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Research Article

Asymptotic Expansions for Higher-Order Scalar Difference Equations

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We give an asymptotic expansion of the solutions of higher-order Poincaré difference equation in terms of the characteristic solutions of the limiting equation. As a consequence, we obtain an asymptotic description of the solutions approaching a hyperbolic equilibrium of a higher-order nonlinear difference equation with sufficiently smooth nonlinearity. The proof is based on the inversion formula for the z-transform and the residue theorem.

1. Introduction

Our principal interest in this paper is the asymptotic behavior of the solutions of higher-order nonlinear difference equations in a neighborhood of an equilibrium. In the specific case of rational difference equations, this problem has been studied in several recent papers (see, e.g., [1–3] and the references therein). In this paper, we will deal with general nonlinear difference equations. Our main result (Theorem 3.1) applies in the case when the equilibrium is hyperbolic and the nonlinearity is sufficiently smooth. Roughly speaking, it says that in this case, the solutions of the nonlinear equation approach the equilibrium along the solutions of the corresponding linearized equation. This result is in fact a consequence of the asymptotic expansion of the solutions of Poincaré difference equation established in Theorem 2.3. Our results may be viewed as the discrete analogs of similar qualitative results known for ordinary and functional differential equations (see [4, Chapter 13, Theorem 4.5], [5, Proposition 7.2], or [6, Theorem 3.1]). The simple short proof presented below is based on the inversion formula for the z-transform and the residue theorem. Finally, we mention the recent remarkable work of Matsunaga and Murakami [7] which is relevant to our study. In this paper, the authors described the structure of the solutions of nonlinear functional difference equations in a neighborhood of an equilibrium. However, their results do not yield explicit asymptotic formulas for the solutions.

The paper is organized as follows. In Section 2, we study the asymptotic behavior of the solutions of Poincaré difference equations. In Section 3, we establish our main theorem about the behavior of the solutions of nonlinear difference equations in a neighborhood of a hyperbolic equilibrium. In Section 4, we apply our result to a second-order rational difference equation. We obtain an asymptotic description of the positive solutions, which improves the recent result due to Kalabušić and Kulenović [1].

2. Asymptotic expansions for Poincaré difference equations

Throughout the paper, we will use the standard notations $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ for the set of integers, real numbers, and complex numbers, respectively. The symbol $\mathbb{Z}^+$ denotes the set of nonnegative integers. For any positive integer k, $\mathbb{R}^k$ is the k-dimensional space of real column vectors with any convenient norm.

Consider the kth-order *Poincaré difference equation*

$$x(n+k)+a_1(n)x(n+k-1)+\cdots+a_k(n)x(n)=0, \quad n\in\mathbb{Z}^+, \tag{2.1}$$

where the coefficients $a_j:\mathbb{Z}^+\to\mathbb{C}$, $1\le j\le k$, are *asymptotically constant* as $n\to\infty$, that is, the limits

$$b_j=\lim_{n\to\infty}a_j(n), \quad 1\le j\le k, \tag{2.2}$$

exist and are finite. It is natural to expect that in this case, the solutions of (2.1) retain some properties of the solutions of the *limiting equation*

$$x(n+k)+b_1x(n+k-1)+\cdots+b_kx(n)=0, \quad n\in\mathbb{Z}^+. \tag{2.3}$$

The following Perron-type theorem established in [8] is a result of this type. It says that the growth rates of the solutions of both (2.1) and (2.3) are equal to the moduli of the roots of the *characteristic equation*

$$\Delta(z)=0, \quad \Delta(z)=z^k+b_1z^{k-1}+\cdots+b_k. \tag{2.4}$$

(For an extension to a more general class of functional difference equations, see [9]. For further related results on Poincaré difference equations, see the monographs [10, Section 2.13] and [11, Chapter 8].)

Theorem 2.1 [8, Theorem 2]. *Suppose (2.2) holds. If $x:\mathbb{Z}^+\to\mathbb{C}$ is a solution of (2.1), then either*

(i) $x(n)=0$ *for all large n, or*

(ii) *the quantity*

$$\mu=\mu(x)=\limsup_{n\to\infty}\sqrt[n]{|x(n)|} \tag{2.5}$$

is equal to the modulus of one of the characteristic roots of (2.3), that is, the set

$$\Lambda(\mu) = \{z \in \mathbb{C} \mid \Delta(z) = 0,\ |z| = \mu\} \tag{2.6}$$

is nonempty.

Remark 2.2. Alternative (i) of the above theorem can be excluded in the following sense. If

$$a_k(n) \neq 0, \quad n \in \mathbb{Z}^+, \tag{2.7}$$

then the only solution of (2.1) satisfying alternative (i) is the trivial solution $x(n) = 0$ identically for $n \in \mathbb{Z}^+$. Indeed, if $x : \mathbb{Z}^+ \to \mathbb{C}$ is a nontrivial solution of (2.1) such that $x(n) = 0$ for all large n and m is the greatest integer for which $x(m) \neq 0$, then (2.1) and (2.7) yield

$$x(m) = a_k(m)^{-1}\big[-x(m+k) - a_1(m)x(m+k-1) - \cdots - a_{k-1}(m)x(m+1)\big] = 0, \tag{2.8}$$

a contradiction.

In this section, we will show that if the limits in (2.2) are approached at an exponential rate, then the conclusion of Theorem 2.1 can be substantially strengthened. First, we introduce some notations and definitions. If λ is a characteristic root of (2.3), m_λ will denote the multiplicity of $z = \lambda$ as a zero of the characteristic polynomial Δ given by (2.4). It is well known that in this case for any complex-valued polynomial p of order less than m_λ, the function

$$y(n) = p(n)\lambda^n, \quad n \in \mathbb{Z}, \tag{2.9}$$

is a solution of (2.3). We will refer to such solutions as a *characteristic solution of (2.3) corresponding to* λ. More generally, if Λ is a nonempty set of characteristic roots of (2.3), then by a *characteristic solution corresponding to the set* Λ we mean a finite sum of characteristic solutions for values $\lambda \in \Lambda$.

The main result of this section is the following theorem.

Theorem 2.3. *Suppose that the convergence in (2.2) is exponentially fast, that is, for some* $\eta \in (0,1)$,

$$a_j(n) = b_j + O(\eta^n), \quad n \longrightarrow \infty,\ 1 \le j \le k. \tag{2.10}$$

Assume also that

$$b_k \neq 0. \tag{2.11}$$

If $x:\mathbb{Z}^+\to\mathbb{C}$ is a solution of (2.1), then either

(i) *$x(n)=0$ for all large n, or*

(ii) *there exists $\mu\in(0,\infty)$ such that the set of characteristic roots $\Lambda(\mu)$ given by (2.6) is nonempty, and for some $\epsilon\in(0,\mu)$, one has the asymptotic expansion*

$$x(n)=y(n)+O((\mu-\epsilon)^n),\quad n\longrightarrow\infty, \tag{2.12}$$

where y is a nontrivial characteristic solution of (2.3) corresponding to the set $\Lambda(\mu)$.

As a preparation for the proof of Theorem 2.3, we establish a useful technical result. Recall that if g is a meromorphic function in a region of the complex plane and λ is a pole of g of order m, then the *residue of g at λ*, denoted by $\operatorname{Res}(g;\lambda)$, is the coefficient c_{-1} of the term $(z-\lambda)^{-1}$ in the *Laurent series*

$$g(z)=\sum_{j=-m}^{\infty}c_j(z-\lambda)^j. \tag{2.13}$$

Lemma 2.4. *If λ is a nonzero characteristic root of (2.3) and f is holomorphic in a neighborhood of $z=\lambda$, then the function*

$$y(n)=\operatorname{Res}(\mathrm{id}^{n-1}\Delta^{-1}f;\lambda),\quad n\in\mathbb{Z}, \tag{2.14}$$

is a characteristic solution of (2.3) corresponding to λ. (Here $\mathrm{id}^{n-1}(z)=z^{n-1}$.)

Proof. The characteristic polynomial Δ can be written as

$$\Delta(z)=(z-\lambda)^{m_\lambda}q(z), \tag{2.15}$$

where q is a polynomial and $q(\lambda)\neq 0$. For each $n\in\mathbb{Z}$, the function $\mathrm{id}^{n-1}f$ is holomorphic in a neighborhood of $z=\lambda$. Consequently, the function $g=\mathrm{id}^{n-1}\Delta^{-1}f$ is holomorphic in a deleted neighborhood of $z=\lambda$ and its Laurent series has the form (2.13) with $m=m_\lambda$. By the calculus of residues, we have that

$$\begin{aligned} y(n)=\operatorname{Res}(g;\lambda)&=\frac{1}{(m_\lambda-1)!}\frac{d^{m_\lambda-1}}{dz^{m_\lambda-1}}\bigg|_{z=\lambda}((z-\lambda)^{m_\lambda}g(z))\\ &=\frac{1}{(m_\lambda-1)!}\frac{d^{m_\lambda-1}}{dz^{m_\lambda-1}}\bigg|_{z=\lambda}(z^{n-1}h(z)), \end{aligned} \tag{2.16}$$

where $h=f/q$ with q as in (2.15). By the Leibniz rule, we have

$$\begin{aligned} &\frac{d^{m_\lambda-1}}{dz^{m_\lambda-1}}\bigg|_{z=\lambda}(z^{n-1}h(z))=\lambda^{n-1}h^{(m_\lambda-1)}(\lambda)\\ &\qquad+\sum_{j=1}^{m_\lambda-1}\binom{m_\lambda-1}{j}(n-1)(n-2)\cdots(n-j)\lambda^{n-j-1}h^{(m_\lambda-1-j)}(\lambda). \end{aligned} \tag{2.17}$$

Thus, y has the form (2.9), where p is a polynomial of order less than m_λ. □

We now give the proof of Theorem 2.3.

Proof of Theorem 2.3. Suppose that x is a solution of (2.1) for which alternative (i) does not hold. By Theorem 2.1, the quantity μ defined by (2.5) is equal to the modulus of one of the characteristic roots of (2.3). Thus, $\Lambda(\mu)$ is nonempty. By virtue of (2.11), the characteristic roots are different from zero, and hence $\mu > 0$. Rewrite (2.1) as

$$x(n+k)+\sum_{j=1}^{k} b_j x(n+k-j) = c(n), \quad n \in \mathbb{Z}^+, \tag{2.18}$$

where

$$c(n) = \sum_{j=1}^{k}\left(b_j - a_j(n)\right)x(n+k-j), \quad n \in \mathbb{Z}^+. \tag{2.19}$$

It is well known that the z-transform of x given by

$$\tilde{x}(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \tag{2.20}$$

defines a holomorphic function in the region $|z| > \mu$ with μ as in (2.5). Similarly, $\tilde{c}$, the z-transform of c, is holomorphic for $|z| > \nu$, where

$$\nu = \limsup_{n\to\infty} \sqrt[n]{|c(n)|}. \tag{2.21}$$

It is easily shown, using (2.5) and (2.10), that

$$\nu \le \eta\mu < \mu. \tag{2.22}$$

Taking the z-transform of (2.18) and using the shifting property

$$\sum_{n=0}^{\infty} x(n+l)z^{-n} = z^l\tilde{x}(z) - \sum_{n=0}^{l-1} x(n)z^{l-n} \tag{2.23}$$

for $l \in \mathbb{Z}^+$ and $|z| > \mu$, we find that

$$\Delta(z)\tilde{x}(z) = q(z) + \tilde{c}(z), \quad |z| > \mu, \tag{2.24}$$

where

$$q(z) = \sum_{n=0}^{k-1} x(n)z^{k-n} + \sum_{j=1}^{k} b_j \sum_{n=0}^{k-j-1} x(n)z^{k-j-n}. \tag{2.25}$$

According to the inversion formula for the z-transform, we have that

$$x(n) = \frac{1}{2\pi i}\int_{\gamma} z^{n-1}\tilde{x}(z)\,dz, \quad n \in \mathbb{Z}^+, \tag{2.26}$$

where γ is any positively oriented simple closed curve that lies in the region $|z| > \mu$ and winds around the origin. Choose $\epsilon > 0$ so small that $\mu - 2\epsilon > \nu$ and the set of the roots of the characteristic polynomial Δ belonging to the annulus $\mu - 2\epsilon < |z| < \mu + 2\epsilon$ coincides with the set $\Lambda(\mu)$. From (2.24) and (2.26), we obtain

$$x(n) = \frac{1}{2\pi i}\int_{\gamma_1} z^{n-1}\Delta^{-1}(z)f(z)dz, \quad n \in \mathbb{Z}^+, \tag{2.27}$$

where $f = q + \tilde{c}$ and γ_1 is the circle,

$$\gamma_1(t) = (\mu + \epsilon)e^{it}, \quad 0 \leq t \leq 2\pi. \tag{2.28}$$

Since q is an entire function (polynomial) and $\tilde{c}$ is holomorphic for $|z| > \nu$, the function f is holomorphic in the region $|z| > \nu$. This, together with the choice of ϵ, implies that the function $\mathrm{id}^{n-1}\Delta^{-1}f$ is meromorphic in the region

$$\Omega = \{z \in \mathbb{C} \mid \mu - 2\epsilon < |z| < \mu + 2\epsilon\} \tag{2.29}$$

and its poles in that region belong to the set $\Lambda(\mu)$. Let γ_2 be the positively oriented circle centered at the origin with radius $\mu - \epsilon$, that is,

$$\gamma_2(t) = (\mu - \epsilon)e^{it}, \quad 0 \leq t \leq 2\pi. \tag{2.30}$$

By the residue theorem, we have that

$$\frac{1}{2\pi i}\int_{\gamma_1} z^{n-1}\Delta^{-1}(z)f(z)dz = \frac{1}{2\pi i}\int_{\gamma_2} z^{n-1}\Delta^{-1}(z)f(z)dz + y(n), \tag{2.31}$$

where

$$y(n) = \sum_{\lambda \in A} \mathrm{Res}\left(\mathrm{id}^{n-1}\Delta^{-1}f;\lambda\right), \tag{2.32}$$

A being the set of poles of $\mathrm{id}^{n-1}\Delta^{-1}f$ in the annulus Ω. (This follows from [12, Theorem 10.42] applied to the cycle $\Gamma = \gamma_1 \dotplus \gamma_3$ in $\Omega - A$, where γ_3 is the opposite to γ_2.) Substitution into (2.27) yields

$$x(n) = y(n) + \frac{1}{2\pi i}\int_{\gamma_2} z^{n-1}\Delta^{-1}(z)f(z)dz, \quad n \in \mathbb{Z}^+. \tag{2.33}$$

As noted before, $A \subset \Lambda(\mu)$. Consequently, by Lemma 2.4, y is a characteristic solution of (2.3) corresponding to the set $\Lambda(\mu)$. The integral in (2.33) can be estimated as follows (see [12, Section 10.8]):

$$\left|\int_{\gamma_2} z^{n-1}\Delta^{-1}(z)f(z)dz\right| \leq 2\pi K(\mu - \epsilon)^n, \tag{2.34}$$

where

$$K = \max_{|z|=\mu-\epsilon} \left|\Delta^{-1}(z)f(z)\right|. \tag{2.35}$$

This proves (2.12). Finally, we show that y is a nontrivial solution. Indeed, if $y(n)$ were zero identically for $n \in \mathbb{Z}^+$, then (2.12) would imply that

$$\limsup_{n\to\infty} \sqrt[n]{|x(n)|} \leq \mu - \epsilon < \mu, \tag{2.36}$$

contradicting (2.5). □

Remark 2.5. As shown in the above proof, the positive constant μ in alternative (ii) of Theorem 2.3 is given by formula (2.5).

In most applications, the coefficients of (2.1) are real and only real-valued solutions are of interest. As a simple consequence of Theorem 2.3, we have the following.

Corollary 2.6. *Suppose that the coefficients $a_j : \mathbb{Z}^+ \to \mathbb{R}$, $1 \leq j \leq k$, are real-valued and there exist constants $b_j \in \mathbb{R}$, $1 \leq j \leq k$, and $\eta \in (0,1)$ such that (2.10) and (2.11) hold. If $x : \mathbb{Z}^+ \to \mathbb{R}$ is a real-valued solution of (2.1), then either*
(i) *$x(n) = 0$ for all large n, or*
(ii) *there exists $\mu \in (0,\infty)$ such that the set $\Lambda(\mu)$ is nonempty, and for some $\epsilon \in (0,\mu)$, one has the asymptotic expansion*

$$x(n) = w(n) + O((\mu - \epsilon)^n), \quad n \longrightarrow \infty, \tag{2.37}$$

where w is a nontrivial (real-valued) solution of the limiting equation (2.3) of the form

$$w(n) = \mu^n \sum_{\lambda \in \Lambda(\mu)} q_\lambda(n) \cos(n\theta_\lambda) + r_\lambda(n) \sin(n\theta_\lambda), \tag{2.38}$$

where, for every $\lambda \in \Lambda(\mu)$, $\theta_\lambda = \operatorname{Arg}\lambda$ is the unique number in $(-\pi,\pi]$ for which $\lambda = \mu e^{i\theta_\lambda}$ and q_λ, r_λ are real-valued polynomials of order less than m_λ.

Proof. Clearly, if x is a real-valued solution of (2.1) satisfying the asymptotic relation (2.12) of Theorem 2.3, then (2.37) holds with $w = \operatorname{Re} y$. Since the coefficients of the limiting equation (2.3) are real, the real part of the solution y is also a solution of (2.3). It is an immediate consequence of the definition of the characteristic solutions that $w = \operatorname{Re} y$ has the form (2.38). Finally, if $w(n)$ were zero identically for $n \in \mathbb{Z}^+$, then (2.37) would imply (2.36) contradicting (2.5). □

3. Behavior near equilibria of nonlinear difference equations

In this section, as an application of our previous results on Poincaré difference equations, we establish an asymptotic expansion of the solutions of nonlinear difference equations which tend to a hyperbolic equilibrium. We will deal with the equation

$$x(n+k) = f(x(n), x(n+1), \ldots, x(n+k-1)), \quad n \in \mathbb{Z}^+, \tag{3.1}$$

where k is a positive integer and $f : \Omega \to \mathbb{R}$ is a C^1 function, Ω being a convex open subset of $\mathbb{R}^k$. Recall that $v \in \mathbb{R}$ is an *equilibrium* of (3.1) if (3.1) admits the constant solution

$x(n) = v$ identically for $n \in \mathbb{Z}^+$. Equivalently,

$$\mathbf{v} = (v, v, \ldots, v) \in \Omega, \tag{3.2}$$

$$v = f(\mathbf{v}). \tag{3.3}$$

Associated with (3.1) is the *linearized equation* about the equilibrium v,

$$\begin{gathered} x(n+k) + b_1 x(n+k-1) + \cdots + b_k x(n) = 0, \quad n \in \mathbb{Z}^+, \\ b_j = -D_{k-j+1} f(\mathbf{v}), \quad 1 \le j \le k. \end{gathered} \tag{3.4}$$

The equilibrium v of (3.1) is called *hyperbolic* if the characteristic polynomial Δ of the linearized equation (3.4) given by (2.4) has no root on the unit circle $|z| = 1$.

The main result of this section is the following theorem.

Theorem 3.1. *Let v be a hyperbolic equilibrium of (3.1). Suppose that the partial derivatives $D_j f$, $1 \le j \le k$, are locally Lipschitz continuous on Ω and*

$$D_1 f(\mathbf{v}) \ne 0 \tag{3.5}$$

(with $\mathbf{v}$ as in (3.2)). Let $x : \mathbb{Z}^+ \to \mathbb{R}$ be a solution of (3.1) satisfying

$$\lim_{n\to\infty} x(n) = v. \tag{3.6}$$

Then either

(i) *$x(n) = v$ for all large n, or*

(ii) *there exists $\mu \in (0,1)$ such that the set $\Lambda(\mu)$ given by (2.6) is nonempty, and for some $\epsilon \in (0,\mu)$, one has the asymptotic expansion*

$$x(n) = v + w(n) + O\big((\mu-\epsilon)^n\big), \quad n \longrightarrow \infty, \tag{3.7}$$

where w is a nontrivial solution of the linearized equation (3.4) of the form (2.38).

If, in addition, it is assumed that

$$D_1 f(\mathbf{x}) \ne 0, \quad \mathbf{x} \in \Omega, \tag{3.8}$$

then alternative (i) holds only for the equilibrium solution $x(n) = v$ identically for $n \in \mathbb{Z}^+$.

Remark 3.2. A sufficient condition for the partial derivatives $D_j f$, $1 \le j \le k$, to be locally Lipschitz continuous on Ω is that f is of class C^2 on Ω.

Proof. Let x be a solution of (3.1) satisfying (3.6) for which $x(n) \ne 0$ for infinitely many n. We have to show that x satisfies alternative (ii). Define

$$\begin{gathered} u(n) = x(n) - v, \quad n \in \mathbb{Z}^+, \\ \mathbf{x}_n = \big(x(n), x(n+1), \ldots, x(n+k-1)\big) \in \Omega, \quad n \in \mathbb{Z}^+. \end{gathered} \tag{3.9}$$

Then $u(n) \neq 0$ for infinitely many n, and from (3.1) and (3.3), we find for $n \in \mathbb{Z}^+$ that

$$\begin{aligned} u(n+k) &= x(n+k) - v = f(\mathbf{x}_n) - f(\mathbf{v}) = \int_0^1 \frac{d}{ds} f(s\mathbf{x}_n + (1-s)\mathbf{v})\,ds \\ &= \int_0^1 \sum_{l=1}^{k} D_l f(s\mathbf{x}_n + (1-s)\mathbf{v})(x(n+l-1) - v)\,ds \\ &= \sum_{j=1}^{k} \int_0^1 D_{k-j+1} f(s\mathbf{x}_n + (1-s)\mathbf{v})\,ds\,u(n+k-j). \end{aligned} \tag{3.10}$$

Thus, $u : \mathbb{Z}^+ \to \mathbb{R}$ is a solution of (2.1) with coefficients

$$a_j(n) = -\int_0^1 D_{k-j+1} f(s\mathbf{x}_n + (1-s)\mathbf{v})\,ds, \quad n \in \mathbb{Z}^+,\ 1 \le j \le k. \tag{3.11}$$

By virtue of (3.6), $\mathbf{x}_n \to \mathbf{v}$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} a_j(n) = -D_{k-j+1} f(\mathbf{v}) = b_j, \quad 1 \le j \le k, \tag{3.12}$$

$$b_k = -D_1 f(\mathbf{v}) \neq 0 \tag{3.13}$$

by (3.5). By the application of Theorem 2.1, we conclude that

$$\mu = \limsup_{n\to\infty} \sqrt[n]{|u(n)|} \tag{3.14}$$

is equal to the modulus of one of the roots of the characteristic equation (2.4). Since $u(n) \to 0$ as $n \to \infty$, $\mu \le 1$. Thus, μ is equal to the modulus of one of the roots of (2.4) belonging to the closed unit disk $|z| \le 1$. Since the roots of (2.4) are nonzero (by (3.13)) and they do not lie on the unit circle $|z| = 1$ (by the hyperbolicity of v), we conclude that $\mu \in (0,1)$. Choose $\eta \in (\mu, 1)$. Then (3.14) implies that

$$\sqrt[n]{|u(n)|} < \eta \quad \forall \text{ large } n, \tag{3.15}$$

and hence there exists $K > 0$ such that

$$|u(n)| = |x(n) - v| \le K\eta^n, \quad n \in \mathbb{Z}^+. \tag{3.16}$$

This, together with the Lipschitz continuity of the partial derivatives $D_j f$, $1 \le j \le k$, implies that (2.10) holds. Applying Corollary 2.6 to the solution u of (2.1), we obtain the existence of $\epsilon \in (0, \mu)$ such that

$$u(n) = w(n) + O((\mu - \epsilon)^n), \quad n \longrightarrow \infty, \tag{3.17}$$

where w is a solution of the linearized equation (3.4) of the form (2.38). This proves (3.7).

Suppose now that (3.8) holds and let x be a solution of (3.1) satisfying alternative (i). Then $u(n) = 0$ for all large n. Since $D_1 f$ is continuous on Ω and Ω is convex, (3.8) implies that $D_1 f$ has a constant sign on Ω. As a consequence, (2.7) holds with a_k as in (3.11). According to Remark 2.2, we have that $u(n) = 0$, and hence $x(n) = v$ identically for $n \in \mathbb{Z}^+$. □

4. Application

As an application, consider the second-order rational difference equation

$$x(n+2) = \frac{B}{x(n+1)} + \frac{C}{x(n)}, \quad n \in \mathbb{Z}^+, \tag{4.1}$$

where $B, C \in (0, \infty)$. We are interested in the asymptotic behavior of the positive solutions of (4.1). This equation is a special case of (3.1) when

$$f(x_1, x_2) = \frac{C}{x_1} + \frac{B}{x_2}, \quad (x_1, x_2) \in \Omega = (0, \infty) \times (0, \infty). \tag{4.2}$$

Equation (4.1) has a unique positive equilibrium $v = \sqrt{B+C}$. The linearized equation about the equilibrium v has the form

$$x(n+2) + \frac{B}{B+C} x(n+1) + \frac{C}{B+C} x(n) = 0, \quad n \in \mathbb{Z}^+. \tag{4.3}$$

The characteristic roots of (4.3) are

$$\lambda_{\pm} = \frac{-B \pm \sqrt{B^2 - 4C(B+C)}}{2(B+C)}. \tag{4.4}$$

Depending on the parameters B and C, we have the following three possible cases.

Case 1 ($C < B/(2(1+\sqrt{2}))$). Then λ_+ and λ_- are real and

$$-1 < \lambda_- < \lambda_+ < 0. \tag{4.5}$$

Case 2 ($C = B/(2(1+\sqrt{2}))$). Then

$$-1 < \lambda_+ = \lambda_- = \lambda = -\frac{B}{2(B+C)} < 0. \tag{4.6}$$

Case 3 ($C > B/(2(1+\sqrt{2}))$). Then λ_+ and λ_- are complex conjugate pairs,

$$\lambda_{\pm} = \mu e^{\pm i\theta} \quad \text{for } \mu = |\lambda_+| = |\lambda_-| = \sqrt{\frac{C}{B+C}} \text{ and some } \theta \in (0, \pi). \tag{4.7}$$

In all three cases, both characteristic roots λ_+ and λ_- lie inside the open unit disk $|z| < 1$.

Let us mention two results available for (4.1). Kulenović and Ladas [2] showed that every positive solution $x : \mathbb{Z}^+ \to (0, \infty)$ of (4.1) tends to v as $n \to \infty$. In a recent paper [1], Kalabušić and Kulenović determined the rate of convergence of the positive solutions of (4.1). In [1, Theorem 2.1], they showed that if $x : \mathbb{Z}^+ \to (0, \infty)$ is a positive solution of (4.1) which is eventually different from v, then in Case 1, either

$$\lim_{n \to \infty} \frac{x(n+1) - v}{x(n) - v} = \lambda_+, \tag{4.8}$$

or

$$\lim_{n\to\infty} \frac{x(n+1)-v}{x(n)-v} = \lambda_-, \tag{4.9}$$

while, in Cases 2 and 3, we have that

$$\limsup_{n\to\infty} \sqrt[n]{|x(n)-v|} = |\lambda_+| = |\lambda_-|. \tag{4.10}$$

Theorem 3.1 yields the following more precise result about the asymptotic behavior of the positive solutions of (4.1).

Theorem 4.1. *Every nonconstant positive solution $x : \mathbb{Z}^+ \to (0,\infty)$ of (4.1) has the following asymptotic representation as $n \to \infty$.*

In Case 1, either

$$x(n) = v + K\lambda_+^n + O((|\lambda_+| - \epsilon)^n) \tag{4.11}$$

for some $K \in \mathbb{R} - \{0\}$ and $\epsilon \in (0, |\lambda_+|)$, or

$$x(n) = v + K\lambda_-^n + O((|\lambda_-| - \epsilon)^n) \tag{4.12}$$

for some $K \in \mathbb{R} - \{0\}$ and $\epsilon \in (0, |\lambda_-|)$.

In Case 2,

$$x(n) = v + \lambda^n (K_1 n + K_2) + O((|\lambda| - \epsilon)^n) \tag{4.13}$$

for some $(K_1, K_2) \in \mathbb{R}^2 - \{(0,0)\}$ and $\epsilon \in (0, |\lambda|)$ (with λ as in (4.6)).

In Case 3,

$$x(n) = v + \mu^n (K_1 \cos(n\theta) + K_2 \sin(n\theta)) + O((\mu - \epsilon)^n) \tag{4.14}$$

for some $(K_1, K_2) \in \mathbb{R}^2 - \{(0,0)\}$ and $\epsilon \in (0, \mu)$ (with μ and θ as in (4.7)).

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