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Research Article

q-Bernoulli Numbers Associated with q-Stirling Numbers

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We consider Carlitz q-Bernoulli numbers and q-Stirling numbers of the first and the second kinds. From the properties of q-Stirling numbers, we derive many interesting formulas associated with Carlitz q-Bernoulli numbers. Finally, we will prove $\beta_{n,q} = \sum_{m=0}^{n}\sum_{k=m}^{n} 1/(1-q)^{n+m-k}\sum_{d_0+\cdots+d_k=n-k} q^{\sum_{i=0}^{k} id_i} s_{1,q}(k,m)(-1)^{n-m}((m+1)/[m+1]_q)$, where $\beta_{n,q}$ are called Carlitz q-Bernoulli numbers.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. For d a fixed positive integer with $(p,d)=1$, let

$$\begin{gathered} X = X_d = \varprojlim_{N} \mathbb{Z}\big/ dp^N\mathbb{Z}, \qquad X_1 = \mathbb{Z}_p, \\ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \\ a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{gathered} \tag{1.1}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$, see [1–21]. The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = 1/p$. When one talks about q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < p^{-1/(p-1)}$, so that $q^x = \exp(x\log q)$ for $|x|_p \leq 1$. We use the notation $[x]_q = [x:q] = (1-q^x)/(1-q)$. For $f \in C^{(1)}(\mathbb{Z}_p) = \{f \mid f' \in C(\mathbb{Z}_p)\}$, let us start with the expressions

$$\frac{1}{[p^N]_q}\sum_{0\leq j<p^N} q^j f(j) = \sum_{0\leq j<p^N} f(j)\mu_q(j+p^N\mathbb{Z}_p) \tag{1.2}$$

(see [6, 8]), representing q-analogue of Riemann sums for f. The p-adic q-integral of a function $f \in C^{(1)}(\mathbb{Z}_p)$ is defined by

$$\int_X f(x)d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x \tag{1.3}$$

(see [8, 22, 23]). For $f \in C^{(1)}(\mathbb{Z}_p)$, it is easy to see that

$$\left|\int_{\mathbb{Z}_p} f(x)d\mu_q(x)\right|_p \leq p\|f\|_1 \tag{1.4}$$

(see [6–14]), where $\|f\|_1 = \sup\{|f(0)|_p, \sup_{x\neq y}|(f(x)-f(y))/(x-y)|_p\}$. If $f_n \to f$ in $C^{(1)}(\mathbb{Z}_p)$, namely, $\|f_n - f\|_1 \to 0$, then

$$\int_{\mathbb{Z}_p} f_n(x)d\mu_q(x) \longrightarrow \int_{\mathbb{Z}_p} f(x)d\mu_q(x) \tag{1.5}$$

(see [6–10]). The q-analogue of binomial coefficient was known as $\begin{bmatrix} x \\ n \end{bmatrix}_q = ([x]_q[x-1]_q\cdots[x-n+1]_q)/[n]_q!$, where $[n]_q! = \prod_{i=1}^n [i]_q$ (see [1, 5, 6, 10, 11]). From this definition, we derive

$$\begin{bmatrix} x+1 \\ n \end{bmatrix}_q = \begin{bmatrix} x \\ n-1 \end{bmatrix}_q + q^n\begin{bmatrix} x \\ n \end{bmatrix}_q = q^{x-n}\begin{bmatrix} x \\ n-1 \end{bmatrix}_q + \begin{bmatrix} x \\ n \end{bmatrix}_q \tag{1.6}$$

(cf. [6, 10]). Thus, we have $\int_{\mathbb{Z}_p}\begin{bmatrix} x \\ n \end{bmatrix}_q d\mu_q(x) = ((-1)^n/[n+1]_q)q^{n+1-\binom{n+1}{2}}$. If $f(x) = \sum_{k\geq 0} a_{k,q}\begin{bmatrix} x \\ k \end{bmatrix}_q$ is the q-analogue of Mahler series of strictly differentiable function f, then we see that

$$\int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \sum_{k\geq 0} a_{k,q}\frac{(-1)^k}{[k+1]_q}q^{k+1-\binom{k+1}{2}}. \tag{1.7}$$

Carlitz q-Bernoulli numbers $\beta_{k,q}(=\beta_k(q))$ can be determined inductively by

$$\beta_{0,q} = 1, \qquad q(q\beta+1)^k - \beta_{k,q} = \begin{cases} 1 & \text{if } k=1, \\ 0 & \text{if } k>1, \end{cases} \tag{1.8}$$

with the usual convention of replacing β^i by $\beta_{i,q}$ (see [2–4]). In this paper, we study the q-Stirling numbers of the first and the second kinds. From these q-Stirling numbers, we derive some interesting q-Stirling numbers identities associated with Carlitz q-Bernoulli numbers. Finally, we will prove the following formula:

$$\beta_{n,q} = \sum_{m=q}^{n}\sum_{k=m}^{n}\frac{1}{(1-q)^{n+m-k}}\sum_{d_0+\cdots+d_k=n-k} q^{\sum_{i=0}^k id_i} s_{1,q}(k,m)(-1)^{n-m}\frac{m+1}{[m+1]_q}, \tag{1.9}$$

where $s_{1,q}(k,m)$ is the q-Stirling number of the first kind.

2. q-Stirling numbers and Carlitz q-Bernoulli numbers

For $m \in \mathbb{Z}_+$, we note that

$$\beta_{m,q} = \int_{\mathbb{Z}_p} [x]_q^m d\mu_q(x) = \int_X [x]_q^m d\mu_q(x). \tag{2.1}$$

From this formula, we derive

$$\beta_{0,q} = 1, \qquad q(q\beta + 1)^k - \beta_{k,q} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases} \tag{2.2}$$

with the usual convention of replacing β^i by $\beta_{i,q}$. By the simple calculation of p-adic q-integral on $\mathbb{Z}_p$, we see that

$$\beta_{n,q} = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{i+1}{[i+1]_q}, \tag{2.3}$$

where $\binom{n}{i} = n!/i!(n-i)! = n(n-1)\cdots(n-i+1)/i!$. Let $F(t)$ be the generating function of Carlitz q-Bernoulli numbers. Then we have

$$\begin{aligned} F(t) &= \sum_{n=0}^{\infty} \beta_{n,q} \frac{t^n}{n!} \\ &= \lim_{\rho \to \infty} \frac{1}{[p^\rho]_q} \sum_{x=0}^{p^\rho - 1} q^x e^{[x]_q t} \\ &= \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \left\{ \sum_{k=0}^{\infty} \binom{n}{k} \frac{k+1}{[k+1]_q} (-1)^k \right\} \frac{t^n}{n!} \\ &= e^{t/(1-q)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(1-q)^k} \frac{k+1}{[k+1]_q} \frac{t^k}{k!}. \end{aligned} \tag{2.4}$$

From (2.4) we note that

$$\begin{aligned} F(t) &= e^{t/(1-q)} + e^{t/(1-q)} \sum_{k=1}^{\infty} \frac{(-1)^k}{(1-q)^{k-1}} \left(\frac{k}{1-q^{k+1}} \right) \frac{t^k}{k!} + e^{t/(1-q)} \sum_{k=1}^{\infty} \frac{(-1)^k}{(1-q)^{k-1}} \left(\frac{1}{1-q^{k+1}} \right) \frac{t^k}{k!} \\ &= -t \sum_{n=0}^{\infty} q^{2n} e^{[n]_q t} + (1-q) \sum_{n=0}^{\infty} q^n e^{[n]_q t}. \end{aligned} \tag{2.5}$$

Therefore, we obtain the following.

Lemma 2.1. *Let $F(t) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) (t^n/n!)$. Then one has*

$$F(t) = -t \sum_{n=0}^{\infty} q^{2n} e^{[n]_q t} + (1-q) \sum_{n=0}^{\infty} q^n e^{[n]_q t}. \tag{2.6}$$

The q-Bernoulli polynomials in the variable x in $\mathbb{C}_p$ with $|x|_p \le 1$ are defined by

$$\beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x+t]_q^n d\mu_q(t) = \int_X [x+t]_q^n d\mu_q(x). \tag{2.7}$$

Thus we have

$$\begin{aligned}\int_{\mathbb{Z}_p} [x+t]_q^n d\mu_q(x) &= \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} \int_{\mathbb{Z}_p} [t]_q^k d\mu_q(t) \\ &= \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} \beta_{k,q} \\ &= \left(q^x \beta + [x]_q\right)^n.\end{aligned} \tag{2.8}$$

From (2.7) we derive

$$\int_{\mathbb{Z}_p} [x+t]_q^n d\mu_q(x) = \beta_{n,q}(x) = \frac{1}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{kx} \frac{k+1}{[k+1]_q}. \tag{2.9}$$

Let $F(t,x)$ be the generating function of q-Bernoulli polynomials. By (2.9) we see that

$$\begin{aligned}F(t,x) &= \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!} \\ &= e^{t/(1-q)} \sum_{k=0}^{\infty} \frac{1}{(1-q)^k} q^{kx} (-1)^k \frac{k+1}{[k+1]_q} \frac{t^k}{k!}.\end{aligned} \tag{2.10}$$

From (2.10) we note that

$$F(t,x) = -t \sum_{n=0}^{\infty} q^{2n+x} e^{[n+x]_q t} + (1-q) \sum_{n=0}^{\infty} q^n e^{[n+x]_q t}. \tag{2.11}$$

By (2.7) and (2.11) we easily see that

$$[m]_q^{k-1} \sum_{i=0}^{m-1} q^i \beta_{k,q^m}\left(\frac{x+i}{m}\right) = \beta_{k,q}(x), \quad m \in \mathbb{N},\ k \in \mathbb{Z}_+. \tag{2.12}$$

If we take $x = 0$ in (2.12), then we have

$$[n]_q \beta_{n,q} = \sum_{k=0}^{m} \binom{m}{k} \beta_{k,q^n} [n]_q^k \sum_{j=0}^{n-1} q^{j(k+1)} [j]_q^{n-k}. \tag{2.13}$$

Let us define new q-Bernoulli polynomials, $\beta^*_{n,q}(x)$, as follows:

$$\begin{aligned}F^*(t,x) &= F(t,x) - (1-q) \sum_{n=0}^{\infty} q^n e^{[n+x]_q t} \\ &= -t \sum_{n=0}^{\infty} q^{2n+x} e^{[n+x]_q t} \\ &= \sum_{n=0}^{\infty} \frac{\beta^*_{n,q}(x)}{n!} t^n.\end{aligned} \tag{2.14}$$

In the special case $x = 0$, we can also derive the definition of q-Bernoulli numbers as follows:

$$F^*(t) = F^*(t,0) = \sum_{n=0}^{\infty} \beta_{n,q}^* \frac{t^n}{n!}. \tag{2.15}$$

From these generating functions, we note that

$$-\sum_{l=0}^{\infty} q^{2l+n} e^{[n+l]_q t} + \sum_{l=0}^{\infty} q^{2l} e^{[l]_q t} = \sum_{m=1}^{\infty} \left(m \sum_{l=0}^{n-1} q^{2l} [l]_q^{m-1} \right) \frac{t^{m-1}}{m!}. \tag{2.16}$$

Note that $-\sum_{l=0}^{\infty} q^{2l+n} e^{[n+l]_q t} + \sum_{l=0}^{\infty} q^{2l} e^{[l]_q t} = (1/t)(F^*(t,n) - F^*(t))$. Thus, we have

$$\sum_{m=0}^{\infty} \left(\beta_{m,q}^*(n) - \beta_{m,q}^*\right) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left(m \sum_{l=0}^{n-1} q^{2l} [l]_q^{m-1} \right) \frac{t^m}{m!}. \tag{2.17}$$

By comparing the coefficients on both sides in (2.17), we see that

$$\beta_{m,q}^*(n) - \beta_{m,q}^* = m \sum_{l=0}^{n-1} q^{2l} [l]_q^{m-1}. \tag{2.18}$$

Therefore, we obtain the following.

Proposition 2.2. *For $m, n \in \mathbb{N}$, one has*

$$(q-1) \sum_{l=0}^{n-1} q^l [l]_q^m + \sum_{l=0}^{n-1} q^l [l]_q^{m-1} = \frac{1}{m} \left(\beta_{m,q}^*(n) - \beta_{m,q}^*\right). \tag{2.19}$$

Now we consider the q-analogue of Jordan factor as follows:

$$\begin{aligned} [x]_{k,q} &= [x]_q [x-1]_q \cdots [x-k+1]_q \\ &= \frac{(1-q^x)(1-q^{x-1}) \cdots (1-q^{x-k+1})}{(1-q)^k}. \end{aligned} \tag{2.20}$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(1-q^n)(1-q^{n-1}) \cdots (1-q^{n-k+1})}{(1-q)(1-q^2) \cdots (1-q^k)}, \tag{2.21}$$

where $[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q$. The q-binomial formulas are known as

$$\begin{aligned} \prod_{i=1}^{n} \left(a + b q^{i-1}\right) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} a^{n-k} b^k, \\ \prod_{i=1}^{n} \left(1 - b q^{i-1}\right)^{-1} &= \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q b^k. \end{aligned} \tag{2.22}$$

The q-Stirling numbers of the first kind $s_{1,q}(n,k)$ and the second kind $s_{2,q}(n,k)$ are defined as

$$[x]_{n,q} = q^{-\binom{n}{2}} \sum_{l=0}^{n} s_{1,q}(n,l)[x]_q^l, \quad n = 0,1,2,\dots, \tag{2.23}$$

$$[x]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2}} s_{2,q}(n,k)[x]_{k,q}, \quad n = 0,1,2,\dots \tag{2.24}$$

(see [2, 3, 6]). The values $s_{1,q}(n,1)$, $n = 1,2,3,\dots$, and $s_{2,q}(n,2)$, $n = 2,3,\dots$, may be deduced from the following recurrence relation:

$$s_{1,q}(n,k) = s_{1,q}(n-1,k-1) - [n-1]_q s_{1,q}(n-1,k) \tag{2.25}$$

(see [2, 3, 6]), for $k = 1,2,\dots,n$, $n = 1,2,\dots$, with initial conditions $s_{1,q}(0,0) = 1$, $s_{1,q}(n,k) = 0$ if $k > n$. For $k = 1$, it follows that

$$s_{1,q}(n,1) = -[n-1]_q s_{1,q}(n-1,1), \quad n = 2,3,\dots, \tag{2.26}$$

and since $s_{1,q}(1,1) = 1$, we have $s_{1,q}(n,1) = (-1)^{n-1}[n-1]_q!$, $n = 1,2,3,\dots$. The recurrence relation for $k = 2$ reduces to $s_{1,q}(n,2) + [n-1]_q s_{1,q}(n-1,2) = (-1)^{n-2}[n-2]_q!$, $n = 3,4,\dots$. By simple calculation, we easily see that

$$\begin{aligned}
\frac{(-1)^{n+1} s_{1,q}(n+1,2)}{[n]_q!} - \frac{(-1)^n s_{1,q}(n,2)}{[n-1]_q!} &= (-1)^{n+1} \frac{s_{1,q}(n+1,2) - [n]_q s_{1,q}(n,2)}{[n]_q!} \\
&= (-1)^{n+1} \frac{(-1)^{n+1}[n-1]_q!}{[n]_q!} \\
&= \frac{1}{[n]_q}, \quad n = 2,3,4,\dots.
\end{aligned} \tag{2.27}$$

Thus we have

$$\frac{(-1)^n s_{1,q}(n,2)}{[n-1]_q!} = \sum_{k=1}^{n-1} \frac{1}{[k]_q}. \tag{2.28}$$

This is equivalent to $s_{1,q}(n,2) = (-1)^n [n-1]_q! \sum_{k=1}^{n-1} 1/[k]_q$. It is easy to see that

$$\sum_{m=1}^{n} (-1)^{m+1} q^{\binom{m+1}{2}} \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_q \sum_{k=1}^{m} \frac{1}{[k]_q} = \sum_{k=1}^{n} (-1)^{k+1} q^{\binom{k+1}{2}} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{[k]_q}. \tag{2.29}$$

From this, we derive

$$\begin{aligned}
\sum_{k=1}^{n} (-1)^{k+1} q^{\binom{k+1}{2}} \frac{1}{[k]_q} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \right) &= \sum_{k=1}^{n} (-1)^{k+1} q^{\binom{k+1}{2}} \frac{1}{[k]_q} \left(q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \right) \\
&= \frac{q^n}{[n]_q} \sum_{k=1}^{n} (-1)^{k+1} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \\
&= \frac{q^n}{[n]_q}.
\end{aligned} \tag{2.30}$$

Note that $\sum_{k=1}^{n}(-1)^{k+1}q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q = -\sum_{k=0}^{n}(-1)^{k}q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q + 1 = 1$. Thus, we have

$$\sum_{k=1}^{n}(-1)^{k+1}q^{\binom{k+1}{2}}\frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{[k]_q} = \sum_{k=1}^{n-1}(-1)^{k+1}q^{\binom{k+1}{2}}\frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_q}{[k]_q} + \frac{q^n}{[n]_q}. \tag{2.31}$$

Continuing this process, we see that

$$\sum_{k=1}^{n}(-1)^{k+1}q^{\binom{k+1}{2}}\frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{[k]_q} = \sum_{k=1}^{n}\frac{q^k}{[k]_q}. \tag{2.32}$$

The p-adic q-gamma function is defined as $\Gamma_{p,q}(n) = (-1)^n\prod_{1\le j<n,(j,p)=1}[j]_q$. For all $x \in \mathbb{Z}_p$, we have $\Gamma_{p,q}(x+1) = E_{p,q}(x)\Gamma_{p,q}(x)$, where

$$E_{p,q}(x) = \begin{cases} -[x]_q & \text{if } |x|_p = 1, \\ -1 & \text{if } |x|_p < 1. \end{cases} \tag{2.33}$$

Thus, we easily see that

$$\log\Gamma_{p,q}(x+1) = \log E_{p,q}(x) + \log\Gamma_{p,q}(x). \tag{2.34}$$

From the differentiation on both sides in (2.34), we derive

$$\frac{\Gamma'_{p,q}(x+1)}{\Gamma_{p,q}(x+1)} = \frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)} + \frac{E'_{p,q}(x)}{E_{p,q}(x)}. \tag{2.35}$$

Continuing this process, we have

$$\frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)} = \left(\sum_{j=1}^{x-1}\frac{q^j}{[j]_q}\right)\frac{\log q}{q-1} + \frac{\Gamma'_{p,q}(1)}{\Gamma_{p,q}(1)}. \tag{2.36}$$

The classical Euler constant is known as $\gamma = \Gamma'(1)/\Gamma(1)$. In [15], Kim defined the p-adic q-Euler constant as

$$\gamma_{p,q} = -\frac{\Gamma'_{p,q}(1)}{\Gamma_{p,q}(1)}. \tag{2.37}$$

Therefore, we obtain the following.

Theorem 2.3. *For $x \in \mathbb{Z}_p$, one has*

$$\sum_{k=1}^{x-1}(-1)^{k+1}q^{\binom{k+1}{2}}\frac{\begin{bmatrix} x-1 \\ k \end{bmatrix}_q}{[k]_q} = \frac{q-1}{\log q}\left(\frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)} - \gamma_{p,q}\right). \tag{2.38}$$

From (2.9), (2.21), (2.23), and (2.24), we derive the following theorem.

Theorem 2.4. *For $n, k \in \mathbb{Z}_+$, one has*

$$\beta_{n,q} = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \sum_{k=0}^{l} (q-1)^k \begin{bmatrix} l \\ k \end{bmatrix}_q \sum_{m=0}^{k} s_{1,q}(k,m) \beta_{m,q}, \tag{2.39}$$

where $s_{1,q}(k,m)$ is the q-Stirling number of the first kind.

By simple calculation, we easily see that

$$\begin{aligned} q^{nt} &= \left([t]_q(q-1)+1\right)^n \\ &= \sum_{m=0}^{n} \binom{n}{m} (-1)^m (1-q)^m [t]_q^m \\ &= \sum_{k=0}^{n} (q-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q [t]_{k,q} \\ &= \sum_{k=0}^{n} (q-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{m=0}^{k} s_{1,q}(k,m) [t]_q^m \\ &= \sum_{m=0}^{n} \left(\sum_{k=m}^{n} (q-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q s_{1,q}(k,m) \right) [t]_q^m. \end{aligned} \tag{2.40}$$

Thus we note

$$\int_{\mathbb{Z}_p} q^{nt} d\mu_q(t) = \sum_{m=0}^{n} \left(\sum_{k=m}^{n} (q-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q s_{1,q}(k,m) \right) \beta_{m,q}. \tag{2.41}$$

From the definition of p-adic q-integral on $\mathbb{Z}_p$, we also derive

$$\int_{\mathbb{Z}_p} q^{nt} d\mu_q(t) = \sum_{m=0}^{n} \binom{n}{m} (q-1)^m \beta_{m,q}. \tag{2.42}$$

By comparing the coefficients on both sides of (2.41) and (2.42), we see that

$$\binom{n}{m} (q-1)^m = \sum_{k=m}^{n} (q-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q s_{1,q}(k,m). \tag{2.43}$$

Therefore, we obtain the following.

Theorem 2.5. *For $n \in \mathbb{N}$, $m \in \mathbb{Z}_+$, one has*

$$\binom{n}{m} = \sum_{k=m}^{n} (q-1)^{-m+k} \begin{bmatrix} n \\ k \end{bmatrix}_q s_{1,q}(k,m). \tag{2.44}$$

From Theorem 2.5, we can also derive the following interesting formula for q-Bernoulli numbers.

Theorem 2.6. *For $n \in \mathbb{Z}_+$, one has*

$$\beta_{n,q} = \frac{1}{(1-q)^n}\sum_{m=0}^{n}\left(\sum_{k=m}^{n}(q-1)^{-m+k}\begin{bmatrix} n \\ k \end{bmatrix}_q s_{1,q}(k,m)\right)(-1)^m\frac{m+1}{[m+1]_q}. \tag{2.45}$$

From the definition of q-binomial coefficient, we easily derive

$$\begin{aligned}\begin{bmatrix} x+1 \\ n \end{bmatrix}_q &= \begin{bmatrix} x \\ n-1 \end{bmatrix}_q + q^n\begin{bmatrix} x \\ n \end{bmatrix}_q \\ &= q^{x-n}\begin{bmatrix} x \\ n-1 \end{bmatrix}_q + \begin{bmatrix} x \\ n \end{bmatrix}_q.\end{aligned} \tag{2.46}$$

By (2.46), we see that

$$\int_{\mathbb{Z}_p}\begin{bmatrix} x \\ n \end{bmatrix}_q d\mu_q(x) = \frac{(-1)^n}{[n+1]_q}q^{n+1-\binom{n+1}{2}}. \tag{2.47}$$

From the definition of q-Stirling number of the first kind, we also note that

$$\begin{aligned}\int_{\mathbb{Z}_p}[x]_{n,q}d\mu_q(x) &= [n]_q!\int_{\mathbb{Z}_p}\begin{bmatrix} x \\ n \end{bmatrix}_q d\mu_q(x) \\ &= q^{-\binom{n}{2}}\sum_{k=0}^{n}s_{1,q}(n,k)\beta_{k,q}.\end{aligned} \tag{2.48}$$

By using (2.47) and (2.48), we see

$$(-1)^n\frac{q[n]_q!}{[n+1]_q} = \sum_{k=0}^{n}s_{1,q}(n,k)\beta_{k,q}. \tag{2.49}$$

From (2.24) and (2.48), we derive

$$\beta_{n,q} = q\sum_{k=0}^{n}s_{2,q}(n,k)(-1)^k\frac{[k]_q!}{[k+1]_q}. \tag{2.50}$$

Therefore, we obtain the following.

Theorem 2.7. *For $n \in \mathbb{Z}_+$, one has*

$$\beta_{n,q} = q\sum_{k=0}^{n}s_{2,q}(n,k)(-1)^k\frac{[k]_q!}{[k+1]_q}, \tag{2.51}$$

where $s_{2,q}(n,k)$ is the q-Stirling number of the second kind.

It is easy to see that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{d_0+\cdots+d_k=n-k}q^{\sum_{i=0}^{k}id_i}. \tag{2.52}$$

By Theorem 2.4, we have the following.

Theorem 2.8. *For $n \in \mathbb{Z}_+$, one has*

$$\beta_{n,q} = \sum_{m=0}^{n}\sum_{k=m}^{n}\frac{1}{(1-q)^{n+m-k}}\sum_{d_0+\cdots+d_k=n-k}q^{\sum_{i=0}^{k}id_i}s_{1,q}(k,m)(-1)^{n-m}\frac{m+1}{[m+1]_q}, \tag{2.53}$$

where $s_{1,q}(k,m)$ is the q-Stirling number of the first kind.

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