OSCILLATION OF HIGHER-ORDER DELAY DIFFERENCE EQUATIONS

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The oscillation and asymptotic behavior of the higher-order delay difference equation $\Delta^l x_n + \sum_{i=1}^m p_i(n)x_{n-k_i} = 0$, $n = 0,1,2,\dots$, are investigated. Some sufficient conditions of oscillation and bounded oscillation of the above equation are obtained.

1. Introduction

Consider the following delay difference equation:

$$\triangle^l x_n + \sum_{i=1}^m p_i(n)x_{n-k_i} = 0, \quad n = 0,1,2,\dots, \tag{1.1}$$

and its first-order corresponding inequality

$$\triangle x_n + \sum_{i=1}^m p_i(n)x_{n-k_i} \le 0, \quad n = 0,1,2,\dots, \tag{1.2}$$

where $\{p_i(n)\}$ are sequences of nonnegative real numbers and not identically equal to zero, and k_i is positive integer, $i = 1,2,\dots$, and $\triangle$ is the first-order forward difference operator, $\triangle x_n = x_{n+1} - x_n$, and $\triangle^l x_n = \triangle^{l-1}(\triangle x_n)$ for $l \ge 2$.

By a solution of (1.1) or inequality (1.2), we mean a nontrival real sequence $\{x_n\}$ satisfying (1.1) or inequality (1.2) for $n \ge 0$. A solution $\{x_n\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise. An equation is said to be oscillatory if its every solution is oscillatory.

The oscillatory behavior of difference equations has been intensively studied in recent years. Most of the literature has been concerned with equations of type (1.1) with $l = 1$ (see [1–10] and references cited therein). But very little is known regarding the oscillation of higher-order difference equation similar to (1.1). The purpose of this paper is to study the oscillatory properties of (1.1).

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2. Main results

We need the following several lemmas in order to prove our results.

Lemma 2.1 [5, 8]. *Assume that*

$$\liminf_{n\to\infty}\sum_{i=1}^{m}\left(\frac{k_i+1}{k_i}\right)^{k_i+1}\sum_{s=n+1}^{n+k_i}p_i(s)>1, \tag{2.1}$$

or

$$\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{s=n}^{n+k_i}p_i(s)>1. \tag{2.2}$$

Then inequality (1.2) has no eventually positive solution.

Lemma 2.2 [1]. *Let x_n be defined for $n\ge n_0$ and $x_n>0$ with $\triangle^l x_n$ eventually of one sign and not identically zero. Then there exist an integer j, $0\le j\le l$ with $(l+j)$ odd for $\triangle^l x_n\le 0$ and $(l+j)$ even for $\triangle^l x_n\ge 0$ and an integer $N\ge n_0$, such that for all $n\ge N$,*

$$\begin{gathered}j\le l-1\Longrightarrow(-1)^{j+i}\triangle^i x_n>0,\quad j\le i\le l-1,\\ j\ge 1\Longrightarrow\triangle^i x_n>0,\quad 1\le i\le j-1.\end{gathered} \tag{2.3}$$

Specially, if $\triangle^l x_n\le 0$ for $n\ge n_0$, and $\{x_n\}$ is bounded, then

$$\begin{gathered}(-1)^{i+1}\triangle^{l-i}x_n\ge 0,\quad \forall\ \text{large } n\ge n_0,\ i=1,\dots,l-1,\\ \lim_{n\to\infty}\triangle^i x_n=0,\quad 1\le i\le l-1.\end{gathered} \tag{2.4}$$

Lemma 2.3 [1]. *Let x_n be defined for $n\ge n_0$, and $x_n>0$ with $\triangle^l x_n\le 0$ for $n\ge n_0$ and not identically zero. If x_n is increasing, then there exists a large integer $n_1\ge n_0$ such that*

$$x_n\ge\frac{2^{2-2l}}{(l-1)!}n^{(l-1)}\triangle^{l-1}x_n,\quad \forall n\ge 2^l n_1. \tag{2.5}$$

Specially,

$$x_n\ge\frac{\theta}{(l-1)!}n^{l-1}\triangle^{l-1}x_n,\quad \text{for sufficiently large } n, \tag{2.6}$$

where $0<\theta<1$ with $\lim_{n\to\infty}\theta=1$, and $n^{(t)}=n(n-1)\cdots(n-t+1)$, for every nonnegative integer t, and $n^{(0)}=1$.

Theorem 2.4. *Assume that*

$$\liminf_{n\to\infty}\sum_{i=1}^{m}\left(\frac{k_i+1}{k_i}\right)^{k_i+1}\sum_{s=n+1}^{n+k_i}p_i(s)>(l-1)!. \tag{2.7}$$

Then every solution x_n of (1.1) oscillates, or $x_n\to 0$ $(n\to\infty)$.

Proof. Assume, for the sake of contradiction, that $\{x_n\}$ is an eventually positive solution of (1.1), then there exists a positive integer N_1 such that

$$x_n > 0, \quad x_{n-k_i} > 0, \quad i = 1,\dots,m,\ n \geq N_1. \tag{2.8}$$

Thus,

$$\triangle^l x_n = -\sum_{i=1}^{m} p_i(n)x_{n-k_i} \leq 0, \quad n \geq N_1, \tag{2.9}$$

and $\triangle^l x_n \not\equiv 0$.

By Lemma 2.2, $\triangle^i x_n$ are eventually of one sign for every $i \in \{1,\dots,l-1\}$ and $\triangle^{l-1}x_n > 0$ holds for large n, and there exist two cases to consider: (1) $\triangle x_n > 0$ and (2) $\triangle x_n < 0$.

Case 1. This says that x_n is increasing. Setting $k = \max\{k_1,\dots,k_m\}$, by Lemma 2.3, there exists an integer $N_2 \geq \max\{k, N_1\}$ such that

$$x_n \geq \frac{\theta}{(l-1)!} n^{l-1} \triangle^{l-1} x_n, \quad n \geq N_2, \tag{2.10}$$

$$\begin{aligned} x_{n-k_i} &\geq \frac{\theta}{(l-1)!}(n-k_i)^{l-1}\triangle^{l-1}x_{n-k_i} \\ &\geq \frac{\theta}{(l-1)!}(n-k)^{l-1}\triangle^{l-1}x_{n-k_i}, \quad i = 1,\dots,m,\ n \geq N_2, \end{aligned} \tag{2.11}$$

where $0 < \theta < 1$ and $\lim_{n\to\infty}\theta = 1$.

Letting $y_n = \triangle^{l-1}x_n$, we have

$$y_n > 0, \quad y_{n-k_i} > 0, \quad i = 1,\dots,m,\ n \geq N_2, \tag{2.12}$$

which implies that

$$\triangle y_n + \sum_{i=1}^{m} p_i(n)x_{n-k_i} = 0, \quad n \geq N_2. \tag{2.13}$$

By (2.11), we get

$$\begin{aligned} x_{n-k_i} &\geq \frac{\theta}{(l-1)!}(n-k)^{l-1}y_{n-k_i}, \quad i = 1,\dots,m,\ n \geq N_2, \\ &\geq \frac{\theta}{(l-1)!}y_{n-k_i}, \quad i = 1,\dots,m,\ n \geq N_2. \end{aligned} \tag{2.14}$$

It follows that

$$\triangle y_n + \sum_{i=1}^{m} \widetilde{p}_i(n)y_{n-k_i} \leq 0, \quad n \geq N_2, \tag{2.15}$$

where $\widetilde{p}_i(n) = (\theta/(l-1)!)p_i(n)$, which means that inequality (2.15) has an eventually positive solution.

On the other hand, condition (2.7) implies that

$$\begin{aligned}
&\liminf_{n\to\infty}\sum_{i=1}^{m}\left(\frac{k_i+1}{k_i}\right)^{k_i+1}\sum_{s=n+1}^{n+k_i}\widetilde{p}_i(s)\\
&\quad=\liminf_{n\to\infty}\frac{\theta}{(l-1)!}\sum_{i=1}^{m}\left(\frac{k_i+1}{k_i}\right)^{k_i+1}\sum_{s=n+1}^{n+k_i}p_i(s)>1.
\end{aligned}\tag{2.16}$$

By Lemma 2.1, (2.15) has no eventually positive solution. This is a contradiction.
Case 2. Note that by Lemma 2.2, the case that l is even is impossible. In what follows, we only consider the case that l is odd. Case 2 says that x_n is monotone and bounded, and so x_n converges a constant a. By Lemma 2.2, we get

$$(-1)^{i+1}\triangle^{l-i}x_n>0,\quad i=1,\dots,l-1,\ \forall \text{ large } n\geq N_1,\tag{2.17}$$

$$\lim_{n\to\infty}\triangle^{l-1}x_n=0.\tag{2.18}$$

By (2.18), there exists an integer $N_3\geq N_1$ such that

$$0\leq\triangle^{l-1}x_n\leq\varepsilon,\quad \text{for any } \varepsilon>0,\ n\geq N_3.\tag{2.19}$$

It is obvious that $a\geq 0$. If $a=0$, then the problem is solved. We can assume that $a>0$ in the sequel, which implies that there exists an integer $N_4\geq N_3$ such that

$$x_n>\frac{1}{2}a,\quad x_{n-k_i}>\frac{1}{2}a,\quad i=1,2,\dots,m,\ n\geq N_4.\tag{2.20}$$

Thus, (1.1) implies that

$$\triangle^l x_n+\frac{a}{2}\sum_{i=1}^{m}p_i(n)\leq 0,\quad n\geq N_4.\tag{2.21}$$

Summing both sides of (2.21) from N_4 to n, we obtain

$$\triangle^{l-1}x_{n+1}-\triangle^{l-1}x_{N_4}+\frac{a}{2}\sum_{s=N_4}^{n}\sum_{i=1}^{m}p_i(s)\leq 0,\quad n\geq N_4.\tag{2.22}$$

Letting $n\to\infty$, we have

$$\frac{a}{2}\sum_{i=1}^{m}\sum_{s=N_4}^{n}p_i(s)\leq\varepsilon,\quad \text{for large } n.\tag{2.23}$$

On the other hand, condition (2.7) says that there exists an integer $N_5\geq N_4$ such that

$$\sum_{i=1}^{m}\left(\frac{k_i+1}{k_i}\right)^{k_i+1}\sum_{s=n+1}^{n+k_i}p_i(s)>\frac{(l-1)!}{2},\quad n\geq N_5.\tag{2.24}$$

Noting that $((k_i+1)/k_i)^{k_i+1} \le 2e$, we have

$$\frac{a}{2}\sum_{i=1}^{m}\sum_{s=n+1}^{n+k_i} p_i(s) > \frac{a(l-1)!}{8e}, \quad \text{for large } n, \tag{2.25}$$

which contradicts (2.23) and (2.25). The proof is completed. □

Similar to the proof of Theorem 2.4, we have Theorem 2.5.

Theorem 2.5. *Assume that*

$$\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{s=n}^{n+k_i} p_i(s) > (l-1)!. \tag{2.26}$$

Then every solution x_n of (1.1) is oscillatory, or $x_n \to 0$ $(n \to \infty)$.

In fact, in the proof of Theorem 2.4, the condition (2.26) implies that (2.25) always holds and (2.16) is changed into the following inequality:

$$\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{s=n}^{n+k_i} \widetilde{p}_i(s) > 1. \tag{2.27}$$

The rest of proof is the same as the proof of Theorem 2.4.

Theorem 2.6. *Assume that l is even, and the following condition holds:*

$$\liminf_{n\to\infty}\sum_{i=1}^{m}\left(\frac{k_i+1}{k_i}\right)^{k_i+1}\sum_{s=n+1}^{n+k_i} s^{l-1}p_i(s) > (l-1)!. \tag{2.28}$$

Then every bounded solution x_n of (1.1) oscillates.

Proof. Assume, for the sake of contradiction, that x_n is an eventually positive bounded solution of (1.1). According to the proof of Theorem 2.4, there exists a positive integer N_1 such that (2.8) and (2.9) hold. By Lemma 2.2, we have

$$\triangle x_n > 0, \tag{2.29}$$

which implies that x_n is increasing. In view of the proof of Theorem 2.4, there exists an integer $N_2 \ge N_1$ such that

$$x_{n-k_i} \ge \frac{\theta}{(l-1)!}(n-k)^{l-1}y_{n-k_i}, \quad i = 1,\dots,m,\ n > N_2, \tag{2.30}$$

where $k = \max\{k_1,\dots,k_m\}$, $0 < \theta < 1$ with $\lim_{n\to\infty}\theta = 1$. It follows that

$$\triangle y_n + \sum_{i=1}^{m}\widetilde{p}_i(n)y_{n-k_i} \le 0, \quad n \ge N_2, \tag{2.31}$$

where $\widetilde{p}_i(n) = (\theta/(l-1)!)(n-k)^{l-1}p_i(n)$, $y_n = \triangle^{l-1}x_n$, which implies that (2.31) has an eventually positive solution.

On the other hand, condition (2.28) implies that

$$\begin{aligned}&\liminf_{n\to\infty}\sum_{i=1}^{m}\left(\frac{k_i+1}{k_i}\right)^{k_i+1}\sum_{s=n+1}^{n+k_i}\widetilde{p}_i(s)\\&\quad=\liminf_{n\to\infty}\frac{\theta}{(l-1)!}\sum_{i=1}^{m}\left(\frac{k_i+1}{k_i}\right)^{k_i+1}\sum_{s=n+1}^{n+k_i}(s-k)^{l-1}p_i(s)>1.\end{aligned}\tag{2.32}$$

By Lemma 2.1, (2.31) has no eventually positive solution. This contradiction completes the proof. □

Similarly, we have Theorem 2.7.

Theorem 2.7. *Assume that l is even, and the following condition holds:*

$$\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{s=n}^{n+k_i}s^{l-1}p_i(s)>(l-1)!.\tag{2.33}$$

Then every bounded solution x_n of (1.1) oscillates.

Corollary 2.8. *Assume that l is even. If (2.7) or (2.26) holds, then every bounded solution of (1.1) oscillates.*

In fact, (2.7) implies that (2.28) holds and (2.26) implies that (2.33) holds.

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