b, x_{n-s}) \le f(b, A) = A$.

If $x_{n-s} > A$, then

$$\frac{f(b, x_{n-s})}{x_{n-s}} \le \frac{f(b, A)}{A} = 1, \tag{2.6}$$

which implies $x_{n+1} \le f(b, x_{n-s}) \le x_{n-s}$. Claim 2 is proven. □

Claim 3. (i) There exists a positive integer N_1 such that $x_n \le A$ for all $n \ge N_1$.

(ii) There exists a positive integer N_2 such that $y_n \le B$ for all $n \ge N_2$.

Proof of Claim 3. We only prove (i) (the proof of (ii) is similar). Assume on the contrary that Claim 3 does not hold. Then it follows from Claim 2 that there exists a positive integer R such that $x_{n(s+1)+R} \ge x_{(n+1)(s+1)+R} > A$ for every $n \ge 1$. Let $\lim_{n\to\infty} x_{n(s+1)+R} = A_1$, then $A_1 \ge A$.

We know from Claim 2 that $\{x_n\}$ and $\{y_n\}$ are bounded. Let $c = \lim_{n\to\infty} \sup y_{n(s+1)+R-q-1}$, then $c \ge b$ and there exists a sequence $n_k \to \infty$ such that

$$\lim_{k\to\infty} y_{n_k(s+1)+R-q-1} = c. \tag{2.7}$$

By (1.3) we have that

$$x_{n_k(s+1)+R} = f(y_{n_k(s+1)+R-q-1}, x_{(n_k-1)(s+1)+R}), \tag{2.8}$$

from which it follows that

$$A_1 = f(c, A_1) \le f(b, A_1) = A_1 \frac{f(b, A_1)}{A_1} \le A_1 \frac{f(b, A)}{A} = A_1. \tag{2.9}$$

This with (H_2) and (H_4) implies $c = b$ and $A_1 = A$. Therefore $\lim_{n\to\infty} y_{n(s+1)+R-q-1} = b$.

Since $\{x_n\}$ and $\{y_n\}$ are bounded, we may assume (by taking a subsequence) that there exist a sequence $l_n \to \infty$ and $\alpha, \beta \in E$ such that

$$\lim_{k\to\infty} x_{l_k(s+1)+R-q-t-2} = \alpha, \qquad \lim_{k\to\infty} y_{l_k(s+1)+R-q-p-2} = \beta. \tag{2.10}$$

By (1.3) we have that

$$y_{l_k(s+1)+R-q-1} = g(x_{l_k(s+1)+R-q-t-2}, y_{l_k(s+1)+R-q-p-2}), \tag{2.11}$$

from which it follows that

$$b = g(\alpha, \beta) > g(\alpha + 1, \beta) \ge b. \tag{2.12}$$

This is a contradiction. Claim 3 is proven. □

Let $N = \max\{N_1, N_2\} + 2s + 2p$, then for all $n > N$ we have that

$$\begin{gathered} x_n \le A, \qquad y_n \le B, \\ x_n = f(y_{n-q-1}, x_{n-s-1}) \ge f(B, a), \\ y_n = g(x_{n-t-1}, y_{n-p-1}) \ge g(A, b). \end{gathered} \tag{2.13}$$

Theorem 2.1 is proven. □

Theorem 2.2. *Let $I = [c, d]$ and $J = [\alpha, \beta]$ be intervals of real numbers. Assume that $f \in C(J \times I, I)$ and $g \in C(I \times J, J)$ satisfy the following properties:*

(i) *$f(u, v)$ and $g(u, v)$ are decreasing in u and increasing in v;*

(ii) *if $M_1, m_1 \in I$ with $m_1 \le M_1$ and $M_2, m_2 \in J$ with $m_2 \le M_2$ are a solution of the system*

$$\begin{aligned} M_1 &= f(m_2, M_1), \qquad & m_1 &= f(M_2, m_1), \\ M_2 &= g(m_1, M_2), \qquad & m_2 &= g(M_1, m_2), \end{aligned} \tag{2.14}$$

then $M_1 = m_1$ and $M_2 = m_2$.

Then the system

$$x_{n+1} = f(y_{n-q}, x_{n-s}), \quad y_{n+1} = g(x_{n-t}, y_{n-p}), \quad n = 0, 1, \ldots, \tag{2.15}$$

has a unique equilibrium $(\overline{S}, \overline{T})$ and every solution of (2.15) with the initial values $x_{-s}, x_{-s+1}, \ldots, x_0 \in I$ and $y_{-p}, y_{-p+1}, \ldots, y_0 \in J$ converges to $(\overline{S}, \overline{T})$.

Proof. Let

$$m_1^0 = c, \qquad m_2^0 = \alpha, \qquad M_1^0 = d, \qquad M_2^0 = \beta, \tag{2.16}$$

and for $i = 1, 2, \ldots,$ we define

$$\begin{aligned} M_1^i &= f(m_2^{i-1}, M_1^{i-1}), \qquad & m_1^i &= f(M_2^{i-1}, m_1^{i-1}), \\ M_2^i &= g(m_1^{i-1}, M_2^{i-1}), \qquad & m_2^i &= g(M_1^{i-1}, m_2^{i-1}). \end{aligned} \tag{2.17}$$

It is easy to verify that

$$\begin{aligned} m_1^0 \le m_1^1 = f(M_2^0, m_1^0) \le f(m_2^0, M_1^0) = M_1^1 \le M_1^0, \\ m_2^0 \le m_2^1 = g(M_1^0, m_2^0) \le g(m_1^0, M_2^0) = M_2^1 \le M_2^0. \end{aligned} \tag{2.18}$$

From (i) and (2.18) we obtain

$$\begin{aligned} m_1^1 &= f(M_2^0, m_1^0) \le f(M_2^1, m_1^1) = m_1^2, \\ m_1^2 &= f(M_2^1, m_1^1) \le f(m_2^1, M_1^1) = M_1^2, \\ M_1^2 &= f(m_2^1, M_1^1) \le f(m_2^0, M_1^0) = M_1^1, \\ m_2^1 &= g(M_1^0, m_2^0) \le g(M_1^1, m_2^1) = m_2^2, \\ m_2^2 &= g(M_1^1, m_2^1) \le g(m_1^1, M_2^1) = M_2^2, \\ M_2^2 &= g(m_1^1, M_2^1) \le g(m_1^0, M_2^0) = M_2^1. \end{aligned} \tag{2.19}$$

By induction it follows that for $i = 0, 1, \ldots,$

$$\begin{aligned} m_1^i &\le m_1^{i+1} \le \cdots \le M_1^{i+1} \le M_1^i, \\ m_2^i &\le m_2^{i+1} \le \cdots \le M_2^{i+1} \le M_2^i. \end{aligned} \tag{2.20}$$

On the other hand, we have $x_n \in [m_1^0, M_1^0]$ for any $n \ge -s$ and $y_n \in [m_2^0, M_2^0]$ for any $n \ge -p$ since $x_{-s}, x_{-s+1}, \ldots, x_0 \in [m_1^0, M_1^0]$ and $y_{-p}, y_{-p+1}, \ldots, y_0 \in [m_2^0, M_2^0]$. For any $n \ge 0$, we obtain

$$\begin{aligned} m_1^1 = f(M_2^0, m_1^0) \le x_{n+1} = f(y_{n-q}, x_{n-s}) \le f(m_2^0, M_1^0) = M_1^1, \\ m_2^1 = g(M_1^0, m_2^0) \le y_{n+1} = g(x_{n-t}, y_{n-p}) \le g(m_1^0, M_2^0) = M_2^1. \end{aligned} \tag{2.21}$$

Let $k = \max\{s+1, p+1\}$. It follows that for any $n \ge k$,

$$\begin{aligned} m_1^2 = f(M_2^1, m_1^1) \le x_{n+1} = f(y_{n-q}, x_{n-s}) \le f(m_2^1, M_1^1) = M_1^2, \\ m_2^2 = g(M_1^1, m_2^1) \le y_{n+1} = g(x_{n-t}, y_{n-p}) \le g(m_1^1, M_2^1) = M_2^2. \end{aligned} \tag{2.22}$$

By induction, for $l = 0, 1, \ldots,$ we obtain that for any $n \ge lk$,

$$m_1^{l+1} \le x_{n+1} \le M_1^{l+1}, \qquad m_2^{l+1} \le y_{n+1} \le M_2^{l+1}. \tag{2.23}$$

Let

$$\begin{aligned} \lim_{n\to\infty} m_1^n = m_1, \qquad \lim_{n\to\infty} m_2^n = m_2, \\ \lim_{n\to\infty} M_1^n = M_1, \qquad \lim_{n\to\infty} M_2^n = M_2. \end{aligned} \tag{2.24}$$

By the continuity of f and g, we have from (2.17) that

$$\begin{aligned} M_1 = f(m_2, M_1), \qquad M_2 = g(m_1, M_2), \\ m_2 = g(M_1, m_2), \qquad m_1 = f(M_2, m_1). \end{aligned} \tag{2.25}$$

Using assumption (ii), it follows from (2.23) that

$$\lim_{n\to\infty} x_n = m_1 = M_1 = \overline{S}, \qquad \lim_{n\to\infty} y_n = m_2 = M_2 = \overline{T}. \tag{2.26}$$

Theorem 2.2 is proven. □

THEOREM 2.3. *If (H_1)–(H_5) hold and the system*

$$\begin{aligned} M_1 = f(m_2, M_1), \qquad M_2 = g(m_1, M_2), \\ m_2 = g(M_1, m_2), \qquad m_1 = f(M_2, m_1), \end{aligned} \tag{2.27}$$

with $f(B,a) \le m_1 \le M_1 \le A$ and $g(A,b) \le m_2 \le M_2 \le B$ has the unique solution $m_1 = M_1 = \overline{x}$ and $m_2 = M_2 = \overline{y}$, then every solution of (1.3) converges to the unique positive equilibrium $(\overline{x},\overline{y})$.

Proof. Let $\{(x_n, y_n)\}$ is a positive solution of (1.3). By Theorem 2.1, there exists a positive integer N such that $f(B,a) \le x_n = f(y_{n-q},x_{n-s}) \le A$ and $g(A,b) \le y_n = g(x_{n-t}, y_{n-p}) \le B$ for all $n \ge N$. Since f, g satisfy the conditions (i) and (ii) of Theorem 2.2 in $I = [f(B,a),A]$ and $J = [(A,b),B]$, it follows that $\{(x_n, y_n)\}$ converges to the unique positive equilibrium $(\overline{x},\overline{y})$. □

3. Examples

In this section, we will give two applications of the above results.

Example 3.1. Consider equation

$$x_{n+1} = \frac{c + x_{n-s}}{a + y_{n-q}}, \qquad y_{n+1} = \frac{d + y_{n-p}}{b + x_{n-t}}, \tag{3.1}$$

where $p,q,s,t \in \{0,1,2,\dots\}$ with $s \ge t$ and $p \ge q$, the initial values $x_{-s}, x_{-s+1},\dots,x_0$, $y_{-p}, y_{-p+1},\dots, y_0 \in (0,+\infty)$ and $a,b,c,d \in (0,+\infty)$. If $a > 1$ and $b > 1$, then every positive solution of (3.1) converges to the unique positive equilibrium.

Proof. Let $E = [0,+\infty)$, it is easy to verify that (H_1)–(H_5) hold for (3.1). In addition, if

$$\begin{aligned} M_1 &= \frac{c + M_1}{a + m_2}, \qquad & M_2 &= \frac{d + M_2}{b + m_1}, \\ m_2 &= \frac{d + m_2}{b + M_1}, \qquad & m_1 &= \frac{c + m_1}{a + M_2}, \end{aligned} \tag{3.2}$$

with $0 \le m_1 \le M_1$ and $0 \le m_2 \le M_2$, then we have

$$\begin{aligned} (M_1 - m_1)(a-1) &= m_1 M_2 - M_1 m_2, \\ (M_2 - m_2)(b-1) &= M_1 m_2 - m_1 M_2, \end{aligned} \tag{3.3}$$

from which it follows that $M_1 = m_1$ and $M_2 = m_2$. Moreover, it is easy to verify that (3.2) have the unique solution

$$\begin{aligned} M_1 = m_1 = \overline{x} &= \frac{-(a-1)(b-1) + c - d + \sqrt{[(a-1)(b-1) + d - c]^2 + 4c(a-1)(b-1)}}{2(a-1)}, \\ M_2 = m_2 = \overline{y} &= \frac{-(a-1)(b-1) + d - c + \sqrt{[(a-1)(b-1) + c - d]^2 + 4d(a-1)(b-1)}}{2(b-1)}. \end{aligned} \tag{3.4}$$

It follows from Theorems 2.1 and 2.3 that every positive solution of (3.1) converges to the unique positive equilibrium $(\overline{x},\overline{y})$. □

Example 3.2. Consider equation

$$x_{n+1} = a + \frac{x_{n-s}}{y_{n-q}}, \qquad y_{n+1} = b + \frac{y_{n-p}}{x_{n-t}}, \tag{3.5}$$

where $p,q,s,t \in \{0,1,2,\dots\}$ with $s \geq t$ and $p \geq q$, the initial values $x_{-s}, x_{-s+1}, \dots, x_0$, $y_{-p}, y_{-p+1}, \dots, y_0 \in (0,+\infty)$ and $a,b \in (0,+\infty)$. If $a > 1$ and $b > 1$, then every positive solution of (3.5) converges to the unique positive equilibrium.

Proof. Let $E = (0,+\infty)$, it is easy to verify that (H_1)–(H_5) hold for (3.5). In addition, if

$$\begin{aligned} M_1 &= a + \frac{M_1}{m_2}, \qquad & M_2 &= b + \frac{M_2}{m_1}, \\ m_2 &= b + \frac{m_2}{M_1}, \qquad & m_1 &= a + \frac{m_1}{M_2}, \end{aligned} \tag{3.6}$$

with $0 \leq m_1 \leq M_1$ and $0 \leq m_2 \leq M_2$, then (3.6) have the unique solution

$$\begin{aligned} M_1 &= m_1 = \overline{x} = \frac{ab-1}{b-1}, \\ M_2 &= m_2 = \overline{y} = \frac{ab-1}{a-1}. \end{aligned} \tag{3.7}$$

It follows from Theorems 2.1 and 2.3 that every positive solution of (3.5) converges to the unique positive equilibrium $(\overline{x}, \overline{y}) = ((ab-1)/(b-1), (ab-1)/(a-1))$. □

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