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Research Article

On Nonresonance Problems of Second-Order Difference Systems

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Let T be an integer with $T \geq 3$, and let $\mathbb{T} := \{1, \dots, T\}$. We study the existence and uniqueness of solutions for the following two-point boundary value problems of second-order difference systems: $\Delta^2 u(t-1) + f(t, u(t)) = e(t)$, $t \in \mathbb{T}$, $u(0) = u(T+1) = 0$, where $e : \mathbb{T} \to \mathbb{R}^n$ and $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is a potential function satisfying $f(t, \cdot) \in C^1(\mathbb{R}^n)$ and some nonresonance conditions. The proof of the main result is based upon a mini-max theorem.

1. Introduction

The existence and uniqueness of solutions of nonresonance problems of differential equations have been studied extensively (see [1–5], and the references therein). However, very few results have been established for nonresonance problems of differential equations. Although we have seen some results of the existence of solutions of discrete equations subjected to diverse boundary conditions, such as in [6–13], none of them addresses the nonresonance problems.

In this paper, we consider nonlinear boundary value problems of second-order difference systems of the form

$$\begin{gathered} \Delta^2 u(t-1) + f\big(t, u(t)\big) = e(t), \quad t \in \mathbb{T}, \\ u(0) = u(T+1) = 0, \end{gathered} \tag{1.1}$$

where $e : \mathbb{T} \to \mathbb{R}^n$ and $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is a potential vector-valued function for $t \in \mathbb{T}$.

Why do we pay attention to the discrete problem (1.1)? Note that the continuous eigenvalue problem

$$y''(t) + \eta u(t) = 0, \quad t \in (0,1),$$
$$u(0) = u(1) = 0 \tag{1.2}$$

has a sequence of eigenvalues

$$\eta_1 < \eta_2 < \cdots < \eta_n < \cdots \longrightarrow \infty, \tag{1.3}$$

while the discrete eigenvalue problem

$$\Delta^2 y(t-1) + \mu y(t) = 0, \quad t \in \mathbb{T},$$
$$y(0) = y(T+1) = 0 \tag{1.4}$$

has exactly T real eigenvalues

$$\mu_1 < \mu_2 < \cdots < \mu_T, \tag{1.5}$$

where T is an integer with $T \geq 3$, $\mathbb{T} := \{1, \ldots, T\}$, and $\widehat{\mathbb{T}} := \{0, 1, \ldots, T+1\}$. Thus, the study of nonresonance problems near the largest eigenvalue μ_T is new and interesting.

Furthermore, the eigenspace corresponding to any eigenvalue in (1.5) is one-dimensional; see [14] for more extensive discussion of these topics. For every $e_1 : \mathbb{T} \to \mathbb{R}^1$, the corresponding nonhomogeneous problem

$$\Delta^2 y(t-1) + \mu y(t) = e_1(t), \quad t \in \mathbb{T},$$
$$y(0) = y(T+1) = 0 \tag{1.6}$$

has a unique solution if $\mu \notin \{\mu_1, \ldots, \mu_T\}$. The purpose of this paper is to provide some nonresonance conditions which guarantee the existence and uniqueness of solutions of (1.1). Especially, we allow that the nonlinearity may be superlinear at ∞. The main tool in this paper is a mini-max theorem due to Lazer [1].

2. Statement of the main result

In this section, we state our main result. First, we need to introduce some notations and preliminary results.

Let $\langle \cdot, \cdot \rangle_n$ be the usual scalar product in $\mathbb{R}^n$. Let $\mathrm{LS}(\mathbb{R}^n)$ be the set of all symmetric $n \times n$ real matrices. For $A, B \in \mathrm{LS}(R^n)$, we say that $A \preccurlyeq B$ if

$$\langle (B-A)\xi, \xi \rangle_n \geq 0 \quad \forall \xi \in \mathbb{R}^n. \tag{2.1}$$

Lemma 2.1. *Let $A, B \in \mathrm{LS}(\mathbb{R}^n)$ be two commutative matrices with $A \preccurlyeq B$. Let $\lambda_1^A < \lambda_2^A < \cdots < \lambda_n^A$ be the eigenvalues of A, and let w_k be the eigenvector corresponding to λ_k^A. Then, there exists γ_k*

such that

$$Bw_k = \gamma_k w_k, \quad k = 1, \ldots, n. \tag{2.2}$$

Proof. Since A and B are commutative, we have that

$$A(Bw_k) = B(Aw_k) = B(\lambda_k^A w_k) = \lambda_k^A (Bw_k). \tag{2.3}$$

Then, there exists γ_k such that $Bw_k = \gamma_k w_k$. □

Remark 2.2. It is worth remarking that the conditions of Lemma 2.1 cannot guarantee that

$$\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n. \tag{2.4}$$

Therefore, [4, Assumption (H2.2)] is not suitable.

Lemma 2.3. *Let $A, B \in \mathrm{LS}(\mathbb{R}^n)$ be two commutative matrices with $A \preccurlyeq B$. Let $\lambda_1^A < \lambda_2^A < \cdots < \lambda_n^A$ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the eigenvalues of A and B, respectively. Let w_k be the eigenvector corresponding to both λ_k^A and γ_k. Then,*

$$\lambda_k^A \leq \gamma_k, \quad k = 1, \ldots, n. \tag{2.5}$$

Proof. From the fact that

$$Aw_k = \lambda_k^A w_k, \qquad Bw_k = \gamma_k w_k, \tag{2.6}$$

we have

$$(B - A)w_k = (\gamma_k - \lambda_k^A) w_k. \tag{2.7}$$

Subsequently,

$$0 \leq \langle (B - A)w_k, w_k \rangle_n = \langle (\gamma_k - \lambda_k^A) w_k, w_k \rangle_n = (\gamma_k - \lambda_k^A) \langle w_k, w_k \rangle_n. \tag{2.8}$$

This implies that $\gamma_k \geq \lambda_k^A$ for $k = 1, \ldots, n$. □

Definition 2.4. One says that $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$, $f(t, \xi) = (f_1(t, \xi_1, \ldots, \xi_n), \ldots, f_n(t, \xi_1, \ldots, \xi_n))$, is a potential vector-valued function for $t \in \mathbb{T}$ if there exists a function $G : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^1$ such that

$$f_i(t, \xi_1, \ldots, \xi_n) = \frac{\partial}{\partial \xi_i} G(t, \xi_1, \ldots, \xi_n), \quad i = 1, \ldots, n, \tag{2.9}$$

for all $(t, \xi_1, \ldots, \xi_n) \in \mathbb{T} \times \mathbb{R}^n$.

Let one suppose that

(H1) $f(t, \cdot) \in C^1(\mathbb{R}^n)$ for $t \in \mathbb{T}$;

(H2) $f(t, \xi)$ is a potential function for $t \in \mathbb{T}$.

Denote

$$\mathscr{H}_{f(t,\xi)} = \begin{pmatrix} f_{11}(t,\xi_1,\ldots,\xi_n) & \cdots & f_{1n}(t,\xi_1,\ldots,\xi_n) \\ \vdots & \vdots & \vdots \\ f_{n1}(t,\xi_1,\ldots,\xi_n) & \cdots & f_{nn}(t,\xi_1,\ldots,\xi_n) \end{pmatrix}, \tag{2.10}$$

where $\xi = (\xi_1,\ldots,\xi_n)$, $f_{ij}(t,\xi_1,\ldots,\xi_n) = (\partial/\partial\xi_j) f_i(t,\xi_1,\ldots,\xi_n)$.

The following theorem is our main result.

Theorem 2.5. *Let (H1) and (H2) hold. Assume that*

(H3) *there exist two diagonal matrices A and B:*

$$A = \begin{pmatrix} \lambda_1^A & 0 & \cdots & 0 \\ 0 & \lambda_2^A & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^A \end{pmatrix}, \qquad B = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \gamma_n \end{pmatrix}, \tag{2.11}$$

where $\lambda_1^A < \lambda_2^A < \cdots < \lambda_n^A$ such that one of the following conditions holds:

(a) *$A \preccurlyeq \mathscr{H}_{f(t,\xi)} \preccurlyeq B$, $[\lambda_1^A, \max\{\gamma_k \mid k = 1,\ldots,n\}] \subset (\mu_1, \mu_T)$, and $\cup_{k=1}^n [\lambda_k^A, \gamma_k] \cap \{\mu_1,\ldots,\mu_T\} = \varnothing$;*

(b) *$\mathscr{H}_{f(t,\xi)} \preccurlyeq A$ and $\lambda_n^A < \mu_1$;*

(c) *$B \preccurlyeq \mathscr{H}_{f(t,\xi)}$ and $\min\{\gamma_k \mid k = 1,\ldots,n\} > \mu_T$.*

Then, the boundary value problem (1.1) *has exact one solution for every $e : \mathbb{T} \to \mathbb{R}^n$.*

Remark 2.6. In (a) in (H3), we use the revised interval $[\lambda_k^A, \gamma_k]$ to replace the interval $[\lambda_k^1, \lambda_k^2]$ which was used in [4, Assumption (H2.2)].

Remark 2.7. (c) in (H3) allows that the nonlinearity f may be superlinear at $+\infty$ and $-\infty$.

3. The main tools

Lemma 3.1 (see [1]). *Let X and Y be two closed subspaces of a real Hilbert space H such that X is finite-dimensional and $H = X \oplus Y$. Let $f : H \to \mathbb{R}$ be a functional and let ∇f and $D^2 f$ denote the gradient and Hessian of f, respectively. Suppose that there exist two positive constants m_1 and m_2 such that*

$$\begin{aligned} (D^2 f(u)h, h) &\le -m_1 \|h\|^2, \\ (D^2 f(u)k, k) &\ge m_2 \|k\|^2 \end{aligned} \tag{3.1}$$

for all $u \in H$, $h \in X$, $k \in Y$. Then, f has a unique critical point. Moreover, this critical point of f is characterized by the equality

$$f(v) = \max_{x\in X} \min_{y\in Y} f(x+y). \tag{3.2}$$

In order to introduce the other tool, we give firstly some notations. Let E denote real Banach space with norm $\|\cdot\|_E$. If E^* is the topological dual of E, then the symbol $\langle\cdot,\cdot\rangle$ will denote the duality pair between E and E^*.

Let $\{u_n\}$ be a sequence in E. We say that u_n converges weakly to u, written as $u_n \rightharpoonup u$, if $\langle\psi, u_n - u\rangle \to 0$ as $n \to \infty$ for all $\psi \in E^*$.

Let $g : E \to \mathbb{R}$ be a functional. We say that g is weakly continuous if for every $\{u_n\} \subset E$ with $u_n \rightharpoonup u$, we have that

$$g(u) = \lim_{n\to\infty} g(u_n). \tag{3.3}$$

We say that g is weakly lower semicontinuous if $\{u_n\} \subset E$ and $u_n \rightharpoonup u$ imply that

$$g(u) \le \liminf_{n\to\infty} g(u_n). \tag{3.4}$$

We say that g is weakly upper semicontinuous if $\{u_n\} \subset E$ and $u_n \rightharpoonup u$ imply that

$$g(u) \ge \limsup_{n\to\infty} g(u_n). \tag{3.5}$$

Lemma 3.2 (see [15, Theorem 1.7, page 417]). *Let E be a real reflexive Banach space. Let $f : E \to \mathbb{R}$ be weakly upper semicontinuous (resp., weakly lower semicontinuous) and let it satisfy*

$$\lim_{\|x\|\to\infty} f(x) = +\infty \quad \left(\text{resp.},\ \lim_{\|x\|\to\infty} f(x) = -\infty\right). \tag{3.6}$$

Then, there exists $x_0 \in E$ such that

$$f(x_0) = \min_{x\in E} f(x) \quad \left(\text{resp.},\ f(x_0) = \max_{x\in E} f(x)\right). \tag{3.7}$$

4. Preliminary lemmas

In this section, we give and prove some preliminary lemmas which are necessary for the proof of the main result, Theorem 2.5.

Let

$$\begin{gathered}\widetilde{D} = \{u \mid u : \widehat{\mathbb{T}} \longrightarrow \mathbb{R},\ u(0) = u(T+1) = 0\},\\ D = \{u \mid u : \mathbb{T} \longrightarrow \mathbb{R}\}.\end{gathered} \tag{4.1}$$

Let

$$\begin{gathered}\widetilde{H} = \{y : \widehat{\mathbb{T}} \longrightarrow \mathbb{R}^n \mid y = (y_1, \dots, y_n)^T,\ y_i \in \widetilde{D},\ i = 1, \dots, n\},\\ H = \{z : \mathbb{T} \longrightarrow \mathbb{R}^n \mid z = (z_1, \dots, z_n)^T,\ z_i \in D,\ i = 1, \dots, n\}.\end{gathered} \tag{4.2}$$

For $\widetilde{u}, \widetilde{v} \in \widetilde{H}$ with

$$\widetilde{u}(t) = \begin{pmatrix}\widetilde{u}_1(t)\\ \vdots \\ \widetilde{u}_n(t)\end{pmatrix}, \quad \widetilde{v}(t) = \begin{pmatrix}\widetilde{v}_1(t)\\ \vdots \\ \widetilde{v}_n(t)\end{pmatrix}, \quad t \in \widehat{\mathbb{T}}, \tag{4.3}$$

let us define the inner product

$$\langle\langle \tilde{u}, \tilde{v} \rangle\rangle = \sum_{t=0}^{T+1} \langle \tilde{u}(t), \tilde{v}(t) \rangle_n = \sum_{t=1}^{T} \sum_{j=1}^{n} \tilde{u}_j(t) \tilde{v}_j(t). \tag{4.4}$$

Similarly for $u, v \in H$ with

$$u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}, \quad v(t) = \begin{pmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{pmatrix}, \quad t \in \mathbb{T}, \tag{4.5}$$

we also define the inner product

$$\langle\langle u, v \rangle\rangle = \sum_{t=1}^{T} \langle u(t), v(t) \rangle_n = \sum_{t=1}^{T} \sum_{j=1}^{n} \tilde{u}_j(t) \tilde{v}_j(t). \tag{4.6}$$

Then, both $(\widetilde{H}, \langle\langle \cdot \rangle\rangle)$ and $(H, \langle\langle \cdot \rangle\rangle)$ are Hilbert spaces.

Lemma 4.1. *Let $u, w \in \widetilde{D}$. Then,*

$$\sum_{k=1}^{T} w(k) \Delta^2 u(k-1) = -\sum_{k=0}^{T} \Delta u(k) \Delta w(k). \tag{4.7}$$

Proof. Using "summation by parts"(see [14, Theorem 2.8]), we have

$$\begin{aligned} \sum_{k=1}^{T} w(k) \Delta^2 u(k-1) &= \left[w(k) \Delta u(k-1) \right]_1^{T+1} - \sum_{k=1}^{T} \Delta w(k) \Delta u(k) \\ &= -w(1) \Delta u(0) - \sum_{k=1}^{T} \Delta w(k) \Delta u(k) \\ &= -\Delta w(0) \Delta u(0) - \sum_{k=1}^{T} \Delta w(k) \Delta u(k) \\ &= -\sum_{k=0}^{T} \Delta u(k) \Delta w(k). \end{aligned} \tag{4.8}$$

□

Define a functional $J : \widetilde{H} \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} \sum_{t=0}^{T} \langle \Delta u(t), \Delta u(t) \rangle_n - \sum_{t=1}^{T} G(t, u(t)) + \sum_{t=1}^{T} \langle u(t), e(t) \rangle_n, \tag{4.9}$$

where G satisfies

$$\operatorname{grad} G(t, \xi) = f(t, \xi) = \left(f_1(t, \xi_1, \ldots, \xi_n), \ldots, f_n(t, \xi_1, \ldots, \xi_n) \right)^T. \tag{4.10}$$

Lemma 4.2. *Let (H1) and (H2) hold. Then, $J : \widetilde{H} \to \mathbb{R}^n$ is weakly semicontinuous and $J \in C^2$.*

Proof. The proof is standard; so we omit it. □

Lemma 4.3. *$u \in \widetilde{H}$ is a critical point of J if and only if u is a solution of* (1.1)*.*

Proof. It is an immediate consequence of Lemma 4.1 and the definition of the Gâteaux-differentiation. □

From now on we assume that the eigenfunction φ_k corresponding to the eigenvalue μ_k satisfies

$$\sum_{t=1}^{T}\varphi_k(t)\varphi_k(t) = 1. \tag{4.11}$$

The following result is a special case of [14, Theorem 7.2].

Lemma 4.4. *$\sum_{t=1}^{T}\varphi_k(t)\varphi_j(t) = 0$ for $k, j \in \mathbb{T}$ with $k \neq j$.*

5. Proof of the main result

Now, we give the proof of Theorem 2.5. We divide the proof into three cases.

Case 1 ($[\lambda_1^A, \max\{\gamma_k \mid k = 1,\ldots,n\}] \subset (\mu_1,\ \mu_T)$ and $\cup_{k=1}^{n}[\lambda_k^A,\gamma_k] \cap \{\mu_1,\ldots,\mu_T\} = \varnothing$). For $k \in \{1,\ldots,n\}$, we define two sets Z_k and Y_k as

$$Z_k = \begin{cases} \{0\} & \text{as } [\lambda_k^A,\gamma_k] \subset (-\infty,\mu_1), \\ \operatorname{span}\{\varphi_1,\ldots,\varphi_{m_k}\} & \text{as } [\lambda_k^A,\gamma_k] \subset (\mu_{m_k},\mu_{m_k+1}), \\ \operatorname{span}\{\varphi_1,\ldots,\varphi_T\} & \text{as } [\lambda_k^A,\gamma_k] \subset (\mu_T,\infty), \end{cases}$$
$$Y_k = \begin{cases} \operatorname{span}\{\varphi_1,\ldots,\varphi_T\} & \text{as } [\lambda_k^A,\gamma_k] \subset (-\infty,\mu_1), \\ \operatorname{span}\{\varphi_{m_k+1},\ldots,\varphi_T\} & \text{as } [\lambda_k^A,\gamma_k] \subset (\mu_{m_k},\mu_{m_k+1}), \\ \{0\} & \text{as } [\lambda_k^A,\gamma_k] \subset (\mu_T,\infty). \end{cases} \tag{5.1}$$

For $u \in \widetilde{H}$ with

$$u(t) = \begin{pmatrix} c_{11}\varphi_1 + \cdots + c_{1T}\varphi_T \\ \vdots \\ c_{n1}\varphi_1 + \cdots + c_{nT}\varphi_T \end{pmatrix}, \tag{5.2}$$

we define the orthogonal projectors $P : \widetilde{H} \to Z_1 \times \cdots \times Z_n$ and $Q : \widetilde{H} \to Y_1 \times \cdots \times Y_n$ by

$$Pu = \begin{pmatrix} c_{11}\varphi_1 + \cdots + c_{1m_1}\varphi_{m_1} \\ \vdots \\ c_{n1}\varphi_1 + \cdots + c_{nm_n}\varphi_{m_n} \end{pmatrix}, \qquad Qu = \begin{pmatrix} c_{1m_1+1}\varphi_{m_1+1} + \cdots + c_{1T}\varphi_T \\ \vdots \\ c_{nm_n+1}\varphi_{m_n+1} + \cdots + c_{nT}\varphi_T \end{pmatrix}. \tag{5.3}$$

Let

$$X = \{x \in \widetilde{H} \mid x = Pu\}, \qquad Y = \{y \in \widetilde{H} \mid y = Qu\}. \tag{5.4}$$

By (a) in (H3),

$$\widetilde{H} = X \oplus Y, \qquad X \perp Y. \tag{5.5}$$

Let us consider the functional $J : \widetilde{H} \to \mathbb{R}$ which is defined in (4.9):

$$J(u) = \frac{1}{2}\sum_{t=0}^{T}\langle \Delta u(t), \Delta u(t)\rangle_n - \sum_{t=1}^{T} G(t, u(t)) + \sum_{t=1}^{T}\langle u(t), e(t)\rangle_n. \tag{5.6}$$

It is easy to check that for $h, k \in \widetilde{H}$,

$$\begin{gathered}\langle\langle \nabla J(u(t)), h(t)\rangle\rangle = \sum_{t=1}^{T}\langle -\Delta^2 u(t-1), h(t)\rangle_n - \sum_{t=1}^{T}\langle f(t,u), h(t)\rangle_n + \sum_{t=1}^{T}\langle e(t), h(t)\rangle_n, \\ \langle\langle D^2 f(u(t))k(t), h(t)\rangle\rangle = \sum_{t=1}^{T}\langle -\Delta^2 h(t-1) - \mathscr{H}_{f(t,u(t))}h(t), k(t)\rangle_n.\end{gathered} \tag{5.7}$$

Now, from (a) in (H3) and Lemma 4.4, for $u \in \widetilde{H}$ and $x \in X$ with

$$u(t) = \begin{pmatrix} c_{11}\varphi_1 + \cdots + c_{1T}\varphi_T \\ \vdots \\ c_{n1}\varphi_1 + \cdots + c_{nT}\varphi_T \end{pmatrix}, \qquad x = \begin{pmatrix} c_{11}\varphi_1 + \cdots + c_{1m_1}\varphi_{m_1} \\ \vdots \\ c_{n1}\varphi_1 + \cdots + c_{nm_n}\varphi_{m_n} \end{pmatrix}, \tag{5.8}$$

we have that

$$\begin{gathered} Ax(t) = \begin{pmatrix} \lambda_1^A & 0 & \cdots & 0 \\ 0 & \lambda_2^A & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^A \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{m_1} c_{1j}\varphi_j \\ \vdots \\ \sum_{j=1}^{m_n} c_{nj}\varphi_j \end{pmatrix} = \begin{pmatrix} \lambda_1^A \sum_{j=1}^{m_1} c_{1j}\varphi_j \\ \vdots \\ \lambda_n^A \sum_{j=1}^{m_n} c_{nj}\varphi_j \end{pmatrix}, \\ \sum_{t=1}^{T}\langle Ax(t), x(t)\rangle_n = \sum_{t=1}^{T}\sum_{k=1}^{n}\sum_{j=1}^{m_k} c_{kj}^2 \lambda_k^A. \end{gathered} \tag{5.9}$$

Thus,

$$
\begin{aligned}
&\langle\langle D^2 J(u(t))x(t), x(t)\rangle\rangle \\
&= \sum_{t=1}^{T}\left\langle -\Delta^2 x(t-1) - \mathscr{H}_{f(t,u(t))}x(t),\ x(t)\right\rangle_n \\
&= \sum_{t=1}^{T}\left\langle -\Delta^2 x(t-1), x(t)\right\rangle_n - \sum_{t=1}^{T}\left\langle \mathscr{H}_{f(t,u(t))}x(t),\ x(t)\right\rangle_n \\
&\le \sum_{t=1}^{T}\left\langle -\Delta^2 x(t-1), x(t)\right\rangle_n - \sum_{t=1}^{T}\left\langle Ax(t),\ x(t)\right\rangle_n \\
&\le \sum_{t=1}^{T}\left\langle \begin{pmatrix} \sum_{j=1}^{m_1} c_{1j}\mu_j\varphi_j(t) \\ \vdots \\ \sum_{j=1}^{m_n} c_{nj}\mu_j\varphi_j(t) \end{pmatrix}, \begin{pmatrix} \sum_{j=1}^{m_1} c_{1j}\varphi_j(t) \\ \vdots \\ \sum_{j=1}^{m_n} c_{nj}\varphi_j(t) \end{pmatrix} \right\rangle_n - \sum_{t=1}^{T}\left\langle Ax(t),\ x(t)\right\rangle_n \\
&= \sum_{t=1}^{T}\sum_{k=1}^{n}\sum_{j=1}^{m_k} c_{kj}^2\mu_j - \sum_{t=1}^{T}\sum_{k=1}^{n}\sum_{j=1}^{m_k} c_{kj}^2\lambda_k^A \\
&\le -\delta_1 \sum_{t=1}^{T}\sum_{k=1}^{n}\sum_{j=1}^{m_k} c_{kj}^2 \\
&= -\delta_1 \langle\langle x(t), x(t)\rangle\rangle,
\end{aligned} \tag{5.10}
$$

where $\delta_1 = \min\{\lambda_k^A - \mu_k \mid k = 1,\ldots,T\}$. Similarly, for $u \in \widetilde{H}$ and $y \in Y$, it follows from (a) in (H3) and Lemma 4.4 that

$$\langle\langle D^2 J(u)y, y\rangle\rangle \ge \delta_2 \langle\langle y, y\rangle\rangle, \tag{5.11}$$

where $\delta_2 = \min\{\mu_{k+1} - \gamma_k \mid k = 1,\ldots,T\}$. Now, applying Lemma 3.1, J has a unique critical point $v \in \widetilde{H}$ such that

$$J(v) = \max_{x\in X}\min_{y\in Y} J(x+y). \tag{5.12}$$

Case 2 ($\mathscr{H}_{f(t,\xi)} \preccurlyeq A$ and $\lambda_n^A < \mu_1$). In this case, it is easy to verify that

$$\langle\langle D^2 J(u)h, h\rangle\rangle \ge \left(\mu_1 - \lambda_n^A\right)\langle\langle h, h\rangle\rangle. \tag{5.13}$$

Applying Lemma 3.2, we obtain that J has a unique critical point $v \in \widetilde{H}$ such that

$$J(v) = \min_{h\in H} J(h). \tag{5.14}$$

Case 3 ($B \preccurlyeq \mathscr{H}_{f(t,\xi)}$ and $\min\{\gamma_k \mid k = 1,\ldots,n\} > \mu_T$). In this case, we have that

$$\langle\langle D^2 J(u)h, h\rangle\rangle \le -\left(\min\{\gamma_k \mid k = 1,\ldots,n\} - \mu_T\right)\langle\langle h, h\rangle\rangle. \tag{5.15}$$

Applying Lemma 3.2, J has a unique critical point $v \in \widetilde{H}$ such that

$$J(v) = \max_{h\in H} J(h). \tag{5.16}$$

This completes the proof of Theorem 2.5.

6. An example

Example 6.1. Let us consider the following boundary value problem of second-order difference system:

$$\Delta^2 u(t-1) + f\big(t, u(t)\big) = e(t), \quad t \in \{1,2,3\},$$
$$u(0) = u(4) = 0, \tag{6.1}$$

where $u = (u_1, u_2)^T$, $f(t,u) = (u_1, (17/12)u_2)$. Clearly, the conditions (H1) and (H2) hold, and

$$\mathscr{A}_{f(t,\xi)} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{17}{12} \end{pmatrix}. \tag{6.2}$$

Since the linear eigenvalue problem of difference equation

$$\Delta^2 y(t-1) + \mu y(t) = 0, \quad t \in \{1,2,3\},$$
$$y(0) = y(4) = 0 \tag{6.3}$$

has exactly 3 eigenvalues

$$2 - \sqrt{2}, \quad 2, \quad 2 + \sqrt{2}, \tag{6.4}$$

now choose

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{4}{3} \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{5}{3} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}. \tag{6.5}$$

Then, it is easy to show that the condition (a) in (H3) holds. According to Theorem 2.5, the boundary value problem (6.1) has a unique solution for every $e : \{1,2,3\} \to \mathbb{R}^2$.

Remark 6.2. Note that $\gamma_1 > \gamma_2$ in B; this case cannot be handled in [4].

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