# OSCILLATION OF A LOGISTIC DIFFERENCE EQUATION WITH SEVERAL DELAYS

L. BEREZANSKY AND E. BRAVERMAN

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For a delay difference equation $N(n+1) - N(n) = N(n)\sum_{k=1}^{m} a_k(n)(1 - N(g_k(n))/K)$, $a_k(n) \geq 0$, $g_k(n) \leq n$, $K > 0$, a connection between oscillation properties of this equation and the corresponding linear equations is established. Explicit nonoscillation and oscillation conditions are presented. Positiveness of solutions is discussed.

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## 1. Introduction

Difference equations provide an important framework for analysis of dynamical phenomena in biology, ecology, economics, and so forth. For example, in population dynamics discrete systems adequately describe organisms for which births occur in regular, usually short, breeding seasons.

Recently the problem of oscillation and nonoscillation of solutions for nonlinear delay difference equations has been intensively studied; see monographs [1, 2, 7–9] and references therein for more details.

In this paper we study the following nonlinear difference equation

$$N(n+1) - N(n) = N(n)\sum_{k=1}^{m} a_k(n)\left(1 - \frac{N(g_k(n))}{K}\right), \quad a_k(n) \geq 0,\ g_k(n) \leq n,\ K > 0, \tag{1.1}$$

where the number $g_k(n)$ is an integer (positive or negative) for every $n$ and $k$. Equation (1.1) describes populations that die out completely at each generation and have birth rates that saturate for large population sizes $N = K$. Equation (1.1) is a discrete analogue

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of the well-known logistic differential equation with several delays

$$N'(t) = N(t)\sum_{k=1}^{m} a_k(t)\left(1 - \frac{N(g_k(t))}{K}\right). \tag{1.2}$$

Oscillation properties of (1.2) were considered in [3, 4, 12].

In [9] oscillation properties of another discrete analogue of autonomous equation (1.2)

$$N(n+1) = \frac{bN(n)}{1+\sum_{k=1}^{m} a_k N(n-\sigma_k)} \tag{1.3}$$

were obtained.

In [14] the oscillation properties of the following equation were considered

$$N(n+1) = N(n)\exp\left\{\sum_{k=1}^{m} a_k\left(1 - \frac{N(n-\sigma_k)}{K}\right)\right\}. \tag{1.4}$$

This equation can be treated as another discrete analogue of autonomous equation (1.2).

Note that in the nondelay case ($g_k(n) = n$, $\sigma_k = 0$) all solutions of (1.1), (1.3) and (1.4) are monotone, similar to the nondelay logistic equations (see, e.g., [6, 10]). However, unlike (1.2), solutions of (1.1) can become negative.

Oscillation of (1.1) with a single delay ($m = 1$) was investigated in [13], however conditions for the positiveness of solutions were not discussed. To the best of our knowledge there are no oscillation results for (1.1).

The paper is organized as follows. Section 2 contains some preliminaries and auxiliary results. In Section 3 we reduce oscillation (nonoscillation) of a nonlinear equation which is obtained from (1.1) by the substitution $x(n) = N(n)/K - 1$ to the oscillation (nonoscillation) problem for some linear equation. After applying these results and the developed oscillation theory for linear equations, in Section 4 sufficient conditions for oscillation (nonoscillation) of solutions of (1.1) about equilibrium $K$ are presented. These conditions are sharp for constant parameters and the only delay. The results on the existence of nonoscillatory solutions provide that there exists a positive solution of (1.1). However oscillation conditions do not distinguish between eventually oscillatory solutions and eventually negative solutions (the population extincts at a certain step). Section 5 contains some discussion on the existence of positive solutions and relevant numerical simulations. As expected, if there is no global attractivity but the solution is positive, then we get asymptotically periodic oscillating solutions. It is to be noted that in the nondelay case ($\sigma_k = 0$) with a variable periodic equilibrium ($K = K(n)$) the existence of periodic solutions for (1.4) was studied in [17].

## 2. Preliminaries

In addition to (1.1) we consider the following scalar difference equation

$$x(n+1) - x(n) = -\sum_{k=1}^{m} a_k(n)(1+x(n))x(g_k(n)), \tag{2.1}$$

with initial conditions

$$x(n) = \varphi(n), \quad n \le 0. \tag{2.2}$$

We assume that the following condition is satisfied

(a1) $a_k(n) \ge 0$, $g_k(n) \le n$, $\lim_{n\to\infty} g_k(n) = \infty$.

Equation (2.1) is obtained if we substitute in (1.1) $N(n) = K[x(n)+1]$.

Consider also a linear difference equation

$$y(n+1) - y(n) = -\sum_{k=1}^{l} b_k(n) y(h_k(n)), \tag{2.3}$$

and the corresponding inequalities:

$$y(n+1) - y(n) \le -\sum_{k=1}^{l} b_k(n) y(h_k(n)), \tag{2.4}$$

$$y(n+1) - y(n) \ge -\sum_{k=1}^{l} b_k(n) y(h_k(n)), \tag{2.5}$$

where for parameters of (2.3) conditions (a1) hold.

*Definition 2.1.* The solution $x(n)$ or $y(n)$ of (2.1) or (2.3), respectively, is called *nonoscillatory* (about zero) if it is eventually positive or eventually negative.

If such solution does not exist we say that all solutions of this equation are *oscillatory* (about zero).

Lemma 2.2 [15]. *Equation (2.3) has a nonoscillatory solution if and only if inequality (2.4) has an eventually positive solution and inequality (2.5) has an eventually negative solution.*

*Suppose $c_k(n) \le b_k(n)$ and (2.3) has a nonoscillatory solution. Then the equation*

$$y(n+1) - y(n) = -\sum_{k=1}^{l} c_k(n) y(h_k(n)) \tag{2.6}$$

*also has a nonoscillatory solution.*

Lemma 2.3 [16]. (1) *Suppose*

$$\liminf_{n\to\infty} \sum_{k=1}^{l} b_k(n) > 0, \qquad \liminf_{n\to\infty} \sum_{k=1}^{l} b_k(n) \frac{\left(n-h_k(n)+1\right)^{n-h_k(n)+1}}{\left(n-h_k(n)\right)^{n-h_k(n)}} > 1. \tag{2.7}$$

*Then all solutions of (2.3) are oscillatory.*

(2) *Suppose there exists $\lambda \in (0,1)$, such that*

$$\limsup_{n\to\infty} \sum_{k=1}^{l} b_k(n) \left[\lambda(1-\lambda)^{n-h_k(n)}\right]^{-1} < 1. \tag{2.8}$$

*Then (2.3) has a nonoscillatory solution.*

## 3. Oscillation and nonoscillation conditions

Lemma 3.1. *Suppose*

$$\sum_{n=1}^{\infty}\sum_{k=1}^{m} a_k(n) = \infty. \tag{3.1}$$

*If* $x(n)$ *is a nonoscillatory solution of (2.1), such that* $1+x(n)>0$, *then* $\lim_{n\to\infty} x(n) = 0$.

*Proof.* Without loss of generality we can assume that $x(n)>0$, $n>0$, $\varphi(n)\geq 0$.

Equality (2.1) implies that $0<x(n+1)\leq x(n)$. Then there exists a nonnegative limit $l=\lim_{n\to\infty} x(n)$. Suppose $l>0$. Equality (2.1) also implies

$$x(n+1)-x(0) = -\sum_{i=1}^{n}\sum_{k=1}^{m} a_k(i)\big(1+x(i)\big)x\big(g_k(i)\big). \tag{3.2}$$

The left-hand side of (3.2) tends to $l-x(0)$. Equality (3.1) yields that the right-hand side of (3.2) tends to $-\infty$, which is a contradiction. Then $l=0$. The lemma is proven. □

Theorem 3.2. *Suppose (3.1) holds and for some* $\epsilon>0$ *all solutions of the following linear equation*

$$y(n+1)-y(n) = -\sum_{k=1}^{m} a_k(n)(1-\epsilon)y\big(g_k(n)\big) \tag{3.3}$$

*are oscillatory. Then all solutions of (2.1) satisfying* $x(n)>-1$ *are oscillatory.*

*Proof.* Suppose $x(n)$ is an eventually positive solution of (2.1). Without loss of generality we can assume $x(n)>0$, $n\geq 0$. From equality (2.1) we have

$$x(n+1)-x(n) \leq -\sum_{k=1}^{m} a_k(n)x\big(g_k(n)\big). \tag{3.4}$$

It means that inequality (3.4) has an eventually positive solution. Lemma 2.2 implies that (3.3) has a nonoscillatory solution, which contradicts the hypothesis of the theorem.

Suppose now $x(n)$ is an eventually negative solution of (2.1). Without loss of generality we can assume $x(n)<0$, $n\geq 0$. Lemma 3.1 implies that for some $N>0$, $-\epsilon<x(n)<0$, $n\geq N$. Hence from (2.1) we have

$$x(n+1)-x(n) \geq -\sum_{k=1}^{m} a_k(n)(1-\epsilon)x\big(g_k(n)\big), \tag{3.5}$$

for $n\geq N$. Then difference inequality (3.5) has an eventually negative solution. Lemma 2.2 implies that difference equation (3.3) has a nonoscillatory solution. This contradiction proves the theorem. □

Corollary 3.3. *Suppose (3.1) holds and*

$$\liminf_{n\to\infty}\sum_{k=1}^{m} a_k(n) > 0, \quad \liminf_{n\to\infty}\sum_{k=1}^{m} a_k(n)\frac{(n-g_k(n)+1)^{n-g_k(n)+1}}{(n-g_k(n))^{n-g_k(n)}} > 1. \tag{3.6}$$

*Then all solutions of (2.1) satisfying $x(n) > -1$ are oscillatory.*

*Proof.* Inequality (3.6) implies that for some $\epsilon > 0$ we have

$$\liminf_{n\to\infty}\sum_{k=1}^{m} a_k(n)(1-\epsilon)\frac{(n-g_k(n)+1)^{n-g_k(n)+1}}{(n-g_k(n))^{n-g_k(n)}} > 1. \tag{3.7}$$

By Lemma 2.3 all solutions of (3.3) are oscillatory. The reference to Theorem 3.2 completes the proof. □

Theorem 3.4. *Suppose for some $\epsilon > 0$ the following linear equation*

$$y(n+1) - y(n) = -\sum_{k=1}^{m} a_k(n)(1+\epsilon)y(g_k(n)) \tag{3.8}$$

*has a nonoscillatory solution. Then (2.1) also has a nonoscillatory solution.*

*Proof.* Suppose $y(n)$ is an eventually positive solution of (3.8). Without loss of generality we can assume $y(n) > 0$, $n \ge 0$. Denote

$$u_0(n) = \frac{y(n) - y(n+1)}{y(n)}, \quad n \ge 0,\ u_0(n) = 0,\ n < 0. \tag{3.9}$$

Then $0 \le u_0(n) < 1$ and

$$y(n) = y(0)\prod_{k=0}^{n-1}(1-u_0(k)), \quad n > 0. \tag{3.10}$$

After substitution (3.10) into (3.8) we get an equality which justifies the following inequality

$$u_0(n) \ge \sum_{k=1}^{m} a_k(n)(1+\epsilon)\prod_{i=g_k(n)}^{n-1}(1-u_0(i))^{-1}. \tag{3.11}$$

Consider now for every $n$ two sequences $\{u_l(n)\}$ and $\{v_l(n)\}$, $l = 0,1,2,\dots,$

$$u_{l+1}(n) = \sum_{k=1}^{m} a_k(n)\left[1+\epsilon\prod_{i=0}^{n-1}(1-v_l(i))\right]\prod_{i=g_k(n)}^{n-1}(1-u_l(i))^{-1}, \tag{3.12}$$

$$v_{l+1}(n) = \sum_{k=1}^{m} a_k(n)\left[1+\epsilon\prod_{i=0}^{n-1}(1-u_l(i))\right]\prod_{i=g_k(n)}^{n-1}(1-v_l(i))^{-1}, \tag{3.13}$$

where $u_0(n)$ is denoted by (3.9) and $v_0(n) \equiv 0$, $u_l(n) = v_l(n) = 0$, $n < 0$.

Condition (3.11) implies

$$u_1(n)=\sum_{k=1}^{m}a_k(n)(1+\epsilon)\prod_{i=g_k(n)}^{n-1}(1-u_0(i))^{-1}\le u_0(n). \tag{3.14}$$

We have

$$v_1(n)=\sum_{k=1}^{m}a_k(n)\left(1+\epsilon\prod_{i=0}^{n-1}(1-u_0(i))\right). \tag{3.15}$$

Consequently $0=v_0(n)\le v_1(n)\le\sum_{k=1}^{m}a_k(n)(1+\epsilon)\le u_1(n)\le u_0(n)<1$.
Then by induction

$$0\le v_l(n)\le v_{l+1}(n)\le u_{l+1}(n)\le u_l(n)<1. \tag{3.16}$$

Hence there exist sequences

$$u(n)=\lim_{l\to\infty}u_l(n),\qquad v(n)=\lim_{l\to\infty}v_l(n), \tag{3.17}$$

which implies

$$0\le v_l(n)\le u_l(n)\le u_0(n)<1. \tag{3.18}$$

Hence $0\le v(n)\le u(n)\le u_0(n)<1$, $u(n)=v(n)=0$, $n<0$.
Equalities (3.12)-(3.13) imply

$$u(n)=\sum_{k=1}^{m}a_k(n)\left[1+\epsilon\prod_{i=0}^{n-1}(1-v(i))\right]\prod_{i=g_k(n)}^{n-1}(1-u(i))^{-1}, \tag{3.19}$$

$$v(n)=\sum_{k=1}^{m}a_k(n)\left[1+\epsilon\prod_{i=0}^{n-1}(1-u(i))\right]\prod_{i=g_k(n)}^{n-1}(1-v(i))^{-1}. \tag{3.20}$$

Consider now a nonlinear operator

$$(Tw)(n)=\sum_{k=1}^{m}a_k(n)\left[1+\epsilon\prod_{i=0}^{n-1}(1-w(i))\right]\prod_{i=g_k(n)}^{n}(1-w(i))^{-1},\quad 0\le n\le N,\ w(n)=0,\ n<0 \tag{3.21}$$

in the finite dimensional space $l^\infty(N)$ with the norm

$$\|w\|_{l^\infty(N)}=\max_{0\le n\le N}|w(n)|. \tag{3.22}$$

This operator is compact and for every $w(n)$, such that $0\le v(n)\le w(n)\le u(n)$, we have $v(n)\le(Tw)(n)\le u(n)$. Hence there exists a nonnegative solution $w_0(n)$, $0\le n\le N$, of the equation $w = Tw$. Then

$$w_0(n) = \sum_{k=1}^{m} a_k(n)\left[1+\epsilon\prod_{i=0}^{n-1}(1-w_0(i))\right]\prod_{i=g_k(n)}^{n-1}(1-w_0(i))^{-1}, \quad 0 \le n \le N,\ w_0(n) = 0,\ n \le 0. \tag{3.23}$$

Therefore the function

$$x(n) = \prod_{i=0}^{n-1}(1-w_0(i)), \quad 0 \le n \le N,\ x(n) = 0,\ n < 0,\ x(0) = 1, \tag{3.24}$$

is a positive solution of (2.1) for $0 \le n \le N$. Since $N$ is an arbitrary integer, then this completes the proof. □

Corollary 3.5. *Suppose there exists* $\lambda \in (0,1)$, *such that*

$$\limsup_{n\to\infty}\sum_{k=1}^{m} a_k(n)\left[\lambda(1-\lambda)^{n-g_k(n)}\right]^{-1} < 1. \tag{3.25}$$

*Then (2.1) has a nonoscillatory solution.*

Proof is based on Lemma 2.3 and Theorem 3.4.

## 4. Main oscillation results

Consider now logistic difference equation (1.1), where $K > 0$ and for the functions $a_k(n)$, $g_k(n)$ conditions (a1) hold.

Motivated by applications, in this section we consider only solutions $N(n)$ of (1.1) for which $N(n) > 0$, $n \ge 0$.

We study the oscillation of the solutions of (1.1) about the equilibrium point $K$.

*Definition 4.1.* The solution $N(n)$ of (1.1) is called *nonoscillatory about* $K$ if $N(n) - K$ is eventually positive or eventually negative.

If such solution does not exist we say that all solutions of this equation are *oscillatory about* $K$.

Suppose $N(n)$ is a positive solution of (1.1) and define $x(n) = (N(n)/K) - 1$. Then $x(n)$ is a solution of (2.1) such that $1 + x(n) > 0$. Hence, oscillation (or nonoscillation) of $N(n)$ about $K$ is equivalent to oscillation (or nonoscillation) of $x(n)$ about zero.

By applying Theorems 3.2, 3.4 and Corollaries 3.3, 3.5 we obtain the following results for (1.1).

Theorem 4.2. *Suppose (3.1) holds. If* $N(n)$ *is a nonoscillatory about* $K$ *positive solution of (1.1) then* $\lim_{n\to\infty} N(n) = K$.

Theorem 4.3. *Suppose (3.1) holds and for some* $\epsilon > 0$ *all solutions of linear equation (3.3) are oscillatory. Then all positive solutions of (1.1) are oscillatory about* $K$.

Corollary 4.4. *Suppose (3.1) and (3.6) hold. Then all positive solutions of (1.1) are oscillatory about K.*

Theorem 4.5. *Suppose for some* $\epsilon > 0$ *linear equation (3.8) has a nonoscillatory solution. Then (1.1) also has a positive nonoscillatory about K solution.*

Corollary 4.6. *Suppose there exists* $\lambda \in (0,1)$ *such that (3.25) holds. Then (1.1) has a positive nonoscillatory about K solution.*

## 5. Existence of positive solutions

As it is known [3], for positive initial conditions the solution of delay logistic differential equation (1.2) is positive. The delay logistic difference equations (1.3)-(1.4) enjoy the same property. However for difference equations (1.1) this is not true.

*Example 5.1.* Consider the following equation

$$N(n+1) - N(n) = N(n)\big(1 - N(n-1)\big). \tag{5.1}$$

If $N(-1) = 3$, $N(0) = 1$, then $N(n) < 0$, $n > 0$.

Thus it is interesting to find such constraints on initial conditions and parameters of the equation for which the solution of (1.1) will be positive.

Everywhere above we considered only positive solutions of (1.1). In this section we discuss sufficient conditions for positiveness of solutions and present some results of numerical simulations. To this end let us consider for any number $b$ an auxiliary linear equation

$$y(n+1) - y(n) = -\sum_{k=1}^{m} a_k(n)(1+b)y\big(g_k(n)\big) \tag{5.2}$$

with the initial conditions

$$y(n) = \varphi(n), \quad n \le 0. \tag{5.3}$$

Theorem 5.2. *Suppose (a1) holds, there exists a constant A,* $0 < A < 1$, *such that as far as for the initial condition (5.3) inequality* $|\varphi(n)| < A$ *holds and* $|b| < A$, *then a solution of the linear equation (5.2) satisfies*

$$|\, y(n) \,| < A. \tag{5.4}$$

*Then all solutions of (1.1), with initial conditions*

$$|\, N(n) - K \,| < AK, \quad n \le 0, \tag{5.5}$$

*are positive for any* $n > 0$. *Moreover, the solution of (1.1) satisfies (5.5) for any n.*

*Proof.* After the transformation $x(n) = N(n)/K - 1$ (1.1) turns into (2.1), and the solution of (1.1) is positive if and only if in (2.1) $x(n) > -1$ for any $n$. Under the conditions of the theorem if initial values $x(n)$ $(n \le 0)$ belong to the interval $(-A, A)$ then $-A < x(n) < A$ for any $n$. Since $A < 1$, then $x(n) > -1$, therefore $N(n)$ is positive. $\square$

Corollary 5.3. *Suppose (a1) is satisfied and for some $A$, $0 < A < 1$,*

$$\lambda = (1+A)\sup_n \sum_{k=1}^{m} a_k(n)(n-g_k(n)) < 1. \tag{5.6}$$

*Then any solution of (1.1) satisfying initial condition (5.5) is positive.*

*Proof.* Suppose that for (5.2) with $|b| < A$ for initial condition (5.3) we have $|\varphi(n)| \le A$. [5, Theorem 2.2] and condition (5.6) imply

$$|y(n)| \le \max_{n\le 0} |\varphi(n)| \le A \tag{5.7}$$

for the solution of (5.2). Hence all conditions of Theorem 5.2 are satisfied. Therefore the solution of (1.1) is positive. □

Finally, let us consider the high order difference equation with a constant delay

$$N(n+1) - N(n) = aN(n)[1 - N(n-h)], \tag{5.8}$$

where $h$ is a positive integer. In accordance with Corollary 3.5 and previous results (5.8) has a nonoscillatory about $K = 1$ solution if

$$a < \frac{h^h}{(h+1)^{h+1}}. \tag{5.9}$$

The condition of asymptotic stability of the linear equation

$$y(n+1) - y(n) = -ay(n-h) \tag{5.10}$$

was obtained in [11]: if

$$0 < a < 2\cos\frac{h\pi}{2h+1}, \tag{5.11}$$

then (5.10) is asymptotically stable.

When reviewing [13] Ladas made the following conjecture (see, e.g., MathSciNet for the review of [13]). Under the same condition (5.11) (5.8) will have positive solutions for $|N(n) - 1| < \varepsilon$, $n \le 0$, where $\varepsilon$ is small enough. However this condition is far from being necessary.

It is to be noted that in numerical simulations we could observe that under condition (5.11) solutions are positive for any "reasonable" initial conditions (by reasonable initial conditions we mean initial conditions for which $N(n) > 0$, $-h \le n \le h$, i.e., there is no immediate extinction at the initial segment with the length of delay $h$). There are also values of parameter $a$ for which (5.10) is not asymptotically stable, however the solution of (1.1) does not extinct. In Figure 5.1 we also demonstrate the numerical bounds which where found for the existence of positive solutions (for "reasonable" initial conditions). Above the curve "positive solutions" in Figure 5.1, for arbitrary small initial conditions (not all zeros) the solution eventually becomes less than zero. The numerically found

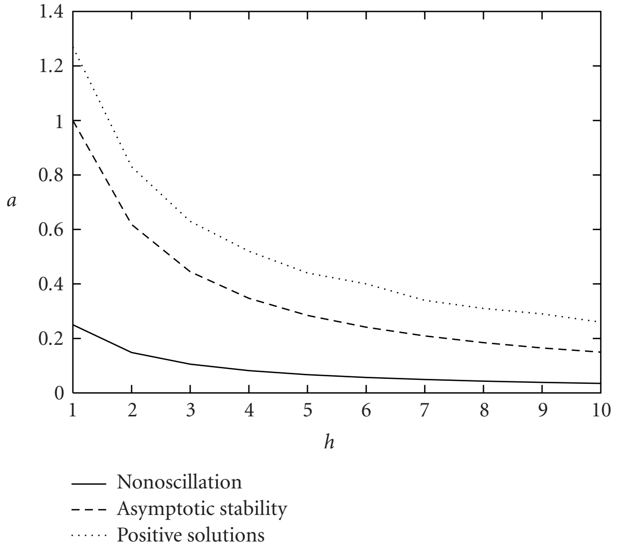


Figure 5.1. Bounds for oscillation, asymptotic stability and existence of positive solutions for (5.8). The first two estimates are found by formulas (5.9), (5.11), while the latter curve is established numerically.

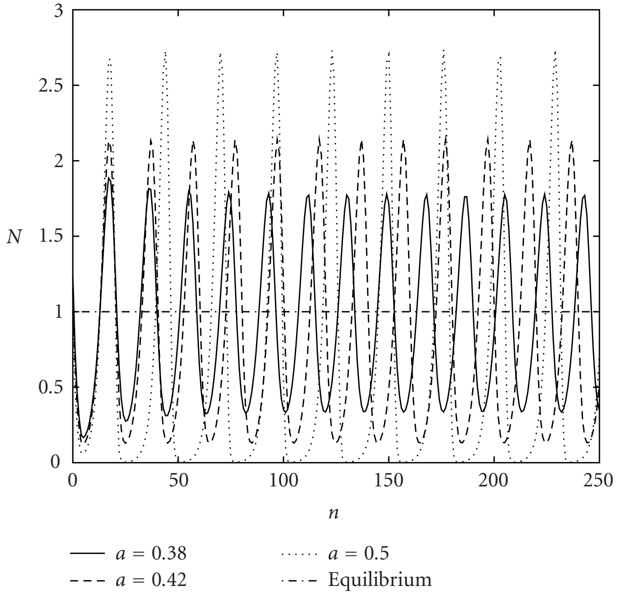


Figure 5.2. The solutions of (5.8) for the initial conditions with the delay $h = 4$ and $a = 0.38$, $a = 0.42$, $a = 0.5$, respectively. The solution is not asymptotically stable. Solutions are asymptotically periodic, with the amplitude growing with the growth of $a$. Here $N(n) = 2$, $n \leq 0$.

constraints are less restrictive compared to oscillation bounds and asymptotic stability conditions.

It is expected that in the range of parameter $a$ between extinction and asymptotic stability we get asymptotically periodic solutions. Figure 5.2 illustrates this fact.

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L. Berezansky: Department of Mathematics, Ben-Gurion University of the Negev,
Beer-Sheva 84105, Israel
*E-mail address*: brznsky@cs.bgu.ac.il

E. Braverman: Department of Mathematics and Statistics, University of Calgary,
2500 University Drive N.W., Calgary, Alberta T2N 1N4, Canada
*E-mail address*: maelena@math.ucalgary.ca