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Research Article

Stability of a Generalized Euler-Lagrange Type Additive Mapping and Homomorphisms in C^*-Algebras

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Let X, Y be Banach modules over a C^*-algebra and let $r_1, \ldots, r_n \in \mathbb{R}$ be given. We prove the generalized Hyers-Ulam stability of the following functional equation in Banach modules over a unital C^*-algebra: $\sum_{j=1}^n f(-r_j x_j + \sum_{1\le i\le n, i\ne j} r_i x_i) + 2\sum_{i=1}^n r_i f(x_i) = nf(\sum_{i=1}^n r_i x_i)$. We show that if $\sum_{i=1}^n r_i \neq 0$, $r_i, r_j \neq 0$ for some $1 \le i < j \le n$ and a mapping $f : X \to Y$ satisfies the functional equation mentioned above then the mapping $f : X \to Y$ is Cauchy additive. As an application, we investigate homomorphisms in unital C^*-algebras.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1 (Th. M. Rassias [4]). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, *where* ϵ *and* p *are constants with* $\epsilon > 0$ *and* $p < 1$. *Then the limit*

$$L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \tag{1.2}$$

exists for all $x \in E$ *and* $L : E \to E'$ *is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p} \|x\|^p \tag{1.3}$$

for all $x \in E$. *If* $p < 0$, *then* (1.1) *holds for* $x, y \neq 0$ *and* (1.3) *for* $x \neq 0$. *Also, if for each* $x \in E$ *the mapping* $t \mapsto f(tx)$ *is continuous in* $t \in \mathbb{R}$, *then* L *is* $\mathbb{R}$-*linear.*

Theorem 1.2 (J. M. Rassias [5–7]). *Let* X *be a real normed linear space and* Y *a real Banach space. Assume that* $f : X \to Y$ *is a mapping for which there exist constants* $\theta \geq 0$ *and* $p, q \in \mathbb{R}$ *such that* $r = p + q \neq 1$ *and* f *satisfies the functional inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \theta \|x\|^p \|y\|^q \tag{1.4}$$

for all $x, y \in X$. *Then there exists a unique additive mapping* $L : X \to Y$ *satisfying*

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^r - 2|} \|x\|^r \tag{1.5}$$

for all $x \in X$. *If, in addition,* $f : X \to Y$ *is a mapping such that the transformation* $t \to f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then* L *is linear.*

The paper of Th. M. Rassias [4] has provided a lot of influence in the development of what we call the *generalized Hyers-Ulam stability* of functional equations. In 1994, a generalization of Theorems 1.1 and 1.2 was obtained by Găvruţa [8], who replaced the bounds $\varepsilon(\|x\|^p + \|y\|^p)$ and $\theta \|x\|^p \|y\|^q$ by a general control function $\varphi(x, y)$.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.6}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [9] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [10] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [11] proved the generalized Hyers-Ulam stability of the quadratic functional equation. J. M. Rassias [12, 13] introduced and investigated the stability problem of Ulam for the Euler-Lagrange quadratic mappings (1.6) and

$$f(a_1 x_1 + a_2 x_2) + f(a_2 x_1 - a_1 x_2) = \left(a_1^2 + a_2^2\right)[f(x_1) + f(x_2)]. \tag{1.7}$$

Grabiec [14] has generalized these results mentioned above. In addition, J. M. Rassias [15] generalized the Euler-Lagrange quadratic mapping (1.7) and investigated its stability problem. Thus these Euler-Lagrange type equations (mappings) are called as Euler-Lagrange-Rassias functional equations (mappings).

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4–8, 12, 13, 15–55]).

Recently, C. Park and J. Park [45] introduced and investigated the following additive functional equation of Euler-Lagrange type:

$$\begin{aligned}\sum_{i=1}^{n} r_i L\left(\sum_{j=1}^{n} r_j (x_i - x_j)\right) + \left(\sum_{i=1}^{n} r_i\right) L\left(\sum_{i=1}^{n} r_i x_i\right) \\ = \left(\sum_{i=1}^{n} r_i\right) \sum_{i=1}^{n} r_i L(x_i), \quad r_1, \dots, r_n \in (0, \infty)\end{aligned} \tag{1.8}$$

whose solution is said to be a *generalized additive mapping of Euler-Lagrange type.*

In this paper, we introduce the following additive functional equation of Euler-Lagrange type which is somewhat different from (1.8):

$$\sum_{j=1}^{n} f\left(-r_j x_j + \sum_{1 \le i \le n, i \ne j} r_i x_i\right) + 2\sum_{i=1}^{n} r_i f(x_i) = n f\left(\sum_{i=1}^{n} r_i x_i\right), \tag{1.9}$$

where $r_1, \dots, r_n \in \mathbb{R}$. Every solution of the functional equation (1.9) is said to be a *generalized Euler-Lagrange type additive mapping.*

We investigate the generalized Hyers-Ulam stability of the functional equation (1.9) in Banach modules over a C^*-algebra. These results are applied to investigate C^*-algebra homomorphisms in unital C^*-algebras.

Throughout this paper, assume that A is a unital C^*-algebra with norm $\|\cdot\|_A$ and unit e, that B is a unital C^*-algebra with norm $\|\cdot\|_B$, and that X and Y are left Banach modules over a unital C^*-algebra A with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let $U(A)$ be the group of unitary elements in A and let $r_1, \dots, r_n \in \mathbb{R}$. For a given mapping $f : X \to Y, u \in U(A)$ and a given $\mu \in \mathbb{C}$, we define $D_{u,r_1,\dots,r_n} f$ and $D_{\mu,r_1,\dots,r_n} f : X^n \to Y$ by

$$\begin{aligned} D_{u,r_1,\dots,r_n} f(x_1, \dots, x_n) &:= \sum_{j=1}^{n} f\left(-r_j u x_j + \sum_{1 \le i \le n, i \ne j} r_i u x_i\right) + 2\sum_{i=1}^{n} r_i u f(x_i) - n f\left(\sum_{i=1}^{n} r_i u x_i\right), \\ D_{\mu,r_1,\dots,r_n} f(x_1, \dots, x_n) &:= \sum_{j=1}^{n} f\left(-\mu r_j x_j + \sum_{1 \le i \le n, i \ne j} \mu r_i x_i\right) + 2\sum_{i=1}^{n} \mu r_i f(x_i) - n f\left(\sum_{i=1}^{n} \mu r_i x_i\right)\end{aligned} \tag{1.10}$$

for all $x_1, \dots, x_n \in X$.

2. Generalized Hyers-Ulam Stability of the Functional Equation (1.9) in Banach Modules Over a C^*-Algebra

Lemma 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and let $r_1, \ldots, r_n$ be real numbers with $\sum_{k=1}^{n} r_k \neq 0$ and $r_i, r_j \neq 0$ for some $1 \leq i < j \leq n$. Assume that a mapping $L : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation (1.9) for all $x_1, \ldots, x_n \in \mathcal{X}$. Then the mapping L is Cauchy additive. Moreover, $L(r_k x) = r_k L(x)$ for all $x \in \mathcal{X}$ and all $1 \leq k \leq n$.*

Proof. Since $\sum_{k=1}^{n} r_k \neq 0$, putting $x_1 = \cdots = x_n = 0$ in (1.9), we get $L(0) = 0$. Without loss of generality, we may assume that $r_1, r_2 \neq 0$. Letting $x_3 = \cdots = x_n = 0$ in (1.9), we get

$$L(-r_1x_1 + r_2x_2) + L(r_1x_1 - r_2x_2) + 2r_1L(x_1) + 2r_2L(x_2) = 2L(r_1x_1 + r_2x_2) \tag{2.1}$$

for all $x_1, x_2 \in \mathcal{X}$. Letting $x_2 = 0$ in (2.1), we get

$$2r_1L(x_1) = L(r_1x_1) - L(-r_1x_1) \tag{2.2}$$

for all $x_1 \in \mathcal{X}$. Similarly, by putting $x_1 = 0$ in (2.1), we get

$$2r_2L(x_2) = L(r_2x_2) - L(-r_2x_2) \tag{2.3}$$

for all $x_1 \in \mathcal{X}$. It follows from (2.1), (2.2) and (2.3) that

$$L(-r_1x_1 + r_2x_2) + L(r_1x_1 - r_2x_2) + L(r_1x_1) + L(r_2x_2) - L(-r_1x_1) - L(-r_2x_2) = 2L(r_1x_1 + r_2x_2) \tag{2.4}$$

for all $x_1, x_2 \in \mathcal{X}$. Replacing x_1 and x_2 by x/r_1 and y/r_2 in (2.4), we get

$$L(-x + y) + L(x - y) + L(x) + L(y) - L(-x) - L(-y) = 2L(x + y) \tag{2.5}$$

for all $x, y \in \mathcal{X}$. Letting $y = -x$ in (2.5), we get that $L(-2x) + L(2x) = 0$ for all $x \in \mathcal{X}$. So the mapping L is odd. Therefore, it follows from (2.5) that the mapping L is additive. Moreover, let $x \in \mathcal{X}$ and $1 \leq k \leq n$. Setting $x_k = x$ and $x_l = 0$ for all $1 \leq l \leq n,\ l \neq k$, in (1.9) and using the oddness of L, we get that $L(r_k x) = r_k L(x)$. □

Using the same method as in the proof of Lemma 2.1, we have an alternative result of Lemma 2.1 when $\sum_{k=1}^{n} r_k = 0$.

Lemma 2.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and let $r_1, \ldots, r_n$ be real numbers with $r_i, r_j \neq 0$ for some $1 \leq i < j \leq n$. Assume that a mapping $L : \mathcal{X} \to \mathcal{Y}$ with $L(0) = 0$ satisfies the functional equation (1.9) for all $x_1, \ldots, x_n \in \mathcal{X}$. Then the mapping L is Cauchy additive. Moreover, $L(r_k x) = r_k L(x)$ for all $x \in \mathcal{X}$ and all $1 \leq k \leq n$.*

We investigate the generalized Hyers-Ulam stability of a generalized Euler-Lagrange type additive mapping in Banach spaces.

Throughout this paper, $r_1, \ldots, r_n$ will be real numbers such that $r_i, r_j \neq 0$ for fixed $1 \leq i < j \leq n$.

Theorem 2.3. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^n \to [0,\infty)$ such that*

$$\widetilde{\varphi_{ij}}(x,y) := \sum_{k=0}^{\infty}\frac{1}{2^k}\varphi\left(0,\ldots,\underbrace{2^k x}_{ith},0,\ldots,\underbrace{2^k y}_{jth},0,\ldots,0\right) < \infty, \tag{2.6}$$

$$\lim_{k\to\infty}\frac{1}{2^k}\varphi(2^k x_1,\ldots,2^k x_n) = 0, \tag{2.7}$$

$$\left\|D_{e,r_1,\ldots,r_n} f(x_1,\ldots,x_n)\right\|_Y \le \varphi(x_1,\ldots,x_n) \tag{2.8}$$

for all $x, x_1,\ldots,x_n \in X$ and $y \in \{0,\pm x\}$. Then there exists a unique generalized Euler-Lagrange type additive mapping $L : X \to Y$ such that

$$\begin{aligned}\|f(x)-L(x)\|_Y \le \frac{1}{4}\Bigg\{&\left[\widetilde{\varphi_{ij}}\left(\frac{x}{r_i},\frac{x}{r_j}\right)+2\widetilde{\varphi_{ij}}\left(\frac{x}{2r_i},-\frac{x}{2r_j}\right)\right]\\&+\left[\widetilde{\varphi_{ij}}\left(\frac{x}{r_i},0\right)+2\widetilde{\varphi_{ij}}\left(\frac{x}{2r_i},0\right)\right]+\left[\widetilde{\varphi_{ij}}\left(0,\frac{x}{r_j}\right)+2\widetilde{\varphi_{ij}}\left(0,-\frac{x}{2r_j}\right)\right]\Bigg\}\end{aligned} \tag{2.9}$$

for all $x \in X$. Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \le k \le n$.

Proof. For each $1 \le k \le n$ with $k \ne i, j$, let $x_k = 0$ in (2.8), tthen we get the following inequality

$$\begin{aligned}&\left\|f(-r_i x_i + r_j x_j) + f(r_i x_i - r_j x_j) - 2f(r_i x_i + r_j x_j) + 2r_i f(x_i) + 2r_j f(x_j)\right\|_Y\\&\quad\le\varphi\left(0,\ldots,0,\underbrace{x_i}_{i\text{th}},0,\ldots,0,\underbrace{x_j}_{j\text{th}},0,\ldots,0\right)\end{aligned} \tag{2.10}$$

for all $x_i, x_j \in X$. For convenience, set

$$\varphi_{ij}(x,y) := \varphi\left(0,\ldots,0,\underbrace{x}_{i\text{th}},0,\ldots,0,\underbrace{y}_{j\text{th}},0,\ldots,0\right) \tag{2.11}$$

for all $x, y \in X$ and all $1 \le i < j \le n$. Letting $x_i = 0$ in (2.10), we get

$$\left\|f(-r_j x_j) - f(r_j x_j) + 2r_j f(x_j)\right\|_Y \le \varphi_{ij}(0,x_j) \tag{2.12}$$

for all $x_j \in X$. Similarly, letting $x_j = 0$ in (2.10), we get

$$\left\|f(-r_i x_i) - f(r_i x_i) + 2r_i f(x_i)\right\|_Y \le \varphi_{ij}(x_i,0) \tag{2.13}$$

for all $x_i \in X$. It follows from (2.10), (2.12) and (2.13) that

$$\begin{aligned} &\left\| f(-r_ix_i + r_jx_j) + f(r_ix_i - r_jx_j) - 2f(r_ix_i + r_jx_j) + f(r_ix_i) + f(r_jx_j) - f(-r_ix_i) - f(-r_jx_j) \right\|_Y \\ &\quad \le \varphi_{ij}(x_i, x_j) + \varphi_{ij}(x_i, 0) + \varphi_{ij}(0, x_j) \end{aligned} \tag{2.14}$$

for all $x_i, x_j \in X$. Replacing x_i and x_j by x/r_i and y/r_j in (2.14), we get that

$$\begin{aligned} &\left\| f(-x + y) + f(x - y) - 2f(x + y) + f(x) + f(y) - f(-x) - f(-y) \right\|_Y \\ &\quad \le \varphi_{ij}\left(\frac{x}{r_i}, \frac{y}{r_j}\right) + \varphi_{ij}\left(\frac{x}{r_i}, 0\right) + \varphi_{ij}\left(0, \frac{y}{r_j}\right) \end{aligned} \tag{2.15}$$

for all $x, y \in X$. Putting $y = x$ in (2.15), we get

$$\left\| 2f(x) - 2f(-x) - 2f(2x) \right\|_Y \le \varphi_{ij}\left(\frac{x}{r_i}, \frac{x}{r_j}\right) + \varphi_{ij}\left(\frac{x}{r_i}, 0\right) + \varphi_{ij}\left(0, \frac{x}{r_j}\right) \tag{2.16}$$

for all $x \in X$. Replacing x and y by $x/2$ and $-x/2$ in (2.15), respectively, we get

$$\left\| f(x) + f(-x) \right\|_Y \le \varphi_{ij}\left(\frac{x}{2r_i}, -\frac{x}{2r_j}\right) + \varphi_{ij}\left(\frac{x}{2r_i}, 0\right) + \varphi_{ij}\left(0, -\frac{x}{2r_j}\right) \tag{2.17}$$

for all $x \in X$. It follows from (2.16) and (2.17) that

$$\left\| f(2x) - 2f(x) \right\|_Y \le \psi(x) \tag{2.18}$$

for all $x \in X$, where

$$\begin{aligned} \psi(x) := \frac{1}{2}\Bigg\{ &\left[\varphi_{ij}\left(\frac{x}{r_i}, \frac{x}{r_j}\right) + 2\varphi_{ij}\left(\frac{x}{2r_i}, -\frac{x}{2r_j}\right)\right] \\ &+ \left[\varphi_{ij}\left(\frac{x}{r_i}, 0\right) + 2\varphi_{ij}\left(\frac{x}{2r_i}, 0\right)\right] + \left[\varphi_{ij}\left(0, \frac{x}{r_j}\right) + 2\varphi_{ij}\left(0, -\frac{x}{2r_j}\right)\right]\Bigg\}. \end{aligned} \tag{2.19}$$

It follows from (2.6) that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{2^k}\psi\left(2^k x\right) = \frac{1}{2}\Bigg\{ &\left[\widetilde{\varphi_{ij}}\left(\frac{x}{r_i}, \frac{x}{r_j}\right) + 2\widetilde{\varphi_{ij}}\left(\frac{x}{2r_i}, -\frac{x}{2r_j}\right)\right] \\ &+ \left[\widetilde{\varphi_{ij}}\left(\frac{x}{r_i}, 0\right) + 2\widetilde{\varphi_{ij}}\left(\frac{x}{2r_i}, 0\right)\right] + \left[\widetilde{\varphi_{ij}}\left(0, \frac{x}{r_j}\right) + 2\widetilde{\varphi_{ij}}\left(0, -\frac{x}{2r_j}\right)\right]\Bigg\} < \infty \end{aligned} \tag{2.20}$$

for all $x \in X$. Replacing x by $2^k x$ in (2.18) and dividing both sides of (2.18) by 2^{k+1}, we get

$$\left\| \frac{1}{2^{k+1}} f(2^{k+1}x) - \frac{1}{2^k} f(2^k x) \right\|_Y \leq \frac{1}{2^{k+1}} \varphi\left(2^k x\right) \tag{2.21}$$

for all $x \in X$ and all $k \in \mathbb{Z}$. Therefore, we have

$$\begin{aligned}
&\left\| \frac{1}{2^{k+1}} f(2^{k+1}x) - \frac{1}{2^m} f(2^m x) \right\|_Y \\
&\quad \leq \sum_{l=m}^{k} \left\| \frac{1}{2^{l+1}} f(2^{l+1}x) - \frac{1}{2^l} f(2^l x) \right\|_Y \leq \frac{1}{2} \sum_{l=m}^{k} \frac{1}{2^l} \varphi\left(2^l x\right)
\end{aligned} \tag{2.22}$$

for all $x \in X$ and all integers $k \geq m$. It follows from (2.20) and (2.22) that the sequence $\{f(2^k x)/2^k\}$ is Cauchy in Y for all $x \in X$, and thus converges by the completeness of Y. Thus we can define a mapping $L : X \to Y$ by

$$L(x) = \lim_{k \to \infty} \frac{f(2^k x)}{2^k} \tag{2.23}$$

for all $x \in X$. Letting $m = 0$ in (2.22) and taking the limit as $k \to \infty$ in (2.22), we obtain the desired inequality (2.9).

It follows from (2.7) and (2.8) that

$$\begin{aligned}
\|D_{e,r_1,\ldots,r_n} L(x_1, \ldots, x_n)\|_Y &= \lim_{k \to \infty} \frac{1}{2^k} \left\| D_{e,r_1,\ldots,r_n} f(2^k x_1, \ldots, 2^k x_n) \right\|_Y \\
&\leq \lim_{k \to \infty} \frac{1}{2^k} \varphi\left(2^k x_1, \ldots, 2^k x_n\right) = 0
\end{aligned} \tag{2.24}$$

for all $x_1, \ldots, x_n \in X$. Therefore, the mapping $L : X \to Y$ satisfies (1.9) and $L(0) = 0$. Hence by Lemma 2.2, L is a generalized Euler-Lagrange type additive mapping and $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \leq k \leq n$.

To prove the uniqueness, let $T : X \to Y$ be another generalized Euler-Lagrange type additive mapping with $T(0) = 0$ satisfying (2.9). By Lemma 2.2, the mapping T is additive. Therefore, it follows from (2.9) and (2.20) that

$$\begin{aligned}
\|L(x) - T(x)\|_Y &= \lim_{k \to \infty} \frac{1}{2^k} \left\| f(2^k x) - T(2^k x) \right\|_Y \leq \frac{1}{2} \lim_{k \to \infty} \frac{1}{2^k} \sum_{l=0}^{\infty} \frac{1}{2^l} \varphi\left(2^{l+k} x\right) \\
&= \frac{1}{2} \lim_{k \to \infty} \sum_{l=k}^{\infty} \frac{1}{2^l} \varphi\left(2^l x\right) = 0.
\end{aligned} \tag{2.25}$$

So $L(x) = T(x)$ for all $x \in X$. □

Theorem 2.4. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^n \to [0,\infty)$ satisfying* (2.6), (2.7) *and*

$$\left\| D_{u,r_1,\ldots,r_n} f(x_1,\ldots,x_n) \right\| \le \varphi(x_1,\ldots,x_n) \tag{2.26}$$

for all $x_1,\ldots,x_n \in X$ and all $u \in U(A)$. Then there exists a unique A-linear generalized Euler-Lagrange type additive mapping $L : X \to Y$ satisfying (2.9) *for all $x \in X$. Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \le k \le n$.*

Proof. By Theorem 2.3, there exists a unique generalized Euler-Lagrange type additive mapping $L : X \to Y$ satisfying (2.9) and moreover $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \le k \le n$.

By the assumption, for each $u \in U(A)$, we get

$$\begin{aligned} \Bigg\| D_{u,r_1,\ldots,r_n} L(0,\ldots,0,\underbrace{x}_{i\text{th}},0\cdots,0) \Bigg\|_Y &= \lim_{k\to\infty} \frac{1}{2^k} \Bigg\| D_{u,r_1,\ldots,r_n} f(0,\ldots,0,\underbrace{2^k x}_{i\text{th}},0\cdots,0) \Bigg\|_Y \\ &\le \lim_{k\to\infty} \frac{1}{2^k} \varphi\Bigg(0,\ldots,0,\underbrace{2^k x}_{i\text{th}},0\cdots,0 \Bigg) = 0 \end{aligned} \tag{2.27}$$

for all $x \in X$. So

$$r_i u L(x) = L(r_i u x) \tag{2.28}$$

for all $u \in U(A)$ and all $x \in X$. Since $L(r_i x) = r_i L(x)$ for all $x \in X$ and $r_i \neq 0$,

$$L(ux) = uL(x) \tag{2.29}$$

for all $u \in U(A)$ and all $x \in X$.

By the same reasoning as in the proofs of [41, 43],

$$L(ax+by) = L(ax) + L(by) = aL(x) + bL(y) \tag{2.30}$$

for all $a,b \in A$ $(a,b \neq 0)$ and all $x,y \in X$. Since $L(0x) = 0 = 0L(x)$ for all $x \in X$, the unique generalized Euler-Lagrange type additive mapping $L : X \to Y$ is an A-linear mapping. □

Corollary 2.5. *Let $\delta \ge 0$, $\{\epsilon_k\}_{k\in J}$ and $\{p_k\}_{k\in J}$ be real numbers such that $\epsilon_k \ge 0$ and $0 < p_k < 1$ for all $k \in J$, where $J \subseteq \{1,2,\ldots,n\}$. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\left\| D_{u,r_1,\ldots,r_n} f(x_1,\ldots,x_n) \right\|_Y \le \delta + \sum_{k\in J} \epsilon_k \|x_k\|_X^{p_k} \tag{2.31}$$

for all $x_1, \dots, x_n \in X$ and all $u \in U(A)$. Then there exists a unique A-linear generalized Euler-Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \begin{cases} M_{ij}(x), & i, j \in J; \\ M_i(x), & i \in J,\ j \notin J; \\ M_j(x), & j \in J,\ i \notin J; \\ M, & i, j \notin J. \end{cases} \tag{2.32}$$

for all $x \in X$, where

$$\begin{aligned} M_{ij}(x) &= \frac{9}{2}\delta + \sum_{k \in \{i,j\}} \frac{(1 + 2^{1-p_k})\epsilon_k}{(2 - 2^{p_k}) r_k^{p_k}} \|x\|_X^{p_k}, \\ M_i(x) &= \frac{9}{2}\delta + \frac{(1 + 2^{1-p_i})\epsilon_i}{(2 - 2^{p_i}) r_i^{p_i}} \|x\|_X^{p_i}, \\ M_j(x) &= \frac{9}{2}\delta + \frac{(1 + 2^{1-p_j})\epsilon_j}{(2 - 2^{p_j}) r_j^{p_j}} \|x\|_X^{p_j}, \qquad M = \frac{9}{2}\delta. \end{aligned} \tag{2.33}$$

Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \leq k \leq n$.

Proof. Define $\varphi(x_1, \dots, x_n) := \delta + \sum_{k \in J} \epsilon_k \|x_k\|_X^{p_k}$, and apply Theorem 2.4. □

Corollary 2.6. *Let $\delta, \epsilon \geq 0$, $p, q > 0$ with $\lambda = p + q < 1$. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_{u, r_1, \dots, r_n} f(x_1, \dots, x_n)\|_Y \leq \delta + \epsilon \|x_i\|_X^p \|x_j\|_X^q \tag{2.34}$$

for all $x_1, \dots, x_n \in X$ and all $u \in U(A)$. Then there exists a unique A-linear generalized Euler-Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{9}{2}\delta + \frac{(1 + 2^{1-\lambda})\epsilon}{2(2 - 2^{\lambda}) r_i^p r_j^q} \|x\|_X^{\lambda} \tag{2.35}$$

for all $x \in X$. Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \leq k \leq n$.

Proof. Define $\varphi(x_1, \dots, x_n) := \delta + \epsilon \|x_i\|_X^p \|x_j\|_X^q$. Applying Theorem 2.4, we obtain the desired result. □

Theorem 2.7. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\phi : X^n \to [0,\infty)$ such that*

$$\widetilde{\phi_{ij}}(x,y) := \sum_{k=1}^{\infty} 2^k \phi\left(0,\ldots,\underbrace{\frac{x}{2^k}}_{ith},0,\ldots,\underbrace{\frac{y}{2^k}}_{jth},0,\ldots,0\right) < \infty, \tag{2.36}$$

$$\lim_{k\to\infty} 2^k \phi\left(\frac{x_1}{2^k},\ldots,\frac{x_n}{2^k}\right) = 0, \tag{2.37}$$

$$\left\| D_{e,r_1,\ldots,r_n} f(x_1,\ldots,x_n)\right\|_Y \le \phi(x_1,\ldots,x_n) \tag{2.38}$$

for all $x, x_1, \ldots, x_n \in X$ and $y \in \{0, \pm x\}$. Then there exists a unique generalized Euler-Lagrange type additive mapping $L : X \to Y$ such that

$$\begin{aligned}\|f(x) - L(x)\|_Y \le \frac{1}{4}\Bigg\{&\left[\widetilde{\phi_{ij}}\left(\frac{x}{r_i},\frac{x}{r_j}\right) + 2\widetilde{\phi_{ij}}\left(\frac{x}{2r_i},-\frac{x}{2r_j}\right)\right]\\ &+\left[\widetilde{\phi_{ij}}\left(\frac{x}{r_i},0\right) + 2\widetilde{\phi_{ij}}\left(\frac{x}{2r_i},0\right)\right] + \left[\widetilde{\phi_{ij}}\left(0,\frac{x}{r_j}\right) + 2\widetilde{\phi_{ij}}\left(0,-\frac{x}{2r_j}\right)\right]\Bigg\}\end{aligned} \tag{2.39}$$

for all $x \in X$. Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \le k \le n$.

Proof. By a similar method to the proof of Theorem 2.3, we have the following inequality

$$\|f(2x) - 2f(x)\|_Y \le \Psi(x) \tag{2.40}$$

for all $x \in X$, where

$$\begin{aligned}\Psi(x) := \frac{1}{2}\Bigg\{&\left[\phi_{ij}\left(\frac{x}{r_i},\frac{x}{r_j}\right) + 2\phi_{ij}\left(\frac{x}{2r_i},-\frac{x}{2r_j}\right)\right]\\ &+\left[\phi_{ij}\left(\frac{x}{r_i},0\right) + 2\phi_{ij}\left(\frac{x}{2r_i},0\right)\right] + \left[\phi_{ij}\left(0,\frac{x}{r_j}\right) + 2\phi_{ij}\left(0,-\frac{x}{2r_j}\right)\right]\Bigg\}.\end{aligned} \tag{2.41}$$

It follows from (2.36) that

$$\begin{aligned}\sum_{k=1}^{\infty} 2^k \Psi\left(\frac{x}{2^k}\right) = \frac{1}{2}\Bigg\{&\left[\widetilde{\phi_{ij}}\left(\frac{x}{r_i},\frac{x}{r_j}\right) + 2\widetilde{\phi_{ij}}\left(\frac{x}{2r_i},-\frac{x}{2r_j}\right)\right]\\ &+\left[\widetilde{\phi_{ij}}\left(\frac{x}{r_i},0\right) + 2\widetilde{\phi_{ij}}\left(\frac{x}{2r_i},0\right)\right] + \left[\widetilde{\phi_{ij}}\left(0,\frac{x}{r_j}\right) + 2\widetilde{\phi_{ij}}\left(0,-\frac{x}{2r_j}\right)\right]\Bigg\} < \infty\end{aligned} \tag{2.42}$$

for all $x \in X$. Replacing x by $x/2^{k+1}$ in (2.40) and multiplying both sides of (2.40) by 2^k, we get

$$\left\| 2^{k+1} f\left(\frac{x}{2^{k+1}}\right) - 2^k f\left(\frac{x}{2^k}\right) \right\|_Y \le 2^k \Psi\left(\frac{x}{2^{k+1}}\right) \tag{2.43}$$

for all $x \in X$ and all $k \in \mathbb{Z}$. Therefore, we have

$$\begin{aligned} &\left\| 2^{k+1} f\left(\frac{x}{2^{k+1}}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\|_Y \\ &\quad \le \sum_{l=m}^{k} \left\| 2^{l+1} f\left(\frac{x}{2^{l+1}}\right) - 2^l f\left(\frac{x}{2^l}\right) \right\|_Y \le \sum_{l=m}^{k} 2^l \Psi\left(\frac{x}{2^{l+1}}\right) \end{aligned} \tag{2.44}$$

for all $x \in X$ and all integers $k \ge m$. It follows from (2.42) and (2.44) that the sequence $\{2^k f(x/2^k)\}$ is Cauchy in Y for all $x \in X$, and thus converges by the completeness of Y. Thus we can define a mapping $L : X \to Y$ by

$$L(x) = \lim_{k \to \infty} 2^k f\left(\frac{x}{2^k}\right) \tag{2.45}$$

for all $x \in X$. Letting $m = 0$ in (2.44) and taking the limit as $k \to \infty$ in (2.44), we obtain the desired inequality (2.39).

The rest of the proof is similar to the proof of Theorem 2.3. □

Theorem 2.8. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\phi : X^n \to [0, \infty)$ satisfying* (2.36), (2.37) *and*

$$\left\| D_{u,r_1,\ldots,r_n} f(x_1, \ldots, x_n) \right\| \le \phi(x_1, \ldots, x_n) \tag{2.46}$$

for all $x_1, \ldots, x_n \in X$ and all $u \in U(A)$. Then there exists a unique A-linear generalized Euler-Lagrange type additive mapping $L : X \to Y$ satisfying (2.39) *for all $x \in X$. Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \le k \le n$.*

Proof. The proof is similar to the proof of Theorem 2.4. □

Corollary 2.9. *Let $\{\epsilon_k\}_{k \in J}$ and $\{p_k\}_{k \in J}$ be real numbers such that $\epsilon_k \ge 0$ and $p_k > 1$ for all $k \in J$, where $J \subseteq \{1, 2, \ldots, n\}$. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\left\| D_{u,r_1,\ldots,r_n} f(x_1, \ldots, x_n) \right\|_Y \le \sum_{k \in J} \epsilon_k \|x_k\|_X^{p_k} \tag{2.47}$$

for all $x_1, \ldots, x_n \in X$ and all $u \in U(A)$. Then there exists a unique A-linear generalized Euler-Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \le \begin{cases} N_{ij}(x), & i, j \in J; \\ N_i(x), & i \in J,\ j \notin J; \\ N_j(x), & j \in J,\ i \notin J; \\ N, & i, j \notin J. \end{cases} \tag{2.48}$$

for all $x \in X$, where

$$\begin{aligned} N_{ij}(x) &= \sum_{k\in\{i,j\}} \frac{(1+2^{1-p_k})\epsilon_k}{(2^{p_k}-2)r_k^{p_k}} \|x\|_X^{p_k}, \\ N_i(x) &= \frac{(1+2^{1-p_i})\epsilon_i}{(2^{p_i}-2)r_i^{p_i}} \|x\|_X^{p_i}, \\ N_j(x) &= \frac{(1+2^{1-p_j})\epsilon_j}{(2^{p_j}-2)r_j^{p_j}} \|x\|_X^{p_j}. \end{aligned} \tag{2.49}$$

Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \le k \le n$.

Proof. Define $\phi(x_1, \ldots, x_n) := \sum_{k\in J} \epsilon_k \|x_k\|_X^{p_k}$. Applying Theorem 2.8, we obtain the desired result. □

Corollary 2.10. *Let $\epsilon \ge 0$, $p, q > 0$ with $\lambda = p + q > 1$. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_{u,r_1,\ldots,r_n} f(x_1, \ldots, x_n)\|_Y \le \epsilon \|x_i\|_X^p \|x_j\|_X^q \tag{2.50}$$

for all $x_1, \ldots, x_n \in X$ and all $u \in U(A)$. Then there exists a unique A-linear generalized Euler-Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \le \frac{(1+2^{1-\lambda})\epsilon}{2(2^\lambda - 2) r_i^p r_j^q} \|x\|_X^\lambda \tag{2.51}$$

for all $x \in X$. Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \le k \le n$.

Proof. Define $\phi(x_1, \ldots, x_n) := \epsilon \|x_i\|_X^p \|x_j\|_X^q$. Applying Theorem 2.8, we obtain the desired result. □

Remark 2.11. In Theorems 2.7 and 2.8 and Corollaries 2.9 and 2.10 one can assume that $\sum_{k=1}^n r_k \ne 0$ instead of $f(0) = 0$.

For the case $p_1 = \cdots = p_n = 1$ in Corollaries 2.5 and 2.9, using an idea from the example of Gajda [56], we have the following counterexample.

Example 2.12. Let $\phi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\phi(x) := \begin{cases} x & \text{for } |x| < 1; \\ 1 & \text{otherwise.} \end{cases} \tag{2.52}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ by the formula

$$f(x) := \sum_{n=0}^{\infty} 2^{-n} \phi(2^n x). \tag{2.53}$$

It is clear that f is continuous and bounded by 2 on $\mathbb{C}$. We prove that

$$\left| D_{\mu, r_1, \ldots, r_n} f(x_1, \ldots, x_n) \right| \leq 8 \left(n + \sum_{i=1}^{n} |r_i| \right) \sum_{i=1}^{n} (|r_i| + 1)|x_i| \tag{2.54}$$

for all $x_1, \ldots, x_n \in \mathbb{C}$ and all $\mu \in U(\mathbb{C}) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. If $\sum_{i=1}^{n} (|r_i| + 1)|x_i| = 0$ or $\sum_{i=1}^{n} (|r_i| + 1)|x_i| \geq 1$, then

$$\left| D_{\mu, r_1, \ldots, r_n} f(x_1, \ldots, x_n) \right| \leq 4n + 4 \sum_{i=1}^{n} |r_i| \leq 4 \left(n + \sum_{i=1}^{n} |r_i| \right) \sum_{i=1}^{n} (|r_i| + 1)|x_i|. \tag{2.55}$$

Now suppose that $0 < \sum_{i=1}^{n} (|r_i| + 1)|x_i| < 1$. Then there exists a nonnegative integer k such that

$$\frac{1}{2^{k+1}} \leq \sum_{i=1}^{n} (|r_i| + 1)|x_i| < \frac{1}{2^k}. \tag{2.56}$$

Therefore

$$2^k \left| -\mu r_j x_j + \sum_{1 \leq i \leq n, i \neq j} \mu r_i x_i \right|, \; 2^k \left| \sum_{i=1}^{n} \mu r_i x_i \right|, \; 2^k |x_1|, \ldots, 2^k |x_n| \in (-1, 1). \tag{2.57}$$

Hence

$$2^m \left| -\mu r_j x_j + \sum_{1 \leq i \leq n, i \neq j} \mu r_i x_i \right|, \; 2^m \left| \sum_{i=1}^{n} \mu r_i x_i \right|, \; 2^m |x_1|, \ldots, 2^m |x_n| \in (-1, 1) \tag{2.58}$$

for all $m = 0, 1, \ldots, k$. From the definition of f and (2.56), we have

$$\begin{aligned}\left|D_{\mu,r_1,\ldots,r_n} f(x_1,\ldots,x_n)\right| &\le 4\left(n + \sum_{i=1}^{n} |r_i|\right) \sum_{m=k+1}^{\infty} \frac{1}{2^m} \\ &= 8\left(n + \sum_{i=1}^{n} |r_i|\right) \frac{1}{2^{k+1}} \\ &\le 8\left(n + \sum_{i=1}^{n} |r_i|\right) \sum_{i=1}^{n} (|r_i| + 1)|x_i|.\end{aligned} \tag{2.59}$$

Therefore f satisfies (2.54). Let $L : \mathbb{C} \to \mathbb{C}$ be an additive mapping such that

$$\left|f(x) - L(x)\right| \le \beta |x| \tag{2.60}$$

for all $x \in \mathbb{C}$. Then there exists a constant $c \in \mathbb{C}$ such that $L(x) = cx$ for all rational numbers x. So we have

$$\left|f(x)\right| \le (\beta + |c|)|x| \tag{2.61}$$

for all rational numbers x. Let $m \in \mathbb{N}$ with $m > \beta + |c|$. If x is a rational number in $(0, 2^{1-m})$, then $2^n x \in (0, 1)$ for all $n = 0, 1, \ldots, m-1$. So

$$f(x) \ge \sum_{n=0}^{m-1} 2^{-n} \phi(2^n x) = mx > \left(\beta + |c|\right)|x| \tag{2.62}$$

which contradicts with (2.61).

3. Homomorphisms in Unital C^*-Algebras

In this section, we investigate C^*-algebra homomorphisms in unital C^*-algebras.

We will use the following lemma in the proof of the next theorem.

Lemma 3.1 (see [43]). *Let $f : A \to B$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in \mathbb{S}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Then the mapping $f : A \to B$ is $\mathbb{C}$-linear.*

Theorem 3.2. *Let $\epsilon \geq 0$ and $\{p_k\}_{k\in J}$ be real numbers such that $p_k > 0$ for all $k \in J$, where $J \subseteq \{1, 2, \ldots, n\}$ and $|J| \geq 3$. Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : A^n \to [0, \infty)$ satisfying* (2.7) *and*

$$\|D_{\mu,r_1,\ldots,r_n} f(x_1, \ldots, x_n)\|_B \leq \epsilon \prod_{k\in J} \|x_k\|_A^{p_k}, \tag{3.1}$$

$$\left\| f(2^k u^*) - f(2^k u)^* \right\|_B \leq \varphi\Bigg(\underbrace{2^k u, \ldots, 2^k u}_{n \text{ times}} \Bigg), \tag{3.2}$$

$$\left\| f(2^k ux) - f(2^k u) f(x) \right\|_B \leq \varphi\Bigg(\underbrace{2^k ux, \ldots, 2^k ux}_{n \text{ times}} \Bigg) \tag{3.3}$$

for all $x, x_1, \ldots, x_n \in A$, for all $u \in U(A)$, all $k \in \mathbb{N}$ and all $\mu \in \mathbb{S}^1$. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Proof. Since $|J| \geq 3$, letting $\mu = 1$ and $x_k = 0$ for all $1 \leq k \leq n$, $k \neq i, j$, in (3.1), we get

$$f(-r_i x_i + r_j x_j) + f(r_i x_i - r_j x_j) + 2r_i f(x_i) + 2r_j f(x_j) = 2f(r_i x_i + r_j x_j) \tag{3.4}$$

for all $x_i, x_j \in A$. By the same reasoning as in the proof of Lemma 2.1, the mapping f is additive and $f(r_k x) = r_k f(x)$ for all $x \in A$ and $k = i, j$. So by letting $x_i = x$ and $x_k = 0$ for all $1 \leq k \leq n$, $k \neq i$, in (3.1), we get that $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in \mathbb{S}^1$. Therefore, by Lemma 3.1, the mapping f is $\mathbb{C}$-linear. Hence it follows from (2.7), (3.2) and (3.3) that

$$\begin{aligned}
\|f(u^*) - f(u)^*\|_B &= \lim_{k\to\infty} \frac{1}{2^k} \left\| f(2^k u^*) - f(2^k u)^* \right\|_B \\
&\leq \lim_{k\to\infty} \frac{1}{2^k} \varphi\Bigg(\underbrace{2^k u, \ldots, 2^k u}_{n \text{ times}} \Bigg) = 0, \\
\|f(ux) - f(u)f(x)\|_B &= \lim_{k\to\infty} \frac{1}{2^k} \|f\left(2^k ux\right) - f\left(2^k u\right) f(x)\|_B \\
&\leq \lim_{k\to\infty} \frac{1}{2^k} \varphi\Bigg(\underbrace{2^k ux, \ldots, 2^k ux}_{n \text{ times}} \Bigg) = 0
\end{aligned} \tag{3.5}$$

for all $x \in A$ and all $u \in U(A)$. So $f(u^*) = f(u)^*$ and $f(ux) = f(u)f(x)$ for all $x \in A$ and all $u \in U(A)$. Since f is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements (see [57]), that is, $x = \sum_{k=1}^{m} \lambda_k u_k$, where $\lambda_k \in \mathbb{C}$ and $u_k \in U(A)$ for all $1 \le k \le n$, we have

$$\begin{aligned}
f(x^*) &= f\left(\sum_{k=1}^{m} \overline{\lambda_k} u_k^*\right) = \sum_{k=1}^{m} \overline{\lambda_k} f(u_k^*) = \sum_{k=1}^{m} \overline{\lambda_k} f(u_k)^* \\
&= \left(\sum_{k=1}^{m} \lambda_k f(u_k)\right)^* = f\left(\sum_{k=1}^{m} \lambda_k u_k\right)^* = f(x)^*, \\
f(xy) &= f\left(\sum_{k=1}^{m} \lambda_k u_k y\right) = \sum_{k=1}^{m} \lambda_k f(u_k y) \\
&= \sum_{k=1}^{m} \lambda_k f(u_k) f(y) = f\left(\sum_{k=1}^{m} \lambda_k u_k\right) f(y) = f(x) f(y)
\end{aligned} \tag{3.6}$$

for all $x, y \in A$. Therefore, the mapping $f : A \to B$ is a C^*-algebra homomorphism, as desired. □

The following theorem is an alternative result of Theorem 3.2.

Theorem 3.3. *Let $\epsilon \ge 0$ and $\{p_k\}_{k \in J}$ be real numbers such that $p_k > 0$ for all $k \in J$, where $J \subseteq \{1, 2, \ldots, n\}$ and $|J| \ge 3$. Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : A^n \to [0, \infty)$ satisfying* (2.37) *and*

$$\begin{aligned}
&\left\| D_{\mu, r_1, \ldots, r_n} f(x_1, \ldots, x_n) \right\|_B \le \epsilon \prod_{k \in J} \|x_k\|_A^{p_k} \\
&\left\| f\left(\frac{u^*}{2^k}\right) - f\left(\frac{u}{2^k}\right)^* \right\|_B \le \phi\left(\underbrace{\frac{u}{2^k}, \ldots, \frac{u}{2^k}}_{n \text{ times}}\right), \\
&\left\| f\left(\frac{ux}{2^k}\right) - f\left(\frac{u}{2^k}\right) f(x) \right\|_B \le \phi\left(\underbrace{\frac{ux}{2^k}, \ldots, \frac{ux}{2^k}}_{n \text{ times}}\right)
\end{aligned} \tag{3.7}$$

for all $x, x_1, \ldots, x_n \in A$, for all $u \in U(A)$, all $k \in \mathbb{N}$ and all $\mu \in \mathbb{S}^1$. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Remark 3.4. In Theorems 3.2 and 3.3, one can assume that $\sum_{k=1}^{n} r_k \ne 0$ instead of $f(0) = 0$.

Theorem 3.5. *Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : A^n \to [0, \infty)$ satisfying* (2.6), (2.7), (3.2), (3.3) *and*

$$\left\| D_{\mu, r_1, \ldots, r_n} f(x_1, \ldots, x_n) \right\|_B \le \varphi(x_1, \ldots, x_n), \tag{3.8}$$

for all $x_1, \ldots, x_n \in A$ and all $\mu \in \mathbb{S}^1$. Assume that $\lim_{k \to \infty}(1/2^k) f(2^k e)$ *is invertible. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.*

Proof. Consider the C^*-algebras A and B as left Banach modules over the unital C^*-algebra $\mathbb{C}$. By Theorem 2.4, there exists a unique $\mathbb{C}$-linear generalized Euler-Lagrange type additive mapping $H : A \to B$ defined by

$$H(x) = \lim_{k\to\infty} \frac{1}{2^k} f\left(2^k x\right) \tag{3.9}$$

for all $x \in A$. Therefore, by (2.7), (3.2) and (3.3), we get

$$\begin{aligned}
\left\|H(u^*) - H(u)^*\right\|_B &= \lim_{k\to\infty} \frac{1}{2^k} \left\| f(2^k u^*) - f\left(2^k u\right)^* \right\|_B \\
&\le \lim_{k\to\infty} \frac{1}{2^k} \varphi\left(\underbrace{2^k u, \ldots, 2^k u}_{n \text{ times}}\right) = 0, \\
\left\|H(ux) - H(u) f(x)\right\|_B &= \lim_{k\to\infty} \frac{1}{2^k} \left\| f(2^k ux) - f(2^k u) f(x) \right\|_B \\
&\le \lim_{k\to\infty} \frac{1}{2^k} \varphi\left(\underbrace{2^k ux, \ldots, 2^k ux}_{n \text{ times}}\right) = 0
\end{aligned} \tag{3.10}$$

for all $u \in U(A)$ and for all $x \in A$. So $H(u^*) = H(u)^*$ and $H(ux) = H(u)f(x)$ for all $u \in U(A)$ and all $x \in A$. Therefore, by the additivity of H we have

$$H(ux) = \lim_{k\to\infty} \frac{1}{2^k} H\left(2^k ux\right) = H(u) \lim_{k\to\infty} \frac{1}{2^k} f\left(2^k x\right) = H(u)H(x) \tag{3.11}$$

for all $u \in U(A)$ and all $x \in A$. Since H is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements, that is, $x = \sum_{k=1}^{m} \lambda_k u_k$, where $\lambda_k \in \mathbb{C}$ and $u_k \in U(A)$ for all $1 \le k \le n$, it follows from (3.11) that

$$\begin{aligned}
H(xy) &= H\left(\sum_{k=1}^{m} \lambda_k u_k y\right) = \sum_{k=1}^{m} \lambda_k H(u_k y) \\
&= \sum_{k=1}^{m} \lambda_k H(u_k) H(y) = H\left(\sum_{k=1}^{m} \lambda_k u_k\right) H(y) = H(x)H(y), \\
H(x^*) &= H\left(\sum_{k=1}^{m} \overline{\lambda_k} u_k^*\right) = \sum_{k=1}^{m} \overline{\lambda_k} H(u_k^*) = \sum_{k=1}^{m} \overline{\lambda_k} H(u_k)^* \\
&= \left(\sum_{k=1}^{m} \lambda_k H(u_k)\right)^* = H\left(\sum_{k=1}^{m} \lambda_k u_k\right)^* = H(x)^*
\end{aligned} \tag{3.12}$$

for all $x, y \in A$. Since $H(e) = \lim_{k\to\infty}(1/2^k) f(2^k e)$ is invertible and

$$H(e)H(y) = H(ey) = H(e)f(y) \tag{3.13}$$

for all $y \in A, H(y) = f(y)$ for all $y \in A$, therefore, the mapping $f : A \to B$ is a C^*-algebra homomorphism. □

The following theorem is an alternative result of Theorem 3.5.

Theorem 3.6. *Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there is a function $\phi : A^n \to [0, \infty)$ satisfying* (2.36), (2.37), (3.7) *and*

$$\left\| D_{\mu,r_1,\ldots,r_n} f(x_1, \ldots, x_n) \right\|_B \le \phi(x_1, \ldots, x_n), \tag{3.14}$$

for all $x_1, \ldots, x_n \in A$ and all $\mu \in \mathbb{S}^1$. Assume that $\lim_{k\to\infty} 2^k f(e/2^k)$ is invertible. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Corollary 3.7. *Let $\{\epsilon_k\}_{k\in J}$ and $\{p_k\}_{k\in J}$ be real numbers such that $\epsilon_k \ge 0$ and $p_k > 1$ $(0 < p_k < 1)$ for all $k \in J$, where $J \subseteq \{1, 2, \ldots, n\}$. Assume that a mapping $f : A \to B$ with $f(0) = 0$ satisfies the inequalities*

$$\begin{gathered}
\left\| D_{\mu,r_1,\ldots,r_n} f(x_1, \ldots, x_n) \right\|_B \le \sum_{k\in J} \epsilon_k \|x_k\|_A^{p_k}, \\
\left\| f\left(\frac{u^*}{2^m}\right) - f\left(\frac{u}{2^m}\right)^* \right\|_B \le \sum_{k\in J} \frac{\epsilon_k}{2^{mp_k}} \\
\left(\text{resp.}, \left\| f(2^m u^*) - f(2^m u)^* \right\|_B \le \sum_{k\in J} \epsilon_k 2^{mp_k} \right), \\
\left\| f\left(\frac{ux}{2^m}\right) - f\left(\frac{u}{2^m}\right) f(x) \right\|_B \le \sum_{k\in J} \frac{\epsilon_k}{2^{mp_k}} \|x\|_A^{p_k} \\
\left(\text{resp.}, \left\| f(2^m ux) - f(2^m u) f(x) \right\|_B \le \sum_{k\in J} \epsilon_k 2^{mp_k} \|x\|_A^{p_k} \right),
\end{gathered} \tag{3.15}$$

for all $x_1, \ldots, x_n \in A$, all $u \in U(A)$, all $m \in \mathbb{N}$ and all $\mu \in \mathbb{S}^1$. Assume that $\lim_{k\to\infty} 2^k f(e/2^k)$ (resp., $\lim_{k\to\infty}(1/2^k) f(2^k e)$) is invertible. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Proof. The result follows from Theorem 3.6 (resp., Theorem 3.5). □

Remark 3.8. In Theorem 3.6 and Corollary 3.7, one can assume that $\sum_{k=1}^n r_k \neq 0$ instead of $f(0) = 0$.

Theorem 3.9. *Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : A^n \to [0, \infty)$ satisfying* (2.6), (2.7), (3.2), (3.3) *and*

$$\left\| D_{\mu,r_1,\ldots,r_n} f(x_1, \ldots, x_n) \right\|_B \le \varphi(x_1, \ldots, x_n), \tag{3.16}$$

for $\mu = i, 1$ *and all* $x_1, \ldots, x_n \in A$. *Assume that* $\lim_{k\to\infty}(1/2^k) f(2^k e)$ *is invertible and for each fixed* $x \in A$ *the mapping* $t \mapsto f(tx)$ *is continuous in* $t \in \mathbb{R}$. *Then the mapping* $f : A \to B$ *is a* C^**-algebra homomorphism.*

Proof. Put $\mu = 1$ in (3.16). By the same reasoning as in the proof of Theorem 2.3, there exists a unique generalized Euler-Lagrange type additive mapping $H : A \to B$ defined by

$$H(x) = \lim_{k\to\infty} \frac{f(2^k x)}{2^k} \tag{3.17}$$

for all $x \in A$. By the same reasoning as in the proof of [4], the generalized Euler-Lagrange type additive mapping $H : A \to B$ is $\mathbb{R}$-linear.

By the same method as in the proof of Theorem 2.4, we have

$$\begin{aligned} &\Bigg\| D_{\mu, r_1, \ldots, r_n} H(0, \ldots, 0, \underbrace{x}_{j\text{th}}, 0, \ldots, 0) \Bigg\|_Y \\ &\quad = \lim_{k\to\infty} \frac{1}{2^k} \Bigg\| D_{\mu, r_1, \ldots, r_n} f(0, \ldots, 0, \underbrace{2^k x}_{j\text{th}}, 0, \ldots, 0) \Bigg\|_Y \\ &\quad \le \lim_{k\to\infty} \frac{1}{2^k} \varphi\Bigg(0, \ldots, 0, \underbrace{2^k x}_{j\text{th}}, 0, \ldots, 0\Bigg) = 0 \end{aligned} \tag{3.18}$$

for all $x \in A$. So

$$r_j \mu H(x) = H(r_j \mu x) \tag{3.19}$$

for all $x \in A$. Since $H(r_j x) = r_j H(x)$ for all $x \in X$ and $r_j \neq 0$,

$$H(\mu x) = \mu H(x) \tag{3.20}$$

for $\mu = i, 1$ and for all $x \in A$.

For each element $\lambda \in \mathbb{C}$ we have $\lambda = s + it$, where $s, t \in \mathbb{R}$. Thus

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) \\ &= sH(x) + itH(x) = (s + it)H(x) = \lambda H(x) \end{aligned} \tag{3.21}$$

for all $\lambda \in \mathbb{C}$ and all $x \in A$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y) \tag{3.22}$$

for all $\zeta, \eta \in \mathbb{C}$ and all $x, y \in A$. Hence the generalized Euler-Lagrange type additive mapping $H : A \to B$ is $\mathbb{C}$-linear. The rest of the proof is the same as in the proof of Theorem 3.5. $\square$

The following theorem is an alternative result of Theorem 3.9.

Theorem 3.10. *Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there is a function $\phi : A^n \to [0, \infty)$ satisfying* (2.36), (2.37), (3.7) *and*

$$\left\| D_{\mu, r_1, \ldots, r_n} f(x_1, \ldots, x_n) \right\|_B \le \phi(x_1, \ldots, x_n), \tag{3.23}$$

for $\mu = i, 1$ and all $x, x_1, \ldots, x_n \in A$. Assume that $\lim_{k\to\infty} 2^k f(e/2^k)$ is invertible and for each fixed $x \in A$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Remark 3.11. In Theorem 3.10, one can assume that $\sum_{k=1}^{n} r_k \neq 0$ instead of $f(0) = 0$.

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