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Research Article

On the Integrability of Quasihomogeneous Systems and Quasidegenerate Infinity Systems

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The integrability of quasihomogeneous systems is considered, and the properties of the first integrals and the inverse integrating factors of such systems are shown. By solving the systems of ordinary differential equations which are established by using the vector fields of the quasihomogeneous systems, one can obtain an inverse integrating factor of the systems. Moreover, the integrability of a class of systems (quasidegenerate infinity systems) which generalize the so-called degenerate infinity vector fields is considered, and a method how to obtain an inverse integrating factor of the systems from the first integrals of the corresponding quasihomogeneous systems is shown.

1. Introduction

We consider quasihomogeneous autonomous systems, which are also called similarity invariant systems or weighted homogeneous systems, that is, the following nth order autonomous system of differential equations:

$$\frac{dx_i}{dt} = X_i(x), \quad i = 1,2,\dots,n, \tag{1.1}$$

where $x = (x_1,x_2,\dots,x_n) \in D \subset \mathbb{R}^n$ (or $\mathbb{C}^n$), $X_i : D \to \mathbb{R}$ (or $\mathbb{C}$), $X_i \in C_\infty(D)$, and $t \in \mathbb{R}$ (or $\mathbb{C}$). System (1.1) is invariant under the similarity transformation $x = (x_1,x_2,\dots,x_n,t) \to (\alpha^{p_1}x_1,\alpha^{p_2}x_2,\dots,\alpha^{p_n}x_n,\alpha^{-l}t)$ for all $\alpha \in \mathbb{R}\setminus\{0\}$, where $p_1,p_2,\dots,p_n$ and l are positive integers. In other words, $X_i(x)$ are p_i ($i = 1,2,\dots,n$) quasihomogeneous functions of weighted degrees $p_i + l$, respectively, that is,

$$X_i(\alpha^{p_1}x_1,\dots,\alpha^{p_n}x_n) = \alpha^{p_i+l}X_i(x_1,\dots,x_n) \tag{1.2}$$

for all $\alpha \in \mathbb{R} \setminus \{0\}$. We also say that system (1.1) is p_i $(i = 1,2,\dots,n)$ quasihomogeneous of weighted degree l.

Notice that if p_i $(i = 1,2,\dots,a)$ are even and p_i $(i = a+1,a+2,\dots,n)$ and l are odd, then the p_i $(i = 1,2,\dots,n)$ quasihomogeneous systems include some class of the reversible systems which are invariant under the symmetry $(x_1,\dots,x_n,t) \to (x_1,\dots,x_a,-x_{a+1},\dots,-x_n,t)$. Moreover, in the particular case p_i $(i = 1,2,\dots,n) = 1$, the quasihomogeneous systems reduce to classical homogeneous systems.

Motion equations of many important problems of dynamics are of the quasihomogeneous form, for example, Euler-Poisson equations, Kirchhoff equations, and so forth. Recently, several works have studied the integrability of autonomous systems and quasihomogeneous polynomial systems; for more details see [1–6]. In [5], several techniques for searching first integrals of nth autonomous systems by using Lie groups admitted by the systems are proposed. The integrability of quasihomogeneous planar systems is studied in [1, 3], and the existence of a link between the Kowalevskaya exponents of quasihomogeneous systems and the degree of their quasihomogeneous polynomial first integrals is studied in [2, 4]. There exist some methods for studying the integrability of autonomous systems by using Lie group admitted by the systems [5, 7, 8] and using by the invariant manifolds of the systems [6]. As we know, the existence of inverse integrating factors gives a lot of information on dynamics, integrability of the systems and so on. In [9], the relationship between the property of a Darboux first integral and the existence of a polynomial inverse integrating factor of a polynomial differential systems was studied. However, generally, it is difficult to search for inverse integrating factors. Searching for first integrals of a system plays a very important role for integrating the system. In this paper, we study the integrability of nth order qusaihomogeneous systems. First, we show the properties of the first integrals and the inverse integrating factors of such systems. Then, we propose a method to obtain an inverse integrating factor of the systems by solving the ordinary differential equations systems established by using the vector fields of the quasihomogeneous systems. System (1.1) with $n = 2$ is called degenerate infinity system if it satisfies $X_1 = x_1 A$, $X_2 = x_2 A$ for some homogeneous polynomial $A(x_1,x_2)$. Degenerate infinity systems have attracted the attention of many authors, see [10, 11]. In this paper, we also consider the integrability of a class of systems which generalize the so-called degenerate infinity vector fields, that is,

$$\frac{dx_i}{dt} = X_i(x) + p_i x_i A(x_1,x_2,\dots,x_n), \quad i = 1,2,\dots,n, \tag{1.3}$$

where $X_i(x)$ $(i = 1,2,\dots,n)$ are p_i $(i = 1,2,\dots,n)$ quasihomogeneous function of weighted degrees $p_i + l$ of system (1.1), respectively. $A(x_1,x_2,\dots,x_n)$ is given a p_i $(i = 1,2,\dots,n)$ quasihomogeneous polynomial of weighted degree α. We call system (1.3) a quasidegenerate infinity system. We propose a method to obtain inverse integrating factors of system (1.3) from the first integrals of the corresponding quasihomogeneous system (1.1) by using the Darboux's theory of integrability.

2. On the integrability of quasihomogeneous systems

Let X be the vector field associated with system (1.1), that is,

$$X = X_1(x)\frac{\partial}{\partial x_1} + X_2(x)\frac{\partial}{\partial x_2} + \cdots + X_n(x)\frac{\partial}{\partial x_n}. \tag{2.1}$$

Let G be a one-parameter Lie group with an associated infinitesimal generator V defined as

$$V = \xi_1(x)\frac{\partial}{\partial x_1} + \xi_2(x)\frac{\partial}{\partial x_2} + \cdots + \xi_n(x)\frac{\partial}{\partial x_n}, \tag{2.2}$$

where $\xi_i(x) \in C^1(D)$, $i = 1,2,\ldots,n$. A Lie group admitted by (in fact an infinitesimal symmetry) system (1.1) is defined to be a group of transformations with infinitesimal generator V such that under the action of this group, a solution curve of system (1.1) is mapped into another solution curve of system (1.1).

Proposition 2.1 (see [7]). *Let G be the one-parameter Lie group with infinitesimal generator V, then G is a Lie group admitted by system (1.1) if and only if*

$$[X,V] = B(x_1,x_2,\ldots,x_n)X \tag{2.3}$$

is satisfied for some smooth scalar function $B(x_1,x_2,\ldots,x_n)$, where $[X,V] := XV - VX$ is the Lie bracket of the C^1-vector fields of X and V.

Definition 2.2. Let $\mathcal{V}$ be an open subset of D. A nonzero function $\mu \in C^1(\mathcal{V}) : \mathcal{V} \to \mathbb{R}$, satisfying the linear partial differential equation $X\mu = \operatorname{div}(X)\mu$, or equivalently,

$$X_1(x)\frac{\partial \mu}{\partial x_1} + X_2(x)\frac{\partial \mu}{\partial x_2} + \cdots + X_n(x)\frac{\partial \mu}{\partial x_n} = \left(\frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right)\mu, \tag{2.4}$$

is called an inverse integrating factor of system (1.1) on $\mathcal{V}$. It is well known, if $n = 2$, system (1.1) has two autonomous differential equations and admits a Lie group G, then the system (1.1) has the following inverse integrating factor defined on $\mathcal{V}$:

$$\mu(x_1,x_2) = X_1(x_1,x_2)\xi_2(x_1,x_2) - X_2(x_1,x_2)\xi_1(x_1,x_2) \tag{2.5}$$

provided that $\mu(x_1,x_2) \neq 0$ (see [8]).

Theorem 2.3. *System (1.1) admits the Lie group G with the following infinitesimal generator V:*

$$V = p_1x_1\frac{\partial}{\partial x_1} + \cdots + p_nx_n\frac{\partial}{\partial x_n}. \tag{2.6}$$

Proof. One can obtain the result by straightforward computing by using Proposition 2.1.

For example, the system of Euler-Poisson equations is a quasihomogeneous system with

$$p_1 = p_2 = p_3 = 1, \qquad p_4 = p_5 = p_6 = 2, \qquad l = 1, \tag{2.7}$$

and it admits Lie group with infinitesimal generator V,

$$V = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + 2x_4\frac{\partial}{\partial x_4} + 2x_5\frac{\partial}{\partial x_5} + 2x_6\frac{\partial}{\partial x_6}. \tag{2.8}$$

In [5], some first integrals of the Euler-Poisson equations system are obtained by using the quasihomogeneous property of the system.

It is well known that, given a polynomial $f \in R[x_1,x_2,\dots,x_n]$, we can split it in the form $f = f_m + f_{m+1} + \cdots + f_{m+r}$, where f_k $(k = m,m+1,\dots,m+r)$ is a p_i $(i = 1,2,\dots,n)$ quasihomogeneous polynomial of weighted degree k, that is,

$$f_k(\alpha^{p_1}x_1,\dots,\alpha^{p_n}x_n) = \alpha^k f_k(x_1,\dots,x_2) \tag{2.9}$$

for $k = m,m+1,\dots,m+r$. We have the following result. □

Theorem 2.4. *Let f be a polynomial in the variables $x_1,x_2,\dots,x_n$ and let*

$$f = f_m + f_{m+1} + \cdots + f_{m+r} \tag{2.10}$$

be its decomposition into p_i $(i = 1,2,\dots,n)$ quasihomogeneous polynomial of weighted degree $m+i$ for $i = 0,1,\dots,r$, then f is either a polynomial first integral or a polynomial inverse integrating factor of system (1.1) if and only if each quasihomogeneous polynomial f_{m+i} is either a first integral or an integrating factor of system (1.1) for $i = 0,1,\dots,r$, respectively.

Proof. If f is a polynomial first integral, the result is proved in [4]. Hence we will proof the case in which f is a polynomial inverse integrating factor of system (1.1).

The sufficiency is obvious. So we will only prove the necessity. From Definition 2.2, we have

$$X_1(x)\frac{\partial f}{\partial x_1} + X_2(x)\frac{\partial f}{\partial x_2} + \cdots + X_n(x)\frac{\partial f}{\partial x_n} = \left(\frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right)f, \tag{2.11}$$

that is,

$$\sum_{i=0}^{r}\left(X_1(x)\frac{\partial f_{m+i}}{\partial x_1} + X_2(x)\frac{\partial f_{m+i}}{\partial x_2} + \cdots + X_n(x)\frac{\partial f_{m+i}}{\partial x_n}\right) = \sum_{i=0}^{r}\left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n}\right)f_{m+i}. \tag{2.12}$$

Since $X_j(x)$ $(j = 1,2,\dots,n)$ have weight degrees $p_j + l$ $(j = 1,2,\dots,n)$, then the divergence of system (1.1)

$$\operatorname{div} X = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n}, \tag{2.13}$$

has weighted degree l. Similarly, $\partial f_{m+i}/\partial x_j$ $(j = 1,2,\dots,n)$ have weighted degrees $m+i-p_j$ $(j = 1,2,\dots,n)$, respectively. So, from the quasihomogeneous polynomial components on the left- and right-hand sides of being of weighted degree $l+m+i$, we can obtain

$$X_1(x)\frac{\partial f_{m+i}}{\partial x_1} + X_2(x)\frac{\partial f_{m+i}}{\partial x_2} + \cdots + X_n(x)\frac{\partial f_{m+i}}{\partial x_n} = \left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n}\right)f_{m+i}, \tag{2.14}$$

where $i = 0, 1, \dots, r$. Consequently, f_{m+i} is an inverse integrating factor of system (1.1) and hence this completes the proof. □

From Theorem 2.4, in order to study either polynomial first integrals or polynomial inverse integrating factors of quasihomogeneous polynomial system, we need only to consider quasihomogeneous polynomial functions.

THEOREM 2.5. *Any inverse integrating factor of system (1.1) is a quasihomogeneous function. Moreover, if*

$$\begin{gathered} \overline{X_i} - w_i \overline{X_1} \frac{p_i}{p_1} \neq 0 \quad (i = 2, 3, \dots, n), \\ \operatorname{div} \overline{X} - \overline{X_1} \frac{m}{p_1} \neq 0, \end{gathered} \tag{2.15}$$

where $\overline{X_i} = X_i(1, w_2, w_3, \dots, w_n)$, then $w_1^{m/p_1} \overline{f_m}$ is an inverse integrating factor of weighted degree m of system (1.1), where $\overline{f_m} = f_m(1, w_2, w_3, \dots, w_n)$ satisfies the following equations:

$$\begin{aligned} \frac{dw_2}{\overline{X_2} - (p_2/p_1) w_2 \overline{X_1}} &= \frac{dw_3}{\overline{X_3} - (p_3/p_1) w_3 \overline{X_1}} \\ &= \cdots = \frac{dw_n}{\overline{X_n} - (p_n/p_1) w_n \overline{X_1}} \\ &= \frac{\overline{f_m}}{\operatorname{div} \overline{X} - (m/p_1) \overline{X_1}}. \end{aligned} \tag{2.16}$$

Proof. Let $f(x_1, \dots, x_n)$ be an inverse integrating factor of system (1.1), that is,

$$X_1 \frac{\partial f}{\partial x_1} + X_2 \frac{\partial f}{\partial x_2} + \cdots + X_n \frac{\partial f}{\partial x_n} = \left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n} \right) f, \tag{2.17}$$

because $X_1, \dots, X_n$ and $\operatorname{div} X$ are quasihomogeneous functions of weighted degrees $p_1 + l, \dots, p_n + l$ and l, respectively. It is not difficult to obtain that (2.17) is invariant under a change of $(x_1, x_2, \dots, x_n) \to (\alpha^{p_1} x_1, \alpha^{p_2} x_2, \dots, \alpha^{p_n} x_n)$. Consequently, their solutions are also invariant, that is,

$$f(\alpha^{p_1} x_1, \dots, \alpha^{p_n} x_n) = f(x_1, \dots, x_n) \tag{2.18}$$

or

$$f(\alpha^{p_1} x_1, \dots, \alpha^{p_n} x_n) = \alpha^m f(x_1, \dots, x_n). \tag{2.19}$$

So f is a $p_1, \dots, p_n$ quasihomogeneous function (of weighted degree m).

Letting $f_m(x_1, \dots, x_n)$ be a quasihomogeneous function of weighted degree m, we have $f(\alpha^{p_1} x_1, \dots, \alpha^{p_n} x_n) = \alpha^m f(x_1, \dots, x_n)$. If f_m is an inverse integrating factor of system (1.1), then the following equation holds:

$$X_1 \frac{\partial f_m}{\partial x_1} + X_2 \frac{\partial f_m}{\partial x_2} + \cdots + X_n \frac{\partial f_m}{\partial x_n} = (\operatorname{div} X) f_m. \tag{2.20}$$

Now, let

$$w_1 = x_1, \qquad w_2 = \frac{x_2}{x_1^{p_2/p_1}}, \dots, w_n = \frac{x_n}{x_1^{p_n/p_1}}, \tag{2.21}$$

then,

$$\begin{aligned} X_i(x_1,\dots,x_n) &= X_i\left(w_1, w_2 w_1^{p_2/p_1}, \dots, w_n w_1^{p_n/p_1}\right) \\ &= w_1^{(p_i+l)/p_1} X_i(1, w_2, \dots, w_n) = w_1^{(p_i+l)/p_1}\overline{X_i}, \quad i = 1,2,\dots,n, \\ \operatorname{div} X(x_1,\dots,x_n) &= \operatorname{div} X\left(w_1, w_2 w_1^{p_2/p_1}, \dots, w_n w_1^{p_n/p_1}\right) \\ &= w_1^{l/p_1} \operatorname{div} X(1, w_2, \dots, w_n) = w_1^{l/p_1} \operatorname{div} \overline{X}. \end{aligned} \tag{2.22}$$

On the other hand, by the chain rule of the derivative, in the new variables $w_1, w_2, \dots, w_n$, (2.20) becomes

$$\begin{aligned} & w_1^{(p_1+l)/p_1}\overline{X_1}\frac{\partial f_m\left(w_1, w_2 w_1^{p_2/p_1}, \dots, w_n w_1^{p_n/p_1}\right)}{\partial x_1} \\ &\quad + w_1^{(p_2+l)/p_1}\overline{X_2}\frac{\partial f_m\left(w_1, w_2 w_1^{p_2/p_1} + \cdots + w_n w_1^{p_n/p_1}\right)}{\partial x_2} \\ &\quad + \cdots + w_1^{(p_n+l)/p_1}\overline{X_n}\frac{\partial f_m\left(w_1, w_2 w_1^{p_2/p_1}, \dots, w_n w_1^{p_n/p_1}\right)}{\partial x_n} \\ &= \left(w_1^{l/p_1} \operatorname{div}\overline{X}\right) f_m\left(w_1, w_2 w_1^{p_2/p_1}, \dots, w_n w_1^{p_n/p_1}\right). \end{aligned} \tag{2.23}$$

Based on the following formulas:

$$f_m\left(w_1, w_2 w_1^{p_2/p_1}, \dots, w_n w_1^{p_n/p_1}\right) = w_1^{m/p_1} f_m(1, w_2, \dots, w_n) = w_1^{m/p_1}\overline{f_m}, \tag{2.24}$$

$$\begin{aligned} \frac{\partial f_m}{\partial x_1} &= \frac{m}{p_1} w^{(m/p_1)-1}\overline{f_m} - \frac{p_2}{p_1}\frac{w_2}{w_1} w_1^{m/p_1}\frac{\partial \overline{f_m}}{\partial w_2} - \cdots - \frac{p_n}{p_1}\frac{w_n}{w_1} w_1^{m/p_1}\frac{\partial \overline{f_m}}{\partial w_n}, \\ \frac{\partial f_m}{\partial x_i} &= w_1^{-(m-p_i)/P_1}\frac{\partial \overline{f_m}}{\partial w_i}, \quad i = 2,\dots,n, \end{aligned} \tag{2.25}$$

(2.23) becomes

$$\left(\overline{X_2} - \frac{p_2}{p_1} w_2 \overline{X_1}\right)\frac{\partial \overline{f_m}}{\partial w_2} + \cdots + \left(\overline{X_n} - \frac{p_n}{p_1} w_n \overline{X_1}\right)\frac{\partial \overline{f_m}}{\partial w_n} = \left(\operatorname{div}\overline{X} - \frac{m}{p_1}\overline{X_1}\right)\overline{f_m}. \tag{2.26}$$

Obviously, its characteristic equation is (2.16). So $\overline{f_m}$ satisfies (2.16). According to the formula (2.24), this completes the proof. □

Example 2.6. We consider the following system:

$$\frac{dx}{dt} = axy, \qquad \frac{dy}{dt} = bx^3 + cy^2. \tag{2.27}$$

This system is a $p_1 = 2$, $p_2 = 3$ quasihomogeneous polynomial system of weighted degree 3, and it is invariant under the similarity transformation

$$(x, y, t) \longrightarrow (\alpha^2 x, \alpha^3 y, \alpha^{-3} t). \tag{2.28}$$

It is easy to get the following formulas:

$$\begin{aligned} \overline{X_1} &= X_1(1, w_2) = a w_2, \\ \overline{X_2} &= X_2(1, w_2) = b + c w_2^2, \\ \operatorname{div} \overline{X} &= \operatorname{div} X(1, w_2) = (a + 2c) w_2. \end{aligned} \tag{2.29}$$

From (2.16), we have

$$\frac{dw_2}{b + (c - (3/2)a) w_2^2} = \frac{d\overline{f_m}}{(-ma/2 - a - 2c) w_2 \overline{f_m}}. \tag{2.30}$$

Its solution is

$$\overline{f_m} = c\left(b + \left(c - \frac{3}{2}a\right) w_2^2\right)^{(-ma/2 - a - 2c)/(2c - 3a)}. \tag{2.31}$$

So

$$f_m\left(1, \frac{y}{x^{3/2}}\right) = c\left(b + \left(c - \frac{3}{2}a\right) \frac{y^2}{x^3}\right)^{(-ma/2 - a - 2c)/(2c - 3a)}. \tag{2.32}$$

Based on Theorem 2.5, we can get an inverse integrating factor

$$x^{m/2}\left(b + \left(c - \frac{3}{2}a\right) \frac{y^2}{x^3}\right)^{(-ma/2 - a - 2c)/(2c - 3a)} \tag{2.33}$$

of the system. Specially, when $m = 2$, the inverse integrating factor is

$$x\left(b + \left(c - \frac{3}{2}a\right) \frac{y^2}{x^3}\right)^{(-2a - 2c)/(2c - 3a)}. \tag{2.34}$$

3. On the integrability of quasidegenerate infinity systems

We consider the quasidegenerate infinity system (1.3).

Lemma 3.1. *Let* $X^* = (X_1 + p_1 x_1 A(x_1, x_2, \ldots, x_n))(\partial/\partial x_1) + \cdots + (X_n + p_n x_n A(x_1, x_2, \ldots, x_n))(\partial/\partial x_n)$ *be the vector field associated with system (1.3) and let* $\Omega(x_1, x_2, \ldots, x_n)$ *be a quasihomogeneous first integral of weighted degree* d *of system (1.1), then*

$$X^* \Omega = d A(x_1, x_2, \ldots, x_n) \Omega. \tag{3.1}$$

Proof. The derivative of $\Omega(x_1,x_2,\dots,x_n)$ along the orbits of system (1.3) is

$$\begin{aligned} X^*\Omega &= \frac{\partial\Omega}{\partial x_1}(X_1+p_1x_1A)+\frac{\partial\Omega}{\partial x_2}(X_2+p_2x_2A)+\cdots+\frac{\partial\Omega}{\partial x_n}(X_n+p_nx_nA) \\ &= \left(\frac{\partial\Omega}{\partial x_1}X_1+\frac{\partial\Omega}{\partial x_2}X_2+\cdots+\frac{\partial\Omega}{\partial x_n}X_n\right)+A\left(p_1x_1\frac{\partial\Omega}{\partial x_1}+\cdots+p_nx_n\frac{\partial\Omega}{\partial x_n}\right) \\ &= A\left(p_1x_1\frac{\partial\Omega}{\partial x_1}+\cdots+p_nx_n\frac{\partial\Omega}{\partial x_n}\right). \end{aligned} \tag{3.2}$$

Based on the generalized Euler's theorem for quasihomogeneous function, we have

$$p_1x_1\frac{\partial\Omega}{\partial x_1}+\cdots+p_nx_n\frac{\partial\Omega}{\partial x_n}=d\Omega. \tag{3.3}$$

So, (3.2) becomes

$$X^*\Omega = dA\Omega. \tag{3.4}$$

□

Lemma 3.2. *Let $f(x_1,\dots,x_n)$ be a quasihomogeneous inverse integrating factor of weighted degree m of system (1.1), then $f(x_1,\dots,x_n)$ is a quasihomogeneous invariant manifold of system (1.3).*

Proof. Because $f(x_1,\dots,x_n)$ is an inverse integrating factor of system (1.1), we have

$$X_1\frac{\partial f}{\partial x_1}+\cdots+X_n\frac{\partial f}{\partial x_n}=\left(\frac{\partial X_1}{\partial x_1}+\cdots+\frac{\partial X_n}{\partial x_n}\right)f. \tag{3.5}$$

The derivative of $f(x_1,\dots,x_n)$ along the orbits of system (1.3) is

$$\begin{aligned} X^*f &= \frac{\partial f}{\partial x_1}(X_1+p_1x_1A)+\frac{\partial f}{\partial x_2}(X_2+p_2x_2A)+\cdots+\frac{\partial f}{\partial x_n}(X_n+p_nx_nA) \\ &= \left(\frac{\partial f}{\partial x_1}X_1+\frac{\partial f}{\partial x_2}X_2+\cdots+\frac{\partial f}{\partial x_n}X_n\right)+A\left(p_1x_1\frac{\partial f}{\partial x_1}+\cdots+p_nx_n\frac{\partial f}{\partial x_n}\right) \\ &= (\operatorname{div}X)f+mAf. \end{aligned} \tag{3.6}$$

The last term of the above expression can be obtained by using the generalized Euler's theorem for quasihomogeneous function. So

$$X^*f=(\operatorname{div}X+mA)f, \tag{3.7}$$

that is, $f(x_1,x_2,\dots,x_n)=0$ is an invariant manifold of system (1.3). □

Theorem 3.3. *Let $\Omega(x_1,x_2,\dots,x_n)$ be a quasihomogeneous first integral of weighted degree d of system (1.1), then $\Omega^{(\alpha-l)/d}f$ is an inverse integrating factor of system (1.3).*

Proof. First, we calculate the divergence of system (1.3):

$$\begin{aligned}\operatorname{div} X^* &= \frac{\partial}{\partial x_1}(X_1 + p_1x_1A) + \frac{\partial}{\partial x_2}(X_2 + p_2x_2A) + \cdots + \frac{\partial}{\partial x_n}(X_n + p_nx_nA)\\ &= \left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n}\right) + \left(p_1x_1\frac{\partial A}{\partial x_1} + \cdots + p_nx_n\frac{\partial A}{\partial x_n}\right) + A(p_1 + p_2 + \cdots + p_n)\\ &= \operatorname{div} X + A(p_1 + p_2 + \cdots + p_n + \alpha).\end{aligned} \tag{3.8}$$

On the other hand, from the proves of Lemmas 3.1 and 3.2, we have

$$\begin{gathered} X^*\Omega = dA\Omega,\\ X^* f = (\operatorname{div} X + mA)f. \end{gathered} \tag{3.9}$$

Let

$$\begin{gathered} K_1(x_1,x_2,\ldots,x_n) = dA,\\ K_2(x_1,x_2,\ldots,x_n) = \operatorname{div} X + A(p_1 + p_2 + \cdots + p_n + l). \end{gathered} \tag{3.10}$$

So, we can find two constants λ_1 and λ_2 such that

$$\sum_{i=1}^{n} \lambda_i K_i(x_1,x_2,\ldots,x_n) = \operatorname{div} X^*, \tag{3.11}$$

that is, $\lambda_1 = (\alpha - l)/d$ and $\lambda_2 = 1$. Therefore, applying the Darboux's theory of integrability (see [12]), we obtain that the function $\Omega^{(\alpha-l)/d} f$ is an inverse integrating factor of system (1.3). □

4. Conclusion

In this paper, we have studied the integrability of quasihomogeneous systems. From the above investigation, we see that the properties of quasihomogeneous systems may help us in studying the integrability of the systems. We need only to consider quasihomogeneous polynomial functions in order to study either polynomial first integrals or polynomial inverse integrating factors of quasihomogeneous systems. Specially, we have proposed a method to obtain an inverse integrating factor of the systems on the base of the systems of ordinary differential equations established by using the quasihomogeneous vector fields. Moreover, we also have considered quasidegenerate infinity systems, and shown how to obtain an inverse integrating factor from the first integrals of the corresponding quasihomogeneous systems by using Darboux's theory of integrability.

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