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Research Article

Existence of Periodic and Subharmonic Solutions for Second-Order p-Laplacian Difference Equations

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We obtain a sufficient condition for the existence of periodic and subharmonic solutions of second-order p-Laplacian difference equations using the critical point theory.

1. Introduction

In this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ the set of all natural numbers, integers, and real numbers, respectively. For $a,b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \dots\}$, $\mathbb{Z}(a,b) = \{a, a+1, \dots, b\}$ when $a \le b$.

Consider the nonlinear second-order difference equation

$$\Delta(\varphi_p(\Delta x_{n-1})) + f(n, x_{n+1}, x_n, x_{n-1}) = 0, \quad n \in \mathbb{Z}, \tag{1.1}$$

where Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$, $\Delta^2 x_n = \Delta(\Delta x_n)$, $\varphi_p(s)$ is p-Laplacian operator $\varphi_p(s) = |s|^{p-2}s$ $(1 < p < \infty)$, and $f : \mathbb{Z} \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous functional in the second, the third, and fourth variables and satisfies $f(t+m, u, v, w) = f(t, u, v, w)$ for a given positive integer m.

We may think of (1.1) as being a discrete analogue of the second-order functional differential equation

$$[\varphi_p(x')]' + f(t, x(t+1), x(t), x(t-1)) = 0, \quad t \in \mathbb{R} \tag{1.2}$$

which includes the following equation:

$$c^2 y''(x) = v'[y(x+1) - y(x)] - v'[y(x) - y(x-1)]. \tag{1.3}$$

Equations similar in structure to (1.3) arise in the study of the existence of solitary waves of lattice differential equations, see [1] and the references cited therein.

Some special cases of (1.1) have been studied by many researchers via variational methods, see [2–7]. However, to our best knowledge, no similar results are obtained in the literature for (1.1). Since f in (1.1) depends on x_{n+1} and x_{n-1}, the traditional ways of establishing the functional in [2–7] are inapplicable to our case. The main purpose of this paper is to give some sufficient conditions for the existence of periodic and subharmonic solutions of (1.1) using the critical point theory.

2. Some basic lemmas

To apply critical point theory to study the existence of periodic solutions of (1.1), we will state some basic notations and lemmas (see [5, 8]), which will be used in the proofs of our main results.

Let S be the set of sequences, $x = (\dots, x_{-n}, \dots, x_{-1}, x_0, x_1, \dots, x_n, \dots) = \{x_n\}_{-\infty}^{+\infty}$, that is, $S = \{x = \{x_n\} : x_n \in \mathbb{R},\ n \in \mathbb{Z}\}$. For a given positive integer q and m, E_{qm} is defined as a subspace of S by

$$E_{qm} = \{x = \{x_n\} \in S \mid x_{n+qm} = x_n,\ n \in \mathbb{Z}\}. \tag{2.1}$$

For any $x, y \in S$, $a, b \in \mathbb{R}$, $ax + by$ is defined by

$$ax + by = \{ax_n + by_n\}_{n=-\infty}^{+\infty}. \tag{2.2}$$

Then S is a vector space. Clearly, E_{qm} is isomorphic to $\mathbb{R}^{qm}$, E_{qm} can be equipped with inner product

$$\langle x, y\rangle_{E_{qm}} = \sum_{j=1}^{qm} x_j y_j, \quad \forall x, y \in E_{qm}, \tag{2.3}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|x\| = \left(\sum_{j=1}^{qm} x_j^2\right)^{1/2}, \quad \forall x \in E_{qm}. \tag{2.4}$$

It is obvious that E_{qm} with the inner product in (2.3) is a finite dimensional Hilbert space and linearly homeomorphic to $\mathbb{R}^{qm}$.

On the other hand, we define the norm $\|\cdot\|_p$ on E_{qm} as follows:

$$\|x\|_p = \left(\sum_{i=1}^{qm} |x_i|^p\right)^{1/p}, \tag{2.5}$$

for all $x \in E_{qm}$ and $p > 1$. Clearly, $\|x\| = \|x\|_2$. Since $\|\cdot\|_p$ and $\|\cdot\|_2$ are equivalent, there exist constants C_1, C_2, such that $C_2 \ge C_1 > 0$, and

$$C_1\|x\|_p \le \|x\|_2 \le C_2\|x\|_p, \quad \forall x \in E_{qm}. \tag{2.6}$$

Define the functional J on E_{qm} as follows:

$$J(x) = \sum_{n=1}^{qm} \left[\frac{1}{p} \left| \Delta x_n \right|^p - F(n, x_{n+1}, x_n) \right], \tag{2.7}$$

where

$$\begin{gathered} f(t,u,v,w) = F_2'(t-1,v,w) + F_3'(t,u,v), \\ F_2'(t-1,v,w) = \frac{\partial F(t-1,v,w)}{\partial v}, \qquad F_3'(t,u,v) = \frac{\partial F(t,u,v)}{\partial v}, \end{gathered} \tag{2.8}$$

then

$$f(n, x_{n+1}, x_n, x_{n-1}) = F_3'(n, x_{n+1}, x_n) + F_2'(n-1, x_n, x_{n-1}). \tag{2.9}$$

Clearly, $J \in C^1(E_{qm}, \mathbb{R})$ and for any $x = \{x_n\}_{n\in\mathbb{Z}} \in E_{qm}$, by using $x_0 = x_{qm}$, $x_1 = x_{qm+1}$, we can compute the partial derivative as

$$\frac{\partial J}{\partial x_n} = -\left(\Delta\left(\varphi_p(\Delta x_{n-1})\right) + f(n, x_{n+1}, x_n, x_{n-1})\right), \quad n \in \mathbb{Z}(1, qm). \tag{2.10}$$

By the periodicity of $\{x_n\}$ and $f(t,u,v,w)$ in the first variable t, we reduce the existence of periodic solutions of (1.1) to the existence of critical points of J on E_{qm}. That is, the functional J is just the variational framework of (1.1).

For convenience, we identify $x \in E_{qm}$ with $x = (x_1, x_2, \ldots, x_{qm})^T$.

Let X be a real Hilbert space, $I \in C^1(X, \mathbb{R})$, which means that I is a continuously *Fréchet* differentiable functional defined on X. I is said to satisfy Palais-Smale condition (P-S condition for short) if any sequence $\{u_n\} \subset X$ for which $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$, as $n \to \infty$, possesses a convergent subsequence in X.

Let B_ρ be the open ball in X with radius ρ and centered at 0 and let ∂B_ρ denote its boundary.

Lemma 2.1 (linking theorem) [8, Theorem 5.3]. *Let X be a real Hilbert space, $X = X_1 \oplus X_2$, where X_1 is a finite-dimensional subspace of X. Assume that $I \in C^1(X, \mathbb{R})$ satisfies the P-S condition and*

(A_1) *there exist constants $\sigma > 0$ and $\rho > 0$, such that $I|_{\partial B_\rho \cap X_2} \geq \sigma$;*

(A_2) *there is an $e \in \partial B_1 \cap X_2$ and a constant $R_1 > \rho$, such that $I|_{\partial Q} \leq 0$, where $Q = (\overline{B}_{R_1} \cap X_1) \oplus \{re \mid 0 < r < R_1\}$.*

Then, I possesses a critical value $c \geq \sigma$, where

$$c = \inf_{h\in\Gamma} \max_{u\in Q} I(h(u)), \qquad \Gamma = \{h \in C(\overline{Q}, X) \mid h|_{\partial Q} = \mathrm{id}\} \tag{2.11}$$

and id *denotes the identity operator.*

3. Main results

Theorem 3.1. *Assume that the following conditions are satisfied:*

(H_1) $f(t,u,v,w) \in C(\mathbb{R}^4,\mathbb{R})$ *and there exists a positive integer* m*, such that for every* $(t,u,v,w) \in \mathbb{R}^4$, $f(t+m,u,v,w) = f(t,u,v,w)$*;*

(H_2) *there exists a functional* $F(t,u,v) \in C^1(\mathbb{R}^3,\mathbb{R})$ *with* $F(t,u,v) \geq 0$ *and it satisfies*

$$F_2'(t-1,v,w) + F_3'(t,u,v) = f(t,u,v,w),$$
$$\lim_{\rho\to 0}\frac{F(t,u,v)}{\rho^p} = 0, \quad \rho = \sqrt{u^2+v^2}; \tag{3.1}$$

(H_3) *there exist constants* $\beta \geq p+1$, $a_1 > 0$, $a_2 > 0$*, such that*

$$F(t,u,v) \geq a_1\left(\sqrt{u^2+v^2}\right)^{\beta} - a_2, \quad \forall (t,u,v) \in \mathbb{R}^3. \tag{3.2}$$

Then, for a given positive integer q*, (1.1) has at least two nontrivial* qm*-periodic solutions.*

First, we prove two lemmas which are useful in the proof of Theorem 3.1.

Lemma 3.2. *Assume that* $f(t,u,v,w)$ *satisfies condition* (H_3) *of Theorem 3.1, then the functional* $J(x) = \sum_{n=1}^{qm}[1/p|\Delta x_n|^p - F(n,x_{n+1},x_n)]$ *is bounded from above on* E_{qm}*.*

Proof. By (H_3), there exist $a_1 > 0$, $a_2 > 0$, $\beta > p$, such that for all $x \in E_{qm}$,

$$\begin{aligned}
J(x) &= \sum_{n=1}^{qm}\left[\frac{1}{p}|\Delta x_n|^p - F(n,x_{n+1},x_n)\right] \leq \sum_{n=1}^{qm}\left[\frac{2^p}{p}\max\{|x_{n+1}|^p,|x_n|^p\} - F(n,x_{n+1},x_n)\right] \\
&\leq \frac{2^p}{p}\sum_{n=1}^{qm}[|x_{n+1}|^p + |x_n|^p] - a_1\sum_{n=1}^{qm}\left(\sqrt{x_{n+1}^2+x_n^2}\right)^{\beta} + a_2qm \\
&\leq \frac{2^{p+1}}{p}\sum_{n=1}^{qm}|x_n|^p - a_1\sum_{n=1}^{qm}|x_n|^{\beta} + a_2qm = \frac{2^{p+1}}{p}\|x\|_p^p - a_1\|x\|_{\beta}^{\beta} + a_2qm.
\end{aligned} \tag{3.3}$$

In view of (2.6), there exist constants C_1, C_3, such that

$$\|x\|_p \leq \frac{1}{C_1}\|x\|, \qquad \|x\|_{\beta} \geq \frac{1}{C_3}\|x\|. \tag{3.4}$$

So

$$J(x) \leq \frac{2^{p+1}}{p(C_1)^p}\|x\|^p - \frac{a_1}{(C_3)^{\beta}}\|x\|^{\beta} + a_2qm. \tag{3.5}$$

By $\beta > p$ and the above inequality, there exists a constant $M > 0$, such that for every $x \in E_{qm}$, $J(x) \leq M$. The proof is complete. □

Lemma 3.3. *Assume that* $f(t,u,v,w)$ *satisfies condition* (H_3) *of Theorem 3.1, then the functional* J *satisfies P-S condition.*

Proof. Let $x^{(k)} \in E_{qm}$, for all $k \in N$, be such that $\{J(x^{(k)})\}$ is bounded. Then there exists $M_1 > 0$, such that

$$-M_1 \leq J(x^{(k)}) \leq \frac{2^{p+1}}{pC_1^p}||x^{(k)}||^p - \frac{a_1}{C_3{}^\beta}||x^{(k)}||^\beta + a_2 qm, \tag{3.6}$$

that is,

$$\frac{a_1}{C_3{}^\beta}||x^{(k)}||^\beta - \frac{2^{p+1}}{pC_1^p}||x^{(k)}||^p \leq M_1 + a_2 qm. \tag{3.7}$$

By $\beta > p$, there exists $M_2 > 0$ such that for every $k \in N$, $\|x^{(k)}\| \leq M_2$.

Thus, $\{x^{(k)}\}$ is bounded on E_{qm}. Since E_{qm} is finite dimensional, there exists a subsequence of $\{x^{(k)}\}$, which is convergent in E_{qm} and the P-S condition is verified. □

Proof of Theorem 3.1. The proof of Lemma 3.2 implies $\lim_{\|x\|\to\infty} J(x) = -\infty$, then $-J$ is coercive. Let $c_0 = \sup_{x\in E_{qm}} J(x)$. By continuity of J on E_{qm}, there exists $\overline{x} \in E_{qm}$, such that $J(\overline{x}) = c_0$, and $\overline{x}$ is a critical point of J. We claim that $c_0 > 0$. In fact, we have

$$\begin{aligned} J(x) &= \frac{1}{p}\left(\left[\sum_{n=1}^{qm} |\Delta x_n|^p\right]^{1/p}\right)^p - \sum_{n=1}^{qm} F(n, x_{n+1}, x_n) \\ &\geq \frac{1}{p}\left(\frac{1}{C_2}\right)^p \left(\left[\sum_{n=1}^{qm} |\Delta x_n|^2\right]^{1/2}\right)^p - \sum_{n=1}^{qm} F(n, x_{n+1}, x_n) \\ &= \frac{1}{p}\left(\frac{1}{C_2}\right)^p \left[\sum_{n=1}^{qm} 2(x_n^2 - x_n x_{n+1})\right]^{p/2} - \sum_{n=1}^{qm} F(n, x_{n+1}, x_n) \\ &= \frac{1}{p}\left(\frac{1}{C_2}\right)^p (x^T A x)^{p/2} - \sum_{n=1}^{qm} F(n, x_{n+1}, x_n), \end{aligned} \tag{3.8}$$

where $x = (x_1, x_2, \ldots, x_{qm})^T$,

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{qm\times qm}. \tag{3.9}$$

Clearly, 0 is an eigenvalue of A and $\xi = (v, v, \ldots, v)^T \in E_{qm}$ is an eigenvector of A corresponding to 0, where $v \neq 0$, $v \in \mathbb{R}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{qm-1}$ be the other eigenvalues of A. By matrix theory, we have $\lambda_j > 0$, for all $j \in \mathbb{Z}(1, qm-1)$.

Denote $Z = \{(v, v, \ldots, v)^T \in E_{qm} \mid v \in \mathbb{R}\}$ and $Y = Z^\perp$, such that $E_{qm} = Y \oplus Z$.

Set

$$\lambda_{\min} = \min_{j\in\mathbb{Z}(1,qm-1)} \lambda_j > 0, \qquad \lambda_{\max} = \max_{j\in\mathbb{Z}(1,qm-1)} \lambda_j > 0. \tag{3.10}$$

By condition (H_2), we have

$$\lim_{\rho\to 0}\frac{F(t,u,v)}{\rho^p}=0,\quad \rho=\sqrt{u^2+v^2}. \tag{3.11}$$

Choose $\varepsilon = 2^{-p/2-2}(1/p)\lambda_{\min}^{p/2}(C_1/C_2)^p$, there exists $\delta>0$, such that

$$|F(t,u,v)|\le 2^{-p/2-2}\frac{1}{p}\lambda_{\min}^{p/2}\left(\frac{C_1}{C_2}\right)^p\rho^p,\quad \forall\rho\le\delta. \tag{3.12}$$

Therefore, for any $x=(x_1,x_2,\dots,x_{qm})^T$ with $\|x\|\le\delta$, $x\in Y$, we have

$$\begin{aligned}
J(x)&\ge\frac{1}{p}\left(\frac{1}{C_2}\right)^p\left(x^TAx\right)^{p/2}-\sum_{n=1}^{qm}F(n,x_{n+1},x_n)\\
&\ge\frac{1}{p}\lambda_{\min}^{p/2}\left(\frac{1}{C_2}\right)^p\|x\|^p-2^{-p/2-2}\frac{1}{p}\lambda_{\min}^{p/2}\left(\frac{C_1}{C_2}\right)^p\sum_{n=1}^{qm}\left[2^{p/2}\max\left(|x_{n+1}|^p,|x_n|^p\right)\right]\\
&\ge\frac{1}{p}\lambda_{\min}^{p/2}\left(\frac{1}{C_2}\right)^p\|x\|^p-2^{-p/2-2}\frac{1}{p}\lambda_{\min}^{p/2}\left(\frac{C_1}{C_2}\right)^p\sum_{n=1}^{qm}\left[2^{p/2}\left(|x_{n+1}|^p+|x_n|^p\right)\right]\\
&=\frac{1}{p}\lambda_{\min}^{p/2}\left(\frac{1}{C_2}\right)^p\|x\|^p-2^{-p/2-2}\frac{1}{p}\lambda_{\min}^{p/2}\left(\frac{C_1}{C_2}\right)^p2^{p/2+1}\|x\|_p^p\\
&\ge\frac{1}{p}\lambda_{\min}^{p/2}\left(\frac{1}{C_2}\right)^p\|x\|^p-\frac{1}{2p}\lambda_{\min}^{p/2}\left(\frac{C_1}{C_2}\right)^p\left(\frac{1}{C_1}\right)^p\|x\|^p=\frac{1}{2p}\left(\frac{1}{C_2}\right)^p\lambda_{\min}^{p/2}\|x\|^p.
\end{aligned} \tag{3.13}$$

Take $\sigma=1/2p(1/C_2)^p\lambda_{\min}^{p/2}\delta^p$, then

$$J(x)\ge\sigma>0,\quad \forall x\in Y\cap\partial B_\delta. \tag{3.14}$$

So

$$c_0=\sup_{x\in E_{qm}}J(x)\ge\sigma>0, \tag{3.15}$$

which implies that J satisfies the condition (A_1) of the linking theorem.

Noting that $Ax=0$, for all $x\in Z$, we have

$$J(x)\le\frac{1}{p}\left(\frac{1}{C_1}\right)^p\left(x^TAx\right)^{p/2}-\sum_{n=1}^{qm}F(n,x_{n+1},x_n)\le 0. \tag{3.16}$$

Therefore, the critical point associated to the critical value c_0 of J is a nontrivial qm-periodic solution of (1.1). Now, we need to verify other conditions of the linking theorem.

By Lemma 3.3, J satisfies P-S condition. So, it suffices to verify condition (A_2). Take $e \in \partial B_1 \cap Y$, for any $z \in Z$, $r \in \mathbb{R}$, let $x = re + z$, then

$$\begin{aligned}
J(x) &= \frac{1}{p}\sum_{n=1}^{qm} |\Delta x_n|^p - \sum_{n=1}^{qm} F(n, x_{n+1}, x_n) \le \frac{1}{p}\left(\frac{1}{C_1}\right)^p \left(\sum_{n=1}^{qm} |\Delta x_n|^2\right)^{p/2} - \sum_{n=1}^{qm} F(n, x_{n+1}, x_n) \\
&= \frac{1}{p}\left(\frac{1}{C_1}\right)^p (x^T A x)^{p/2} - \sum_{n=1}^{qm} F(n, x_{n+1}, x_n) \\
&= \frac{1}{p}\left(\frac{1}{C_1}\right)^p \langle A(re+z), (re+z)\rangle^{p/2} - \sum_{n=1}^{qm} F(n, re_{n+1} + z_{n+1}, re_n + z_n) \\
&= \frac{1}{p}\left(\frac{1}{C_1}\right)^p \langle Are, re\rangle^{p/2} - \sum_{n=1}^{qm} F(n, re_{n+1} + z_{n+1}, re_n + z_n) \\
&\le \frac{1}{p}\left(\frac{1}{C_1}\right)^p \lambda_{\max}^{p/2} r^p - a_1 \sum_{n=1}^{qm} \left(\sqrt{(re_{n+1} + z_{n+1})^2 + (re_n + z_n)^2}\right)^\beta + a_2 qm \\
&\le \frac{1}{p}\left(\frac{1}{C_1}\right)^p \lambda_{\max}^{p/2} r^p - a_1 \left(\frac{1}{C_3}\right)^\beta \left(\sum_{n=1}^{qm} \left[(re_{n+1} + z_{n+1})^2 + (re_n + z_n)^2\right]\right)^{\beta/2} + a_2 qm \\
&= \frac{1}{p}\lambda_{\max}^{p/2}\left(\frac{1}{C_1}\right)^p r^p - a_1 \left(\frac{1}{C_3}\right)^\beta (2r^2 + 2\|z\|^2)^{\beta/2} + a_2 qm \\
&\le \frac{1}{p}\lambda_{\max}^{p/2}\left(\frac{1}{C_1}\right)^p r^p - a_1 \left(\frac{1}{C_3}\right)^\beta 2^{\beta/2} r^\beta - a_1 \left(\frac{1}{C_3}\right)^\beta 2^{\beta/2}\|z\|^\beta + a_2 qm.
\end{aligned} \tag{3.17}$$

Let

$$g_1(r) = \frac{1}{p}\lambda_{\max}^{p/2}\left(\frac{1}{C_1}\right)^p r^p - a_1 \left(\frac{1}{C_3}\right)^\beta 2^{\beta/2} r^\beta, \qquad g_2(t) = -a_1 \left(\frac{1}{C_3}\right)^\beta 2^{\beta/2} t^\beta + a_2 qm. \tag{3.18}$$

Then

$$\lim_{r \to +\infty} g_1(r) = -\infty, \qquad \lim_{t \to +\infty} g_2(t) = -\infty, \tag{3.19}$$

and $g_1(r)$ and $g_2(t)$ are bounded from above.

Thus, there exists a constant $R_2 > \delta$, such that $J(x) \le 0$, for all $x \in \partial Q$, where

$$Q = (\overline{B}_{R_2} \cap Z) \oplus \{re \mid 0 < r < R_2\}. \tag{3.20}$$

By the linking theorem, J possesses a critical value $c \ge \sigma > 0$, where

$$\begin{gathered}
c = \inf_{h \in \Gamma} \max_{u \in Q} J(h(u)), \\
\Gamma = \{h \in C(\overline{Q}, E_{qm}) \mid h|_{\partial Q} = \mathrm{id}\}.
\end{gathered} \tag{3.21}$$

The rest of the proof is similar to that of [5, Theorem 1.1], but for the sake of completeness, we give the details.

Let $\tilde{x} \in E_{qm}$ be a critical point associated to the critical value c of J, that is, $J(\tilde{x}) = c$. If $\tilde{x} \neq \overline{x}$, then the proof is complete; if $\tilde{x} = \overline{x}$, then $c_0 = J(\overline{x}) = J(\tilde{x}) = c$, that is

$$\sup_{x \in E_{qm}} J(x) = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u)). \tag{3.22}$$

Choose $h = \mathrm{id}$, we have $\sup_{x \in Q} J(x) = c_0$. Since the choice of $e \in \partial B_1 \cap Y$ is arbitrary, we can take $-e \in \partial B_1 \cap Y$. By a similar argument, there exists a constant $R_3 > \delta$, for any $x \in \partial Q_1$, $J(x) \leq 0$, where

$$Q_1 = (\overline{B}_{R_3} \cap Z) \oplus \{-re \mid 0 < r < R_3\}. \tag{3.23}$$

Again, by using the linking theorem, J possesses a critical value $c' \geq \sigma > 0$, where

$$c' = \inf_{h \in \Gamma_1} \max_{u \in Q_1} J(h(u)), \qquad \Gamma_1 = \{h \in C(\overline{Q}_1, E_{qm}) \mid h|_{\partial Q_1} = \mathrm{id}\}. \tag{3.24}$$

If $c' \neq c_0$, then the proof is complete. If $c' = c_0$, then $\sup_{x \in Q_1} J(x) = c_0$. Due to the fact that $J|_{\partial Q} \leq 0$, $J|_{\partial Q_1} \leq 0$, J attains its maximum at some points in the interior of the set Q and Q_1. Clearly, $Q \cap Q_1 = \emptyset$, and for any $x \in Z$, $J(x) \leq 0$. This shows that there must be a point $\hat{x} \in E_{qm}$, such that $\hat{x} \neq \tilde{x}$ and $J(\hat{x}) = c' = c_0$.

The above argument implies that whether or not $c = c_0$, (1.1) possesses at least two nontrivial qm-periodic solutions.

Remark 3.4. when $qm = 1$, (1.1) is reduced to trivial case; when $qm = 2$, A has the following form:

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{3.25}$$

In this case, it is easy to complete the proof of Theorem 3.1.

Finally, we give an example to illustrate Theorem 3.1.

Example 3.5. Assume that

$$f(t,u,v,w) = 2(p+1)v\left[\left(1+\sin^2\frac{\pi t}{m}\right)(u^2+v^2)^p + \left(1+\sin^2\frac{\pi(t-1)}{m}\right)(v^2+w^2)^p\right]. \tag{3.26}$$

Take

$$F(t,u,v) = \left(1+\sin^2\frac{\pi t}{m}\right)(u^2+v^2)^{p+1}. \tag{3.27}$$

Then,

$$\begin{aligned} &F_2'(t-1,v,w) + F_3'(t,u,v) \\ &\quad = 2(p+1)v\left[\left(1+\sin^2\frac{\pi t}{m}\right)(u^2+v^2)^p + \left(1+\sin^2\frac{\pi(t-1)}{m}\right)(v^2+w^2)^p\right]. \end{aligned} \tag{3.28}$$

It is easy to verify that the assumptions of Theorem 3.1 are satisfied and then (1.1) possesses at least two nontrivial qm-periodic solutions. □

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