Research Article

On the Identities of Symmetry for the ζ-Euler Polynomials of Higher Order

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The main purpose of this paper is to investigate several further interesting properties of symmetry for the multivariate p-adic fermionic integral on $\mathbb{Z}_p$. From these symmetries, we can derive some recurrence identities for the ζ-Euler polynomials of higher order, which are closely related to the Frobenius-Euler polynomials of higher order. By using our identities of symmetry for the ζ-Euler polynomials of higher order, we can obtain many identities related to the Frobenius-Euler polynomials of higher order.

1. Introduction/Definition

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integer, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \tag{1.1}$$

(see [1]). Let us define the fermionic p-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f) = \lim_{q \to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) \tag{1.2}$$

(see [1–8]). From (1.2), we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0) \tag{1.3}$$

(see [9, 10]), where $f_1(x) = f(x + 1)$. For $\zeta \in \mathbb{C}_p$ with $|1 - \zeta|_p < 1$, let $f(x) = e^{xt}\zeta^x$. Then, we define the ζ-Euler numbers as follows:

$$\int_{\mathbb{Z}_p} \zeta^x e^{xt} d\mu_{-1}(x) = \frac{2}{\zeta e^t + 1} = \sum_{n=0}^{\infty} E_{n,\zeta} \frac{t^n}{n!}, \tag{1.4}$$

where $E_{n,\zeta}$ are called the ζ-Euler numbers. We can show that

$$\frac{2}{\zeta e^t + 1} = \frac{1 + \zeta^{-1}}{e^t + \zeta^{-1}} \cdot \frac{2}{1 + \zeta} = \frac{2}{1 + \zeta} \sum_{n=0}^{\infty} H_n\left(-\zeta^{-1}\right) \frac{t^n}{n!}, \tag{1.5}$$

where $H_n(-\zeta^{-1})$ are the Frobenius-Euler numbers. By comparing the coefficients on both sides of (1.4) and (1.5), we see that

$$E_{n,\zeta} = \frac{2}{1 + \zeta} H_n\left(-\zeta^{-1}\right). \tag{1.6}$$

Now, we also define the ζ-Euler polynomials as follows:

$$\frac{2}{\zeta e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,\zeta}(x) \frac{t^n}{n!}. \tag{1.7}$$

In the viewpoint of (1.5), we can show that

$$\frac{2}{\zeta e^t + 1} e^{xt} = e^{xt} \frac{1 + \zeta^{-1}}{e^t + \zeta^{-1}} \cdot \frac{2}{1 + \zeta} = \frac{2}{1 + \zeta} \sum_{n=0}^{\infty} H_n\left(-\zeta^{-1}, x\right) \frac{t^n}{n!}, \tag{1.8}$$

where $H_n(-\zeta^{-1}, x)$ are the nth Frobenius-Euler polynomials. From (1.7) and (1.8), we note that

$$E_{n,\zeta}(x) = \frac{2}{1 + \zeta} H_n\left(-\zeta^{-1}, x\right) \tag{1.9}$$

(cf. [1–8, 11–18]). For each positive integer k, let $T_{k,\zeta}(n) = \sum_{\ell=0}^{n} (-1)^{\ell} \zeta^{\ell} \ell^k$. Then we have

$$\sum_{k=0}^{\infty} T_{k,\zeta}(n) \frac{t^k}{k!} = \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{n} (-1)^{\ell} \ell^k \zeta^{\ell}\right) \frac{t^k}{k!} = \sum_{\ell=0}^{n} (-1)^{\ell} \zeta^{\ell} e^{\ell t} = \frac{1 + (-1)^{n+1} e^{(n+1)t}}{\zeta e^t + 1}. \tag{1.10}$$

The ζ-Euler polynomials of order k, denoted $E_{n,\zeta}^{(k)}(x)$, are defined as

$$e^{xt}\left(\frac{2}{\zeta e^t + 1}\right)^k = \left(\frac{2}{\zeta e^t + 1}\right) \times \cdots \times \left(\frac{2}{\zeta e^t + 1}\right)e^{xt} = \sum_{n=0}^{\infty} E_{n,\zeta}^{(k)}(x)\frac{t^n}{n!}. \tag{1.11}$$

Then the values of $E_{n,\zeta}^{(k)}(x)$ at $x = 0$ are called the ζ-Euler numbers of order k. When $k = 1$, the polynomials or numbers are called the ζ-Euler polynomials or numbers. The purpose of this paper is to investigate some properties of symmetry for the multivariate p-adic fermionic integral on $\mathbb{Z}_p$. From the properties of symmetry for the multivariate p-adic fermionic integral on $\mathbb{Z}_p$, we derive some identities of symmetry for the ζ-Euler polynomials of higher order. By using our identities of symmetry for the ζ-Euler polynomials of higher order, we can obtain many identities related to the Frobenius-Euler polynomials of higher order.

2. On the Symmetry for the ζ-Euler Polynomials of Higher Order

Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$ and $w_2 \equiv 1 \pmod 2$. Then we set

$$R^{(m)}(\mathrm{w}_1, w_2) = \frac{\int_{\mathbb{Z}_p^m} e^{w_1(x_1+x_2+\cdots+x_m+w_2x)t}\zeta^{w_1x_1+\cdots+w_1x_m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}e^{w_1w_2xt}d\mu_{-1}(x)}$$

$$\times \int_{\mathbb{Z}_p^m} e^{w_2(x_1+x_2+\cdots+x_m+w_1y)t}\zeta^{w_2x_1+\cdots+w_2x_m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m), \tag{2.1}$$

where

$$\int_{\mathbb{Z}_p^m} f(x_1,\ldots,x_m)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} f(x_1,\ldots,x_m)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m). \tag{2.2}$$

Thus, we note that this expression for $R^{(m)}(w_1, w_2)$ is symmetry in w_1 and w_2. From (2.1), we have

$$R^{(m)}(w_1, w_2) = \left(\int_{\mathbb{Z}_p^m} e^{w_1(x_1+\cdots+x_m)t}\zeta^{w_1x_1+\cdots+w_1x_m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m)\right)e^{w_1w_2xt}$$

$$\times \left(\frac{\int_{\mathbb{Z}_p}e^{w_2x_mt}\zeta^{w_2x_m}d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p}e^{w_1w_2xt}\zeta^{w_1w_2x}d\mu_{-1}(x)}\right) \tag{2.3}$$

$$\times \left(\int_{\mathbb{Z}_p^{m-1}} e^{w_2(x_1+\cdots+x_{m-1})t}\zeta^{w_2x_1+\cdots+w_2x_{m-1}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_{m-1})\right)e^{w_1w_2yt}.$$

We can show that

$$\frac{\int_{\mathbb{Z}_p} e^{xt}\zeta^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} e^{w_1 xt}\zeta^{w_1 x} d\mu_{-1}(x)} = \sum_{\ell=0}^{w_1-1}(-1)^\ell \zeta^\ell e^{\ell t} = \sum_{k=0}^{\infty}(T_{k,\zeta}(w_1-1))\frac{t^k}{k!}. \tag{2.4}$$

By (1.4) and (1.11), we see that

$$\left(\int_{\mathbb{Z}_p^m} e^{w_1(x_1+\cdots+x_m)t}\zeta^{w_1 x_1+\cdots+w_1 x_m} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m)\right)e^{w_1 w_2 xt}$$

$$= \left(\frac{2}{\zeta^{w_1}e^{w_1 t}+1}\right)^m e^{w_1 w_2 xt} \tag{2.5}$$

$$= \sum_{n=0}^{\infty}E_{n,\zeta^{w_1}}^{(m)}(w_2 x)w_1^n \frac{t^n}{n!}.$$

Thus, we have

$$E_{n,\zeta^{w_1}}^{(m)}(w_2 x) = \sum_{\ell=0}^{n}\binom{n}{\ell}E_{\ell,\zeta^{w_1}}^{(m)}w_2^{n-\ell}x^{n-\ell}. \tag{2.6}$$

From (2.3), (2.4), and (2.5), we can derive

$$R^{(m)}(w_1,w_2) = \left(\sum_{\ell=0}^{\infty}E_{\ell,\zeta^{w_1}}^{(m)}(w_2 x)w_1^\ell \frac{t^\ell}{\ell!}\right)\left(\sum_{k=0}^{\infty}T_{k,\zeta^{w_2}}(w_1-1)\frac{w_2^k}{k!}t^k\right)\left(\sum_{i=0}^{\infty}E_{i,\zeta^{w_2}}^{(m-1)}(w_1 y)\frac{w_2^i}{i!}t^i\right)$$

$$= \left(\sum_{\ell=0}^{\infty}E_{\ell,\zeta^{w_1}}^{(m)}(w_2 x)w_1^\ell \frac{t^\ell}{\ell!}\right)\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}T_{k,\zeta^{w_2}}(w_1-1)w_2^k w_2^{j-k}\frac{E_{j-k}^{(m-1)}(w_1 y)}{k!(j-k)!}j!\right)\frac{t^j}{j!}\right)$$

$$= \left(\sum_{\ell=0}^{\infty}E_{\ell,\zeta^{w_1}}^{(m)}(w_2 x)w_1^\ell \frac{t^\ell}{\ell!}\right)\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}T_{k,\zeta^{w_2}}(w_1-1)\binom{j}{k}E_{j-k,\zeta^{w_2}}^{(m-1)}(w_1 y)\right)w_2^j \frac{t^j}{j!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\left(\sum_{k=0}^{j}T_{k,\zeta^{w_2}}(w_1-1)\binom{j}{k}E_{j-k,\zeta^{w_2}}^{(m-1)}(w_1 y)\right)\frac{w_2^j w_1^{n-j}}{(n-j)!j!}E_{n-j,\zeta^{w_1}}^{(m)}(w_2 x)n!\right)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}w_2^j w_1^{n-j}E_{n-j,\zeta^{w_1}}^{(m)}(w_2 x)\sum_{k=0}^{j}T_{k,\zeta^{w_2}}(w_1-1)\binom{j}{k}E_{j-k,\zeta^{w_2}}^{(m-1)}(w_1 y)\right)\frac{t^n}{n!}. \tag{2.7}$$

By the same method, we also see that

$$
\begin{aligned}
R^{(m)}(w_1, w_2) &= \left(\int_{\mathbb{Z}_p^m} e^{w_2(x_1+\cdots+x_m)t} \zeta^{w_2 x_1+\cdots+w_2 x_m} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m) \right) e^{w_1 w_2 xt} \\
&\quad \times \left(\frac{\int_{\mathbb{Z}_p} e^{w_1 x_m t} \zeta^{w_1 x_m} d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 xt} \zeta^{w_1 w_2 x} d\mu_{-1}(x)} \right) \\
&\quad \times \left(\int_{\mathbb{Z}_p^{m-1}} e^{w_1(x_1+\cdots+x_{m-1})t} \zeta^{w_1 x_1+\cdots+w_1 x_{m-1}} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_{m-1}) \right) e^{w_1 w_2 yt} \\
&= \left(\sum_{\ell=0}^{\infty} E_{\ell,\zeta^{w_2}}^{(m)}(w_1 x) w_2^\ell \frac{t^\ell}{\ell!} \right) \left(\sum_{k=0}^{\infty} T_{k,\zeta^{w_1}}(w_2-1) w_1^k \frac{t^k}{k!} \right) \left(\sum_{i=0}^{\infty} E_{i,\zeta^{w_1}}^{(m-1)}(w_2 y) w_1^i \frac{t^i}{i!} \right) \\
&= \left(\sum_{\ell=0}^{\infty} E_{\ell,\zeta^{w_2}}^{(m)}(w_1 x) w_2^\ell \frac{t^\ell}{\ell!} \right) \left(\sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \frac{T_{k,\zeta^{w_1}}(w_2-1)}{k!} \frac{E_{j-k}^{(m-1)}(w_2 y)}{(j-k)!} \right) w_1^j t^j \right) \\
&= \left(\sum_{\ell=0}^{\infty} E_{\ell,\zeta^{w_2}}^{(m)}(w_1 x) w_2^\ell \frac{t^\ell}{\ell!} \right) \left(\sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \frac{T_{k,\zeta^{w_1}}(w_2-1) E_{j-k}^{(m-1)}(w_2 y)}{k!(j-k)!} j! \right) \frac{w_1^j t^j}{j!} \right) \\
&= \left(\sum_{\ell=0}^{\infty} E_{\ell,\zeta^{w_2}}^{(m)}(w_1 x) w_2^\ell \frac{t^\ell}{\ell!} \right) \left(\sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \binom{j}{k} T_{k,\zeta^{w_1}}(w_2-1) E_{j-k,\zeta^{w_1}}^{(m-1)}(w_2 y) \right) w_1^j \frac{t^j}{j!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \left(\sum_{k=0}^{j} \binom{j}{k} T_{k,\zeta^{w_1}}(w_2-1) E_{j-k,\zeta^{w_1}}^{(m-1)}(w_2 y) \right) \frac{w_1^j w_2^{n-j}}{j!(n-j)!} E_{n-j,\zeta^{w_2}}^{(m)}(w_1 x) n! \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} E_{n-j,\zeta^{w_2}}^{(m)}(w_1 x) \sum_{k=0}^{j} \binom{j}{k} T_{k,\zeta^{w_1}}(w_2-1) E_{j-k,\zeta^{w_1}}^{(m-1)}(w_2 y) \right) \frac{t^n}{n!}.
\end{aligned}
\tag{2.8}
$$

By comparing the coefficients on both sides of (2.7) and (2.8), we obtain the following.

Theorem 2.1. *For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 (\mathrm{mod}\, 2)$, $w_2 \equiv 1 (\mathrm{mod}\, 2)$, and $n \geq 0, m \geq 1$, one has*

$$
\begin{aligned}
&\sum_{j=0}^{n} \binom{n}{j} w_2^j w_1^{n-j} E_{n-j,\zeta^{w_1}}^{(m)}(w_2 x) \sum_{k=0}^{j} T_{k,\zeta^{w_2}}(w_1-1) \binom{j}{k} E_{j-k,\zeta^{w_2}}^{(m-1)}(w_1 y) \\
&= \sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} E_{n-j,\zeta^{w_2}}^{(m)}(w_1 x) \sum_{k=0}^{j} \binom{j}{k} T_{k,\zeta^{w_1}}(w_2-1) E_{j-k,\zeta^{w_1}}^{(m-1)}(w_2 y).
\end{aligned}
\tag{2.9}
$$

Let $y = 0$ and $m = 1$ in (2.9). Then we have

$$
\begin{aligned}
\sum_{j=0}^{n} \binom{n}{j} & w_1^{n-j} w_2^{j} E_{n-j,\zeta^{w_1}}(w_2 x) T_{k,\zeta^{w_2}}(w_1 - 1) \\
&= \sum_{j=0}^{n} \binom{n}{j} w_1^{j} w_2^{n-j} E_{n-j,\zeta^{w_2}}(w_1 x) T_{k,\zeta^{w_1}}(w_2 - 1).
\end{aligned}
\tag{2.10}
$$

From (2.10), we note that

$$
\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} & w_1^{i} w_2^{n-i} E_{i,\zeta^{w_1}}(w_2 x) T_{n-i,\zeta^{w_2}}(w_1 - 1) \\
&= \sum_{i=0}^{n} \binom{n}{i} w_1^{n-i} w_2^{i} E_{i,\zeta^{w_2}}(w_1 x) T_{n-i,\zeta^{w_1}}(w_2 - 1).
\end{aligned}
\tag{2.11}
$$

If we take $w_2 = 1$ in (2.11), then we have

$$
E_{n,\zeta}(w_1 x) = \sum_{i=0}^{n} \binom{n}{i} w_1^{i} E_{i,\zeta^{w_1}}(x) T_{n-i,\zeta}(w_1 - 1).
\tag{2.12}
$$

From (2.3), we note that

$$
\begin{aligned}
R^{(m)}(w_1, w_2) = {} & \left(\int_{\mathbb{Z}_p^m} e^{w_1(x_1 + \cdots + x_m)t} \zeta^{w_1 x_1 + \cdots + w_1 x_m} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m) \right) e^{w_1 w_2 x t} \\
& \times \left(\frac{\int_{\mathbb{Z}_p} e^{w_2 x_m t} \zeta^{w_2 x_m} d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} \zeta^{w_1 w_2 x} d\mu_{-1}(x)} \right) \\
& \times \left(\int_{\mathbb{Z}_p^{m-1}} e^{w_2(x_1 + \cdots + x_{m-1})t} \zeta^{w_2 x_1 + \cdots + w_2 x_{m-1}} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_{m-1}) \right) e^{w_1 w_2 y t} \\
= {} & \left(\int_{\mathbb{Z}_p^m} e^{w_1(x_1 + \cdots + x_m)t} \zeta^{w_1 x_1 + \cdots + w_1 x_m} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m) \right) e^{w_1 w_2 x t} \\
& \times \left(\sum_{i=0}^{w_1 - 1} (-1)^i e^{w_2 i t} \zeta^{w_2 i} \right) \\
& \times \left(\int_{\mathbb{Z}_p^{m-1}} e^{w_2(x_1 + \cdots + x_{m-1})t} \zeta^{w_2 x_1 + \cdots + w_2 x_{m-1}} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_{m-1}) \right) e^{w_1 w_2 y t}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\sum_{i=0}^{w_1-1} (-1)^i \zeta^{w_2 i} \int_{\mathbb{Z}_p^m} e^{w_1(x_1+\cdots+x_m+(w_2/w_1)i+w_2 x)t} \zeta^{w_1 x_1 + \cdots + w_1 x_m} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m) \right) \\
&\quad \times \left(\int_{\mathbb{Z}_p^{m-1}} e^{w_2(x_1+\cdots+x_{m-1}+w_1 y)t} \zeta^{w_2 x_1 + \cdots + w_2 x_{m-1}} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_{m-1}) \right) \\
&= \left(\sum_{i=0}^{w_1-1} (-1)^i \zeta^{w_2 i} \sum_{k=0}^{\infty} E_{k,\zeta^{w_1}}^{(m)} \left(\frac{w_2}{w_1} i + w_2 x \right) w_1^k \frac{t^k}{k!} \right) \left(\sum_{\ell=0}^{\infty} E_{\ell,\zeta^{w_2}}^{(m-1)}(w_1 y) w_2^{\ell} \frac{t^{\ell}}{\ell!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(\sum_{i=0}^{w_1-1} (-1)^i \zeta^{w_2 i} E_{k,\zeta^{w_1}}^{(m)} \left(w_2 x + \frac{w_2}{w_1} i \right) \right) \frac{w_1^k}{k!} E_{n-k,\zeta^{w_2}}^{(m-1)}(w_1 y) \frac{w_2^{n-k}}{(n-k)!} n! \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} w_1^k w_2^{n-k} E_{n-k,\zeta^{w_2}}^{(m-1)}(w_1 y) \sum_{i=0}^{w_1-1} (-1)^i \zeta^{w_2 i} E_{k,\zeta^{w_1}}^{(m)} \left(w_2 x + \frac{w_2}{w_1} i \right) \right) \frac{t^n}{n!}.
\end{aligned}
$$

$$(2.13)$$

By the symmetric property of $R^{(m)}(w_1, w_2)$ in w_1, w_2, we also see that

$$
\begin{aligned}
R^{(m)}(w_1, w_2) &= \left(\int_{\mathbb{Z}_p^m} e^{w_2(x_1+\cdots+x_m)t} \zeta^{w_2(x_1+\cdots+x_m)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m) \right) e^{w_1 w_2 x t} \\
&\quad \times \left(\frac{\int_{\mathbb{Z}_p} e^{w_1 x_m t} \zeta^{w_1 x_m} d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} \zeta^{w_1 w_2 x} d\mu_{-1}(x)} \right) \\
&\quad \times \left(\int_{\mathbb{Z}_p^{m-1}} e^{w_1(x_1+\cdots+x_{m-1})t} \zeta^{w_1(x_1+\cdots+x_{m-1})} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_{m-1}) \right) e^{w_1 w_2 y t} \\
&= \left(\int_{\mathbb{Z}_p^m} e^{w_2(x_1+\cdots+x_m)t} \zeta^{w_2(x_1+\cdots+x_m)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m) \right) e^{w_1 w_2 x t} \\
&\quad \times \left(\sum_{i=0}^{w_2-1} (-1)^i e^{w_1 i t} \zeta^{w_1 i} \right) \\
&\quad \times \left(\int_{\mathbb{Z}_p^{m-1}} e^{w_1(x_1+\cdots+x_{m-1}+w_2 y)t} \zeta^{w_1(x_1+\cdots+x_{m-1})} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_{m-1}) \right) \\
&= \sum_{i=0}^{w_2-1} (-1)^i \zeta^{w_1 i} \left(\int_{\mathbb{Z}_p^m} e^{w_2(x_1+\cdots+x_m+(w_1/w_2)i+w_1 x)t} \zeta^{w_2(x_1+\cdots+x_m)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m) \right) \\
&\quad \times \left(\int_{\mathbb{Z}_p^{m-1}} e^{w_1(x_1+\cdots+x_{m-1}+w_2 y)t} \zeta^{w_1 x_1 + \cdots + w_1 x_{m-1}} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_{m-1}) \right) \\
&= \left(\sum_{i=0}^{w_2-1} (-1)^i \zeta^{w_1 i} \sum_{k=0}^{\infty} E_{k,\zeta^{w_2}}^{(m)} \left(\frac{w_1}{w_2} i + w_1 x \right) w_2^k \frac{t^k}{k!} \right) \left(\sum_{\ell=0}^{\infty} E_{\ell,\zeta^{w_1}}^{(m-1)}(w_2 y) w_1^{\ell} \frac{t^{\ell}}{\ell!} \right)
\end{aligned}
$$

$$
= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(\sum_{i=0}^{w_2-1} (-1)^i \zeta^{w_1 i} E_{k,\zeta^{w_2}}^{(m)} \left(w_1 x + \frac{w_1}{w_2} i \right) \right) \frac{w_2^k}{k!} E_{n-k}^{(m-1)} (w_2 y) \frac{w_1^{n-k}}{(n-k)!} n! \right) \frac{t^n}{n!}
$$

$$
= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} w_2^k w_1^{n-k} E_{n-k,\zeta^{w_1}}^{(m-1)} (w_2 y) \sum_{i=0}^{w_2-1} (-1)^i \zeta^{w_1 i} E_{k,\zeta^{w_2}}^{(m)} \left(w_1 x + \frac{w_1}{w_2} i \right) \right) \frac{t^n}{n!}.
$$

$$(2.14)$$

By comparing the coefficients on both sides of (2.13) and (2.14), we obtain the following theorem.

Theorem 2.2. *For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 (\mathrm{mod}\ 2)$ and $w_2 \equiv 1 (\mathrm{mod}\ 2)$, one has*

$$
\sum_{k=0}^{n} \binom{n}{k} w_1^k w_2^{n-k} E_{n-k,\zeta^{w_2}}^{(m-1)} (w_1 y) \sum_{i=0}^{w_1-1} (-1)^i \zeta^{w_2 i} E_{k,\zeta^{w_1}}^{(m)} \left(w_2 x + \frac{w_2}{w_1} i \right)
$$

$$
= \sum_{k=0}^{n} \binom{n}{k} w_2^k w_1^{n-k} E_{n-k,\zeta^{w_1}}^{(m-1)} (w_2 y) \sum_{i=0}^{w_2-1} (-1)^i \zeta^{w_1 i} E_{k,\zeta^{w_2}}^{(m)} \left(w_1 x + \frac{w_1}{w_2} i \right).
$$

$$(2.15)$$

Let $y = 0$ and $m = 1$, we have

$$
w_1^n \sum_{i=0}^{w_1-1} (-1)^i \zeta^{w_2 i} E_{n,\zeta^{w_1}} \left(w_2 x + \frac{w_2}{w_1} i \right) = w_2^n \sum_{i=0}^{w_2-1} (-1)^i \zeta^{w_1 i} E_{n,\zeta^{w_2}} \left(w_1 x + \frac{w_1}{w_2} i \right).
$$

$$(2.16)$$

From (2.16), we can derive

$$
\sum_{i=0}^{w_1-1} (-1)^i \zeta^i E_{n,\zeta^{w_1}} \left(x + \frac{1}{w_1} i \right) = \frac{1}{w_1^n} E_{n,\zeta} (w_1 x).
$$

$$(2.17)$$

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