Research Article

Inequalities among Eigenvalues of Second-Order Difference Equations with General Coupled Boundary Conditions

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This paper studies general coupled boundary value problems for second-order difference equations. Existence of eigenvalues is proved, numbers of their eigenvalues are calculated, and their relationships between the eigenvalues of second-order difference equation with three different coupled boundary conditions are established.

1. Introduction

Consider the second-order difference equation

$$-\nabla\left(p_n \Delta y_n\right) + q_n y_n = \lambda w_n y_n, \quad n \in [0, N-1] \tag{1.1}$$

with the general coupled boundary condition

$$\begin{pmatrix} y_{N-1} \\ \Delta y_{N-1} \end{pmatrix} = e^{i\alpha} K \begin{pmatrix} y_{-1} \\ \Delta y_{-1} \end{pmatrix}, \tag{1.2}$$

where $N \geq 2$ is an integer, Δ is the forward difference operator: $\Delta y_n = y_{n+1} - y_n$, ∇ is the backward difference operator: $\nabla y_n = y_n - y_{n-1}$, and p_n, q_n, and w_n are real numbers with $p_n > 0$ for $n \in [-1, N-1]$, $w_n > 0$ for $n \in [0, N-1]$, and $p_{-1} = p_{N-1} = 1$; λ is the spectral

parameter; the interval $[0, N - 1]$ is the integral set $\{n\}_{n=0}^{N-1}$; α, $-\pi < \alpha \le \pi$ is a constant parameter; $i = \sqrt{-1}$,

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}, \qquad k_{ij} \in \mathbf{R}, \quad i, j = 1, 2, \ \text{with det } K = 1. \tag{1.3}$$

The boundary condition (1.2) contains the periodic and antiperiodic boundary conditions. In fact, (1.2) is the periodic boundary condition in the case where $\alpha = 0$ and $K = I$, the identity matrix, and (1.2) is the antiperiodic condition in the case where $\alpha = \pi$ and $K = I$.

We first briefly recall some relative existing results of eigenvalue problems for difference equations. Atkinson [1, Chapter 6, Section 2] discussed the boundary conditions

$$y_{-1} = \alpha y_{m-1}, \qquad y_m = \beta y_0 \tag{1.4}$$

when he investigated the recurrence formula

$$c_n y_{n+1} = \left(a_n \widehat{\lambda} + b_n\right) y_n - c_{n-1} y_{n-1}, \quad n \in [0, m-1], \tag{1.5}$$

where a_n, b_n, c_n, α, and β are real numbers, subject to $a_n > 0$, $c_n > 0$, and

$$\alpha c_{-1} = \beta c_{m-1}. \tag{1.6}$$

He remarked that all the eigenvalues of the boundary value problem (1.4) and (1.5) are real, and they may not be all distinct. If $c_{-1} = c_{m-1}$ and $\alpha = \beta = 1$, he viewed the boundary conditions (1.4) as the periodic boundary conditions for (1.5). Shi and Chen [2] investigated the more general boundary value problem

$$-\nabla(C_n \Delta x_n) + B_n x_n = \lambda w_n x_n, \quad n \in [1, N], \ N \ge 2, \tag{1.7}$$

$$R \begin{pmatrix} -x_0 \\ x_N \end{pmatrix} + S \begin{pmatrix} C_0 \Delta x_0 \\ C_N \Delta x_N \end{pmatrix} = 0, \tag{1.8}$$

where C_n, B_n, and w_n are $d \times d$ Hermitian matrices; C_0 and C_N are nonsingular; $w_n > 0$ for $n \in [1, N]$; R and S are $2d \times 2d$ matrices. Moreover, R and S satisfy $\text{rank}(R, S) = 2d$ and the self-adjoint condition $RS^* = SR^*$ [2, Lemma 2.1]. A series of spectral results was obtained. We will remark that the boundary condition (1.8) includes the coupled boundary condition (1.2) when $d = 1$, and the boundary conditions (1.4) when (1.6) holds. Agarwal and Wong studied existence of minimal and maximal quasisolutions of a second-order nonlinear periodic boundary value problem [3, Section 4]. In 2005, Wang and Shi [4] considered (1.1) with the periodic and antiperiodic boundary conditions. They found out the following results

(see [4, Theorems 2.2 and 3.1]): the periodic and antiperiodic boundary value problems have exactly N real eigenvalues $\{\lambda_i\}_{i=0}^{N-1}$ and $\{\tilde{\lambda}_i\}_{i=1}^{N}$, respectively, which satisfy

$$
\begin{aligned}
\lambda_0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 < \lambda_1 \leq \lambda_2 < \tilde{\lambda}_3 \leq \tilde{\lambda}_4 < \cdots < \lambda_{N-2} \leq \lambda_{N-1} < \tilde{\lambda}_N, \quad \text{if } N \text{ is odd,} \\
\lambda_0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 < \lambda_1 \leq \lambda_2 < \tilde{\lambda}_3 \leq \tilde{\lambda}_4 < \cdots < \tilde{\lambda}_{N-1} \leq \tilde{\lambda}_N < \lambda_{N-1}, \quad \text{if } N \text{ is even.}
\end{aligned}
\tag{1.9}
$$

These results are similar to those about eigenvalues of periodic and antiperiodic boundary value problems for second-order ordinary differential equations (cf. [5–8]).

Motivated by [4], we compare the eigenvalues of the eigenvalue problem (1.1) with the coupled boundary condition (1.2) as α varies and obtain relationships between the eigenvalues in the present paper. These results extend the above results obtained in [4]. In this paper, we will apply some results obtained by Shi and Chen [2] to prove the existence of eigenvalues of (1.1) and (1.2) to calculate the number of these eigenvalues, and to apply some oscillation results obtained by Agarwal et al. [9] to compare the eigenvalues as α varies.

This paper is organized as follows. Section 2 gives some preliminaries including existence and numbers of eigenvalues of the coupled boundary value problems, and some properties of eigenvalues of a kind of separated boundary value problem, which will be used in the next section. Section 3 pays attention to comparison between the eigenvalues of problem (1.1) and (1.2) as α varies.

2. Preliminaries

Equation (1.1) can be rewritten as the recurrence formula

$$
p_n y_{n+1} = (p_n + p_{n-1} + q_n - \lambda w_n) y_n - p_{n-1} y_{n-1}, \quad n \in [0, N-1].
\tag{2.1}
$$

Clearly, y_n is a polynomial in λ with real coefficients since p_n, q_n, and w_n are all real. Hence, all the solutions of (1.1) are entire functions of λ. Especially, if $y_0 \neq 0$, y_n is a polynomial of degree n in λ for $n \leq N$. However, if $y_{-1} \neq 0$ and $y_0 = 0$, y_n is a polynomial of degree $n-1$ in λ for $n \leq N$.

We now prepare some results that are useful in the next section. The following lemma is mentioned in [4, Theorem 2.1].

Lemma 2.1 ([4, Theorem 2.1]). *Let y and z be any solutions of* (1.1). *Then the Wronskian*

$$
W[y, z](n) = \begin{vmatrix} y_{n+1} & z_{n+1} \\ p_n \Delta y_n & p_n \Delta z_n \end{vmatrix} = -p_n (y_{n+1} z_n - y_n z_{n+1})
\tag{2.2}
$$

is a constant on $[-1, N-1]$.

Theorem 2.2. *If $k_{11} \neq k_{12}$ then the coupled boundary value problem* (1.1) *and* (1.2) *has exactly N real eigenvalues.*

Proof. By setting $d = 1$, $C_n = p_n$, $B_n = q_n$,

$$R = (R_1, R_2) = \begin{pmatrix} e^{i\alpha}k_{11} & 1 \\ e^{i\alpha}k_{21} & 0 \end{pmatrix}, \qquad S = (S_1, S_2) = \begin{pmatrix} -e^{i\alpha}k_{12} & 0 \\ -e^{i\alpha}k_{22} & 1 \end{pmatrix}, \tag{2.3}$$

shifting the whole interval $[1, N]$ left by one unit, and using $p_{-1} = p_{N-1} = 1$, (1.1) and (1.2) are written as (1.7) and (1.8), respectively. It is evident that $\text{rank}(R, S) = 2d$ and $RS^* = SR^*$. Hence, the boundary condition (1.2) is self-adjoint by [2, Lemma 2.1]. In addition, it follows from (2.3) and $C_{-1} = 1$ that

$$(R_1 + S_1 C_{-1}, S_2) = \begin{pmatrix} e^{i\alpha}(k_{11} - k_{12}) & 0 \\ e^{i\alpha}(k_{21} - k_{22}) & 1 \end{pmatrix}. \tag{2.4}$$

By noting that $k_{11} \neq k_{12}$, we get $\text{rank}(R_1 + S_1 C_{-1}, S_2) = 2$. Therefore, by [2, Theorem 4.1], the problem (1.1) and (1.2) has exactly N real eigenvalues. This completes the proof. □

Let $y_n(\lambda)$ be the solution of (1.1) with the initial conditions

$$y_{-1}(\lambda) = 0, \qquad y_0(\lambda) \neq 0. \tag{2.5}$$

Consider the sequence

$$y_0(\lambda), y_1(\lambda), \ldots, y_{N-1}(\lambda). \tag{2.6}$$

If $y_n(\lambda) = 0$ for some $n \in (0, N-1)$, then, we get from (2.1) that $y_{n-1}(\lambda)$ and $y_{n+1}(\lambda)$ have opposite signs. Hence, we say that sequence (2.6) exhibits a change of sign if $y_n(\lambda)y_{n+1}(\lambda) < 0$ for some $n \in [0, N-1)$, or $y_n(\lambda) = 0$ for some $n \in (0, N-1)$. A general zero of the sequence (2.6) is defined as its zero or a change of sign.

Now we consider (1.1) with the following separated boundary conditions:

$$y_{-1} = 0, \qquad k_{12}\Delta y_{N-1} - k_{22}y_{N-1} = 0, \tag{2.7}$$

where k_{12}, k_{22} are entries of K. It follows from (2.1) that the separated boundary value problem (1.1) with (2.7) has a unique solution, and the separated boundary value problem will be used to compare the eigenvalues of (1.1) and (1.2) as α varies in the next section.

In [9], Agarwal et al. studied the following boundary value problem on time scales:

$$y^{\Delta\Delta} + q(t)y^{\sigma} = -\lambda y^{\sigma}, \quad t \in [\rho(a), \rho(b)] \cap \mathbf{T}, \tag{2.8}$$

with the boundary conditions

$$R_a(y) := \alpha y(\rho(a)) + \beta y^{\Delta}(\rho(a)) = 0, \qquad R_b(y) := \gamma y(b) + \delta y^{\Delta}(b) = 0, \tag{2.9}$$

where $\mathbf{T}$ is a time scale, $\sigma(t)$ and $\rho(t)$ are the forward and backward jump operators in $\mathbf{T}$, y^Δ is the delta derivative, and $y^\sigma(t) := y(\sigma(t))$; $q : [\rho(a), \rho(b)] \cap \mathbf{T} \rightarrow \mathbf{R}$ is continuous; $(\alpha^2 + \beta^2)(\gamma^2 + \delta^2) \neq 0$; $a, b \in \mathbf{T}$ with $a < b$. They obtained some useful oscillation results. With a similar argument to that used in the proof of [9, Theorem 1], one can show the following result.

Lemma 2.3. *The eigenvalues of the boundary value problem are*

$$-\left(p(t)y^\Delta(t)\right)^\Delta + q^\sigma(t)y^\sigma(t) = \lambda r^\sigma(t)y^\sigma(t), \quad t \in [\rho(a), \rho(b)] \cap \mathbf{T}, \tag{2.10}$$

with

$$R_a(y) = R_b(y) = 0, \tag{2.11}$$

where p^Δ, q^σ, and r^σ are real and continuous functions in $[\rho(a), \rho(b)] \cap \mathbf{T}$, $p > 0$ over $[\rho(a), b] \cap \mathbf{T}$, $r^\sigma > 0$ over $[\rho(a), \rho(b)] \cap \mathbf{T}, p(\rho(a)) = p(b) = 1$ are arranged as $-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$, and an eigenfunction corresponding to λ_k has exactly k generalized zeros in the open interval (a, b).

By setting $[\rho(a), b] \cap \mathbf{T} = [-1, N-1] := \{n\}_{-1}^{N-1}$, $\alpha = 1$, $\beta = 0$, $\gamma = -k_{22}$, $\delta = k_{12}$, the above boundary value problem can be written as (1.1) with (2.7), then we have the following result.

Lemma 2.4. *The boundary value problem (1.1) and (2.7) has $N-1$ real and simple eigenvalues as $k_{12} = 0$ and N real and simple eigenvalues as $k_{12} \neq 0$, which can be arranged in the increasing order*

$$\mu_0 < \mu_1 < \cdots < \mu_{N_s}, \quad \text{where } N_s := N-2 \text{ or } N-1. \tag{2.12}$$

Let $y_n(\lambda)$ be the solution of (1.1) with the separated boundary conditions (2.7). Then sequence (2.6) exhibits no changes of sign for $\lambda \leq \mu_0$, exactly $k+1$ changes of sign for $\mu_k < \lambda \leq \mu_{k+1}$ ($0 \leq k \leq N_s-1$), and $N_s + 1$ changes of sign for $\lambda > \mu_{N_s}$.

Let φ_n and ψ_n be the solutions of (1.1) satisfying the following initial conditions:

$$\varphi_{-1} = \psi_0 = 1, \qquad \varphi_0 = \psi_{-1} = 0, \tag{2.13}$$

respectively. By Lemma 2.1 and using $p_{N-1} = 1$, we have

$$\Delta\varphi_{N-1}\psi_{N-1} - \varphi_{N-1}\Delta\psi_{N-1} = \varphi_N\psi_{N-1} - \varphi_{N-1}\psi_N = -1. \tag{2.14}$$

Obviously, $\varphi_n(\lambda)$ and $\psi_n(\lambda)$ are two linearly independent solutions of (1.1). The following lemma can be derived from [4, Proposition 3.1].

Lemma 2.5. *Let μ_k $(0 \leq k \leq N_s)$ be the eigenvalues of (1.1) and (2.7) with $k_{12} = 0$ and be arranged as (2.12). Then, $\psi_n(\mu_k)$ is an eigenfunction of the problem (1.1) and (2.7) with respect to μ_k $(0 \leq k \leq N_s)$, that is, for $0 \leq k \leq N_s$, $\psi_n(\mu_k)$ is a nontrivial solution of (1.1) satisfying*

$$\psi_{-1}(\mu_k) = \psi_{N-1}(\mu_k) = 0. \tag{2.15}$$

Moreover, if k is odd, $\psi_N(\mu_k) > 0$ and if k is even, $\psi_N(\mu_k) < 0$ for $2 \leq k \leq N_s$.

A representation of solutions for a nonhomogeneous linear equation with initial conditions is given by the following lemma.

Lemma 2.6 (see [4, Theorem 2.3]). *For any $\{f_n\}_{n=0}^{N-1} \subset \mathbf{C}$ and for any $c_{-1}, c_0 \in \mathbf{C}$, the initial value problem*

$$-\nabla(p_n \Delta z_n) + (q_n - \lambda w_n) z_n = w_n f_n, \quad n \in [0, N-1],$$

$$z_{-1} = c_{-1}, \qquad z_0 = c_0 \tag{2.16}$$

has a unique solution z, which can be expressed as

$$z_n = c_{-1}\varphi_n + c_0\psi_n + \sum_{j=0}^{n-1} w_j(\varphi_n\psi_j - \varphi_j\psi_n) f_j, \quad n \in [-1, N], \tag{2.17}$$

where $\sum_{j=0}^{-2} \cdot = \sum_{j=0}^{-1} \cdot := 0$.

3. Main Results

Let φ_n and ψ_n be defined in Section 2, let μ_k $(0 \leq k \leq N_s)$ be the eigenvalues of the separated boundary value problem (1.1) with (2.7), and let $\lambda_j(e^{i\alpha}K)$ $(0 \leq j \leq N-1)$ be the eigenvalues of the coupled boundary value problem (1.1) and (1.2) and arranged in the nondecreasing order

$$\lambda_0\left(e^{i\alpha}K\right) \leq \lambda_1\left(e^{i\alpha}K\right) \leq \cdots \leq \lambda_{N-1}\left(e^{i\alpha}K\right). \tag{3.1}$$

Clearly, $\lambda_j(K)$ $(0 \leq j \leq N-1)$ denotes the eigenvalue of the problem (1.1) and (1.2) with $\alpha = 0$, and $\lambda_j(-K)$ $(0 \leq j \leq N-1)$ denotes the eigenvalue of the problem (1.1) and (1.2) with $\alpha = \pi$. We now present the main results of this paper.

Theorem 3.1. *Assume that $k_{11} > 0$, $k_{12} \leq 0$ or $k_{11} \geq 0$, $k_{12} < 0$. Then, for every fixed $\alpha \neq 0$, $-\pi < \alpha < \pi$, one has the following inequalities:*

$$\lambda_0(K) < \lambda_0\left(e^{i\alpha}K\right) < \lambda_0(-K) \leq \lambda_1(-K) < \lambda_1\left(e^{i\alpha}K\right) < \lambda_1(K)$$

$$\leq \lambda_2(K) < \lambda_2\left(e^{i\alpha}K\right) < \lambda_2(-K)$$

$$\leq \lambda_3(-K) < \lambda_3\left(e^{i\alpha}K\right) < \lambda_3(K)$$

$$\leq \cdots \leq \lambda_{N-2}(-K) < \lambda_{N-2}\left(e^{i\alpha}K\right) < \lambda_{N-2}(K)$$

$$\leq \lambda_{N-1}(K) < \lambda_{N-1}\left(e^{i\alpha}K\right) < \lambda_{N-1}(-K), \quad \text{if } N \text{ is odd,}$$

$$\lambda_0(K) < \lambda_0\left(e^{i\alpha}K\right) < \lambda_0(-K) \leq \lambda_1(-K) < \lambda_1\left(e^{i\alpha}K\right) < \lambda_1(K)$$

$$\leq \lambda_2(K) < \lambda_2\left(e^{i\alpha}K\right) < \lambda_2(-K)$$

$$\leq \lambda_3(-K) < \lambda_3\left(e^{i\alpha}K\right) < \lambda_3(K)$$

$$\leq \cdots \leq \lambda_{N-2}(K) < \lambda_{N-2}\left(e^{i\alpha}K\right) < \lambda_{N-2}(-K)$$

$$\leq \lambda_{N-1}(-K) < \lambda_{N-1}\left(e^{i\alpha}K\right) < \lambda_{N-1}(K), \quad \text{if } N \text{ is even.}$$

$$(3.2)$$

Remark 3.2. If $k_{11} \leq 0$, $k_{12} > 0$ or $k_{11} < 0$, $k_{12} \geq 0$, a similar result can be obtained by applying Theorem 3.1 to $-K$. In fact, $e^{i\alpha}K = e^{i(\pi+\alpha)}(-K)$ for $\alpha \in (-\pi, 0)$ and $e^{i\alpha}K = e^{i(-\pi+\alpha)}(-K)$ for $\alpha \in (0, \pi)$. Hence, the boundary condition (1.2) in the cases of $k_{11} \leq 0$, $k_{12} > 0$ or $k_{11} < 0$, $k_{12} \geq 0$ and $\alpha \neq 0, -\pi < \alpha < \pi$, can be written as condition (1.2), where α is replaced by $\pi + \alpha$ for $\alpha \in (-\pi, 0)$ and $-\pi + \alpha$ for $\alpha \in (0, \pi)$, and K is replaced by $-K$.

Before proving Theorem 3.1, we prove the following five propositions.

Proposition 3.3. *For $\lambda \in \mathbb{C}$, λ is an eigenvalue of (1.1) and (1.2) if and only if*

$$f(\lambda) = 2\cos\alpha, \tag{3.3}$$

where

$$f(\lambda) := k_{22}\varphi_{N-1}(\lambda) + (k_{11} - k_{12})\Delta\psi_{N-1}(\lambda) - (k_{21} - k_{22})\psi_{N-1}(\lambda) - k_{12}\Delta\varphi_{N-1}(\lambda). \tag{3.4}$$

Moreover, λ is a multiple eigenvalue of (1.1) and (1.2) if and only if

$$\varphi_{N-1}(\lambda) = e^{i\alpha}(k_{11} - k_{12}), \qquad \Delta\varphi_{N-1}(\lambda) = e^{i\alpha}(k_{21} - k_{22}),$$

$$\psi_{N-1}(\lambda) = e^{i\alpha}k_{12}, \qquad \Delta\psi_{N-1}(\lambda) = e^{i\alpha}k_{22}. \tag{3.5}$$

Proof. Since φ_n and ψ_n are linearly independent solutions of (1.1), then λ is an eigenvalue of the problem (1.1) and (1.2) if and only if there exist two constants C_1 and C_2 not both zero such that $C_1\varphi_n + C_2\psi_n$ satisfies (1.2), which yields

$$\begin{pmatrix} \varphi_{N-1}(\lambda) - e^{i\alpha}(k_{11} - k_{12}) & \psi_{N-1}(\lambda) - e^{i\alpha}k_{12} \\ \Delta\varphi_{N-1}(\lambda) - e^{i\alpha}(k_{21} - k_{22}) & \Delta\psi_{N-1}(\lambda) - e^{i\alpha}k_{22} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0. \tag{3.6}$$

It is evident that (3.6) has a nontrivial solution (C_1, C_2) if and only if

$$\det\begin{pmatrix} \varphi_{N-1}(\lambda) - e^{i\alpha}(k_{11} - k_{12}) & \psi_{N-1}(\lambda) - e^{i\alpha}k_{12} \\ \Delta\varphi_{N-1}(\lambda) - e^{i\alpha}(k_{21} - k_{22}) & \Delta\psi_{N-1}(\lambda) - e^{i\alpha}k_{22} \end{pmatrix} = 0 \tag{3.7}$$

which, together with (2.14) and $\det K = 1$, implies that

$$1 + e^{2i\alpha} - e^{i\alpha}f(\lambda) = 0. \tag{3.8}$$

Then (3.3) follows from the above relation and the fact that $e^{-i\alpha} + e^{i\alpha} = 2\cos\alpha$. On the other hand, (1.1) has two linearly independent solutions satisfying (1.2) if and only if all the entries of the coefficient matrix of (3.6) are zero. Hence, λ is a multiple eigenvalue of (1.1) and (1.2) if and only if (3.5) holds. This completes the proof. $\square$

The following result is a direct consequence of the first result of Proposition 3.3.

Corollary 3.4. *For any $\alpha \in (-\pi, \pi]$,*

$$\lambda_j\left(e^{i\alpha}K\right) = \lambda_j\left(e^{-i\alpha}K\right), \quad 0 \le j \le N - 1. \tag{3.9}$$

Proposition 3.5. *Assume that $k_{11} > 0$, $k_{12} \le 0$ or $k_{11} \ge 0$, $k_{12} < 0$. Then one has the following results.*

   (i) *For each k, $0 \le k \le N_s$, $f(\mu_k) \ge 2$ if k is odd, and $f(\mu_k) \le -2$ if k is even.*

   (ii) *There exists a constant $\nu_0 < \mu_0$ such that $f(\nu_0) \ge 2$.*

   (iii) *If the boundary value problem (1.1) and (2.7) has exactly $N - 1$ eigenvalues then there exists a constant ξ_0 such that $\mu_{N-2} < \xi_0$ and $f(\xi_0) \le -2$, where N is odd, and there exists a constant η_0 such that $\mu_{N-2} < \eta_0$ and $f(\eta_0) \ge 2$, where N is even.*

Proof. (i) If $\psi_n(\mu_k)$ is an eigenfunction of the problem (1.1) and (2.7) respect to μ_k then $k_{12}\Delta\psi_{N-1}(\mu_k) - k_{22}\psi_{N-1}(\mu_k) = 0$. By Lemma 2.3 and the initial conditions (2.13), we have that if $k_{12} < 0$ then the sequence $\psi_0(\mu_k), \psi_1(\mu_k), \ldots, \psi_{N-1}(\mu_k)$ exhibits k changes of sign and

$$\mathrm{sgn}\psi_{N-1}(\mu_k) = (-1)^k. \tag{3.10}$$

Case 1. If $k_{12} < 0$ then it follows from $k_{12}\Delta\psi_{N-1}(\mu_k) - k_{22}\psi_{N-1}(\mu_k) = 0$ that

$$\frac{\psi_{N-1}(\mu_k)}{k_{12}} = \frac{\Delta\psi_{N-1}(\mu_k)}{k_{22}}, \qquad k_{11}k_{22}\psi_{N-1}(\mu_k) = k_{11}k_{12}\Delta\psi_{N-1}(\mu_k). \tag{3.11}$$

By (2.14) and the first relation in (3.11), for each k, $0 \leq k \leq N_s$, we have

$$\begin{aligned}
\varphi_{N-1}(\mu_k)&\Delta\psi_{N-1}(\mu_k) - \Delta\varphi_{N-1}(\mu_k)\psi_{N-1}(\mu_k) \\
&= \varphi_{N-1}(\mu_k)\frac{k_{22}}{k_{12}}\psi_{N-1}(\mu_k) - \Delta\varphi_{N-1}(\mu_k)\psi_{N-1}(\mu_k) \\
&= (k_{22}\varphi_{N-1}(\mu_k) - k_{12}\Delta\varphi_{N-1}(\mu_k))\frac{\psi_{N-1}(\mu_k)}{k_{12}} = 1.
\end{aligned} \tag{3.12}$$

By the definition of $f(\lambda)$, (3.11), and det $K = 1$,

$$\begin{aligned}
k_{12}f(\mu_k) &= k_{12}k_{22}\varphi_{N-1}(\mu_k) + k_{12}(k_{11} - k_{12})\Delta\psi_{N-1}(\mu_k) \\
&\quad - k_{12}(k_{21} - k_{22})\psi_{N-1}(\mu_k) - k_{12}^2\Delta\varphi_{N-1}(\mu_k) \\
&= k_{12}k_{22}\varphi_{N-1}(\mu_k) + k_{11}k_{12}\Delta\psi_{N-1}(\mu_k) - k_{12}k_{21}\psi_{N-1}(\mu_k) - k_{12}^2\Delta\varphi_{N-1}(\mu_k) \\
&= k_{12}k_{22}\varphi_{N-1}(\mu_k) + k_{11}k_{22}\psi_{N-1}(\mu_k) - k_{12}k_{21}\psi_{N-1}(\mu_k) - k_{12}^2\Delta\varphi_{N-1}(\mu_k) \\
&= k_{12}k_{22}\varphi_{N-1}(\mu_k) - k_{12}^2\Delta\varphi_{N-1}(\mu_k) + \psi_{N-1}(\mu_k).
\end{aligned} \tag{3.13}$$

Hence,

$$f(\mu_k) = (k_{22}\varphi_{N-1}(\mu_k) - k_{12}\Delta\varphi_{N-1}(\mu_k)) + \frac{\psi_{N-1}(\mu_k)}{k_{12}}. \tag{3.14}$$

Noting $(k_{22}\varphi_{N-1}(\mu_k) - k_{12}\Delta\varphi_{N-1}(\mu_k))(\psi_{N-1}(\mu_k)/k_{12}) = 1$, $k_{12} < 0$, and (3.10), we have that if k is odd then

$$f(\mu_k) = \left(\sqrt{\frac{\psi_{N-1}(\mu_k)}{k_{12}}} - \sqrt{k_{22}\varphi_{N-1}(\mu_k) - k_{12}\Delta\varphi_{N-1}(\mu_k)}\right)^2 + 2 \geq 2, \tag{3.15}$$

and if k is even then

$$f(\mu_k) = -\left(\sqrt{-\frac{\psi_{N-1}(\mu_k)}{k_{12}}} - \sqrt{-(k_{22}\varphi_{N-1}(\mu_k) - k_{12}\Delta\varphi_{N-1}(\mu_k))}\right)^2 - 2 \leq -2. \tag{3.16}$$

Case 2. If $k_{12} = 0$ then it follows from (2.7) and (2.14) that for each k, $0 \leq k \leq N_s$,

$$\varphi_{N-1}(\mu_k)\psi_N(\mu_k) = 1. \tag{3.17}$$

From (2.15) and by the definition of $f(\lambda)$, we get

$$f(\mu_k) = \frac{k_{22}}{\psi_N(\mu_k)} + k_{11}\psi_N(\mu_k). \tag{3.18}$$

Hence, noting $\det K = k_{11}k_{22} = 1$, $k_{11} > 0$, and by **Lemma 2.5**, we have that if k is odd, then

$$f(\mu_k) \geq 2, \tag{3.19}$$

and if k is even, then

$$f(\mu_k) \leq -2. \tag{3.20}$$

(ii) By the discussions in the first paragraph of **Section 2**, $\varphi_{N-1}(\lambda)$ is a polynomial of degree $N - 2$ in λ, $\varphi_N(\lambda)$ is a polynomial of degree $N - 1$ in λ, $\psi_{N-1}(\lambda)$ is a polynomial of degree $N - 1$ in λ, and $\psi_N(\lambda)$ is a polynomial of degree N in λ. Further, $\psi_N(\lambda)$ can be written as

$$\psi_N(\lambda) = (-1)^N A_N \lambda^N + A_{N-1}\lambda^{N-1} + \cdots + A_0, \tag{3.21}$$

where $A_N = w_0 w_1 \cdots w_{N-1}(p_0 p_1 \cdots p_{N-1})^{-1} > 0$ and A_n is a certain constant for $n \in [0, N-1]$. Then

$$f(\lambda) = (-1)^N (k_{11} - k_{12})A_N \lambda^N + h(\lambda), \tag{3.22}$$

where $h(\lambda)$ is a polynomial in λ whose degree is not larger than $N - 1$. Clearly, as $\lambda \to -\infty$, $f(\lambda) \to +\infty$ since $(k_{11} - k_{12}) > 0$. By the first part of this proposition, $f(\mu_0) \leq -2$. So there exists a constant $\nu_0 < \mu_0$ such that $f(\nu_0) \geq 2$.

(iii) It follows from the first part of this proposition that if N is odd, $f(\mu_{N-2}) \geq 2$ and if N is even, $f(\mu_{N-2}) \leq -2$. By (3.22), if N is odd, $f(\lambda) \to -\infty$ as $\lambda \to +\infty$; if N is even, $f(\lambda) \to +\infty$ as $\lambda \to +\infty$. Hence, if N is odd, there exists a constant $\xi_0 > \mu_{N-2}$ such that $f(\xi_0) \leq -2$; if N is even, there exists a constant $\eta_0 > \mu_{N-2}$ such that $f(\eta_0) \geq 2$. This completes the proof. $\square$

Since φ_n and ψ_n are both polynomials in λ, so is $f(\lambda)$. Denote

$$\frac{d}{d\lambda}f(\lambda) := f'(\lambda), \qquad \frac{d^2}{d\lambda^2}f(\lambda) := f''(\lambda). \tag{3.23}$$

Proposition 3.6. *Assume that $k_{11} > 0$, $k_{12} \le 0$ or $k_{11} \ge 0$, $k_{12} < 0$. Equations $f'(\lambda) = 0$ and $f(\lambda) = 2$ or -2 hold if and only if λ is a multiple eigenvalue of (1.1) and (1.2) with $\alpha = 0$ or $\alpha = \pi$. If $f(\lambda) = 2$ or -2 for some $\lambda \ne \mu_k$ $(0 \le k \le N_s)$, then λ is a simple eigenvalue of (1.1) and (1.2) with $\alpha = 0$ or $\alpha = \pi$ and for every $\lambda \ne \mu_k$ $(0 \le k \le N_s)$, with $-2 \le f(\lambda) \le 2$ one has:*

$$f'(\lambda) < 0, \quad \lambda < \mu_0,$$

$$(-1)^k f'(\lambda) > 0, \quad \mu_k < \lambda < \mu_{k+1}, \ 0 \le k \le N - 3, \tag{3.24}$$

$$(-1)^{N-2} f'(\lambda) > 0, \quad \lambda > \mu_{N-2}.$$

Proof. Since φ_n and ψ_n are solutions of (1.1), we have

$$-\nabla(p_n \Delta \varphi_n(\lambda)) + q_n \varphi_n(\lambda) = \lambda w_n \varphi_n(\lambda), \tag{3.25}$$

$$-\nabla(p_n \Delta \psi_n(\lambda)) + q_n \psi_n(\lambda) = \lambda w_n \psi_n(\lambda). \tag{3.26}$$

Differentiating (3.25) and (3.26) with respect to λ, respectively, yields that

$$-\nabla(p_n \Delta \varphi_n'(\lambda)) + (q_n - \lambda w_n)\varphi_n'(\lambda) = w_n \varphi_n(\lambda),$$
$$-\nabla(p_n \Delta \psi_n'(\lambda)) + (q_n - \lambda w_n)\psi_n'(\lambda) = w_n \psi_n(\lambda). \tag{3.27}$$

It follows from (2.13) that

$$\varphi_0' = \varphi_{-1}' = \psi_0' = \psi_{-1}' = 0. \tag{3.28}$$

Thus, by Lemma 2.6 and from (3.27)–(3.28), we have

$$\varphi_n'(\lambda) = \sum_{j=0}^{n-1} w_j \varphi_j(\lambda) (\varphi_n(\lambda)\psi_j(\lambda) - \varphi_j(\lambda)\psi_n(\lambda)),$$

$$\psi_n'(\lambda) = \sum_{j=0}^{n-1} w_j \psi_j(\lambda) (\varphi_n(\lambda)\psi_j(\lambda) - \varphi_j(\lambda)\psi_n(\lambda)). \tag{3.29}$$

It follows from (3.29) that

$$\Delta \varphi_{n-1}'(\lambda) = \sum_{j=0}^{n-1} w_j \varphi_j(\lambda) (\Delta \varphi_{n-1}(\lambda)\psi_j(\lambda) - \varphi_j(\lambda)\Delta \psi_{n-1}(\lambda)),$$

$$\Delta \psi_{n-1}'(\lambda) = \sum_{j=0}^{n-1} w_j \psi_j(\lambda) (\Delta \varphi_{n-1}(\lambda)\psi_j(\lambda) - \varphi_j(\lambda)\Delta \psi_{n-1}(\lambda)). \tag{3.30}$$

Hence, not indicating λ explicitly, we get

$$
\begin{aligned}
f' &= k_{22}\varphi'_{N-1} + (k_{11} - k_{12})\Delta\psi'_{N-1} - (k_{21} - k_{22})\psi'_{N-1} - k_{12}\Delta\varphi'_{N-1} \\
&= k_{22}\sum_{j=0}^{N-2} w_j\varphi_j(\varphi_{N-1}\psi_j - \varphi_j\psi_{N-1}) + (k_{11} - k_{12})\sum_{j=0}^{N-1} w_j\psi_j(\Delta\varphi_{N-1}\psi_j - \varphi_j\Delta\psi_{N-1}) \\
&\quad - (k_{21} - k_{22})\sum_{j=0}^{N-2} w_j\psi_j(\varphi_{N-1}\psi_j - \varphi_j\psi_{N-1}) - k_{12}\sum_{j=0}^{N-1} w_j\varphi_j(\Delta\varphi_{N-1}\psi_j - \varphi_j\Delta\psi_{N-1}) \\
&= \sum_{j=0}^{N-1} w_j\delta_j,
\end{aligned}
\tag{3.31}
$$

where

$$
\begin{aligned}
\delta_j &:= \left((k_{11} - k_{12})\Delta\varphi_{N-1} - (k_{21} - k_{22})\varphi_{N-1}\right)\psi_j^2 \\
&\quad + \left(k_{22}\varphi_{N-1} - (k_{11} - k_{12})\Delta\psi_{N-1} + (k_{21} - k_{22})\psi_{N-1} - k_{12}\Delta\varphi_{N-1}\right)\psi_j\varphi_j \\
&\quad + \left(k_{12}\Delta\psi_{N-1} - k_{22}\psi_{N-1}\right)\varphi_j^2 \\
&= (\psi_j, \varphi_j) I \begin{pmatrix} \psi_j \\ \varphi_j \end{pmatrix},
\end{aligned}
\tag{3.32}
$$

$$
I := \begin{pmatrix} (k_{11} - k_{12})\Delta\varphi_{N-1} - (k_{21} - k_{22})\varphi_{N-1} & \begin{aligned} &\tfrac{1}{2}\big(k_{22}\varphi_{N-1} - (k_{11} - k_{12})\Delta\psi_{N-1} \\ &\quad + (k_{21} - k_{22})\psi_{N-1} - k_{12}\Delta\varphi_{N-1}\big) \end{aligned} \\[2em] \begin{aligned} &\tfrac{1}{2}\big(k_{22}\varphi_{N-1} - (k_{11} - k_{12})\Delta\psi_{N-1} \\ &\quad + (k_{21} - k_{22})\psi_{N-1} - k_{12}\Delta\varphi_{N-1}\big) \end{aligned} & k_{12}\Delta\psi_{N-1} - k_{22}\psi_{N-1} \end{pmatrix},
$$

which is symmetric for any $\lambda \in \mathbf{R}$. Then, we have

$$
\begin{aligned}
\det I(\lambda) &= \left(k_{12}\Delta\psi_{N-1}(\lambda) - k_{22}\psi_{N-1}(\lambda)\right)\left((k_{11} - k_{12})\Delta\varphi_{N-1}(\lambda) - (k_{21} - k_{22})\varphi_{N-1}(\lambda)\right) \\
&\quad - \frac{\left(k_{22}\varphi_{N-1}(\lambda) - (k_{11} - k_{12})\Delta\psi_{N-1}(\lambda) + (k_{21} - k_{22})\psi_{N-1}(\lambda) - k_{12}\Delta\varphi_{N-1}(\lambda)\right)^2}{4} \\
&= -\frac{1}{4}f^2(\lambda) + 1.
\end{aligned}
\tag{3.33}
$$

Hence, if $f(\lambda) = 2$ or -2, we get from (3.33) that $\det I(\lambda) = 0$. Then, for any fixed λ with $f(\lambda) = 2$ or -2, the matrix $I(\lambda)$ is positive semidefinite or negative semidefinite. Therefore, for such a λ, $f'(\lambda)$ cannot vanish unless $\delta_j(\lambda) = 0$ for all $0 \le j \le N - 1$. Because φ_n and ψ_n are

linearly independent, $\delta_j(\lambda)$ is identically zero if and only if all the entries of the matrix $I(\lambda)$ vanish, namely,

$$
\begin{aligned}
k_{12}\Delta\psi_{N-1}(\lambda) - k_{22}\psi_{N-1}(\lambda) &= 0, \\
(k_{11} - k_{12})\Delta\varphi_{N-1}(\lambda) - (k_{21} - k_{22})\varphi_{N-1}(\lambda) &= 0, \\
k_{22}\varphi_{N-1}(\lambda) - (k_{11} - k_{12})\Delta\psi_{N-1}(\lambda) + (k_{21} - k_{22})\psi_{N-1}(\lambda) - k_{12}\Delta\varphi_{N-1}(\lambda) &= 0
\end{aligned}
\tag{3.34}
$$

which, together with $f(\lambda) = 2$ and $\det K = 1$, implies

$$
\begin{aligned}
\varphi_{N-1}(\lambda) &= k_{11} - k_{12}, & \Delta\varphi_{N-1}(\lambda) &= k_{21} - k_{22}, \\
\psi_{N-1}(\lambda) &= k_{12}, & \Delta\psi_{N-1}(\lambda) &= k_{22}.
\end{aligned}
\tag{3.35}
$$

Then by Proposition 3.3, λ is a multiple eigenvalue of (1.1) and (1.2) with $\alpha = 0$. In addition, (3.34), together with $f(\lambda) = -2$ and $\det K = 1$, implies

$$
\begin{aligned}
\varphi_{N-1}(\lambda) &= -(k_{11} - k_{12}), & \Delta\varphi_{N-1}(\lambda) &= -(k_{21} - k_{22}), \\
\psi_{N-1}(\lambda) &= -k_{12}, & \Delta\psi_{N-1}(\lambda) &= -k_{22}.
\end{aligned}
\tag{3.36}
$$

Then by Proposition 3.3, λ is a multiple eigenvalue of (1.1) and (1.2) with $\alpha = \pi$. Conversely, from (3.35) or (3.36), it can be easily verified that (3.34) holds, then $f\prime(\lambda) = 0$. It follows again from (3.35) or (3.36) that $f(\lambda) = 2$ or $f(\lambda) = -2$. Thus $f\prime(\lambda) = 0$ and $f(\lambda) = 2$ or -2 if and only if λ is a multiple eigenvalue of (1.1) and (1.2) with $\alpha = 0$ or $\alpha = \pi$.

Further, for every fixed λ with $f(\lambda) = 2$ or -2, not indicating λ explicitly, (3.33) implies that

$$
\begin{aligned}
&\left(k_{12}\Delta\psi_{N-1} - k_{22}\psi_{N-1}\right)\left((k_{11} - k_{12})\Delta\varphi_{N-1} - (k_{21} - k_{22})\varphi_{N-1}\right) \\
&\quad = \frac{\left(k_{22}\varphi_{N-1} - (k_{11} - k_{12})\Delta\psi_{N-1} + (k_{21} - k_{22})\psi_{N-1} - k_{12}\Delta\varphi_{N-1}\right)^2}{4}.
\end{aligned}
\tag{3.37}
$$

Therefore, from (3.37) and by the definition of δ_j, we have

$$
\begin{aligned}
\delta_j = {}&\left(k_{12}\Delta\psi_{N-1} - k_{22}\psi_{N-1}\right) \\
&\cdot \left(\varphi_j + \frac{k_{22}\varphi_{N-1} - (k_{11} - k_{12})\Delta\psi_{N-1} + (k_{21} - k_{22})\psi_{N-1} - k_{12}\Delta\varphi_{N-1}}{2\left(k_{12}\Delta\psi_{N-1} - k_{22}\psi_{N-1}\right)}\psi_j\right)^2
\end{aligned}
\tag{3.38}
$$

and consequently, not indicating λ explicitly, we have

$$
\begin{aligned}
f' = {}&\left(k_{12}\Delta\psi_{N-1} - k_{22}\psi_{N-1}\right) \\
&\cdot \sum_{j=0}^{N-1} w_j\left(\varphi_j + \frac{k_{22}\varphi_{N-1} - (k_{11} - k_{12})\Delta\psi_{N-1} + (k_{21} - k_{22})\psi_{N-1} - k_{12}\Delta\varphi_{N-1}}{2\left(k_{12}\Delta\psi_{N-1} - k_{22}\psi_{N-1}\right)}\psi_j\right)^2
\end{aligned}
\tag{3.39}
$$

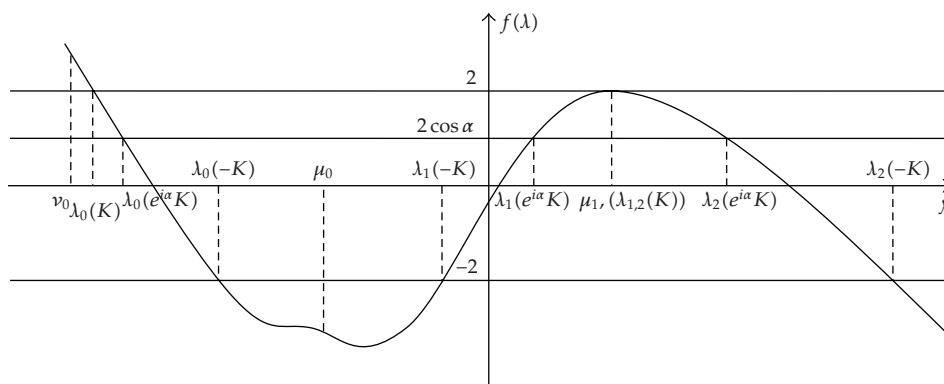for every fixed λ with $f(\lambda) = 2$ or -2.

Figure 1: The graph of $f(\lambda)$.

Suppose that $f(\lambda) = 2$ or -2 for some $\lambda \neq \mu_k$ ($0 \leq k \leq N_s$), we have $k_{12}\Delta\psi_{N-1}(\lambda) - k_{22}\psi_{N-1}(\lambda) \neq 0$. From the above discussions again, λ is a simple eigenvalue of (1.1) and (1.2) with $\alpha = 0$ or $\alpha = \pi$, and δ_j is not identically zero for $0 \leq j \leq N-1$.

For this $\lambda \neq \mu_k$ ($0 \leq k \leq N_s$), (3.39) implies that $f\prime(\lambda) \neq 0$, and from Proposition 3.5 (i), (ii) that $f(\mu_0) \leq -2$, $f(\nu_0) \geq 2$. Hence, $f\prime(\lambda) < 0$, where $\nu_0 < \lambda < \mu_0$. It follows from Proposition 3.5 (i) that $f(\mu_k)f(\mu_{k+1}) \leq -4$ and $(-1)^k f\prime(\lambda) > 0$, where $\mu_k < \lambda < \mu_{k+1}$ ($0 \leq k \leq N-3$). By Proposition 3.5 (i), (iii), $f(\mu_{N-2}) \geq 2$ and there exists $\mu_{N-2} < \xi_0$ such that $f(\xi_0) \leq -2$ if N is odd, and $f(\mu_{N-2}) \leq -2$ and there exists $\mu_{N-2} < \eta_0$ such that $f(\eta_0) \geq 2$ if N is even. Hence, $(-1)^{N-2}f\prime(\lambda) > 0$ where $\mu_{N-2} < \lambda$. This completes the proof. □

Proposition 3.7. *For any fixed $\alpha \neq 0$, $-\pi < \alpha < \pi$, each eigenvalue of (1.1) and (1.2) is simple.*

Proof. Fix α, $-\pi < \alpha < \pi$ with $\alpha \neq 0$. Suppose that λ is an eigenvalue of the problem (1.1) and (1.2). By Proposition 3.3, we have $f^2(\lambda) = 4\cos^2\alpha < 4$. It follows from (3.33) that $\det I(\lambda) > 0$ and the matrix $I(\lambda)$ is positive definite or negative definite. Hence, $\delta_j > 0$ for $0 \leq j \leq N-1$ or $\delta_j < 0$ for $0 \leq j \leq N-1$ since φ_n and ψ_n are linearly independent.

If λ is a multiple eigenvalue of problem (1.1) and (1.2), then (3.5) holds by Proposition 3.3. By using (3.5), it can be easily verified that (3.34) holds, that is, all the entries of the matrix $I(\lambda)$ are zero. Then $\delta_j = 0$ for $0 \leq j \leq N-1$, which is contrary to $\delta_j \neq 0$ for $0 \leq j \leq N-1$. Hence, λ is a simple eigenvalue of (1.1) and (1.2). This completes the proof. □

Proposition 3.8. *Assume that $k_{11} > 0$, $k_{12} \leq 0$ or $k_{11} \geq 0$, $k_{12} < 0$. If k is odd, $f(\mu_k) = 2$, and $f'(\mu_k) = 0$, then $f''(\mu_k) < 0$; if k is even, $f(\mu_k) = -2$, and $f'(\mu_k) = 0$, then $f''(\mu_k) > 0$ for $0 \leq k \leq N-2$.*

Proof. We first prove the first result. Suppose that k is odd, $f(\mu_k) = 2$, and $f\prime(\mu_k) = 0$. Then μ_k is a multiple eigenvalue of (1.1) and (1.2) with $\alpha = 0$ by Proposition 3.6. Then by Proposition 3.3, (3.5) holds for $\lambda = \mu_k$ and $\alpha = 0$, that is,

$$\varphi_{N-1}(\mu_k) = k_{11} - k_{12}, \qquad \Delta\varphi_{N-1}(\mu_k) = k_{21} - k_{22},$$
$$\psi_{N-1}(\mu_k) = k_{12}, \qquad \Delta\psi_{N-1}(\mu_k) = k_{22}. \tag{3.40}$$

Differentiating $f(\lambda)$ with respect to λ two times, we get

$$f''(\mu_k) = k_{22}\varphi''_{N-1}(\mu_k) + (k_{11} - k_{12})\Delta\psi''_{N-1}(\mu_k) - (k_{21} - k_{22})\psi''_{N-1}(\mu_k) - k_{12}\Delta\varphi''_{N-1}(\mu_k).$$

(3.41)

Differentiating (2.14) with respect to λ two times and from (3.40), we get

$$
\begin{aligned}
&- \left(k_{22}\varphi''_{N-1}(\mu_k) + (k_{11} - k_{12})\Delta\psi''_{N-1}(\mu_k) - (k_{21} - k_{22})\psi''_{N-1}(\mu_k) - k_{12}\Delta\varphi''_{N-1}(\mu_k)\right) \\
&+ 2\left(\varphi'_N(\mu_k)\psi'_{N-1}(\mu_k) - \varphi'_{N-1}(\mu_k)\psi'_N(\mu_k)\right) = 0,
\end{aligned}
$$

(3.42)

which, together with (3.41), implies that

$$f''(\mu_k) = 2\left(\varphi'_N(\mu_k)\psi'_{N-1}(\mu_k) - \varphi'_{N-1}(\mu_k)\psi'_N(\mu_k)\right).$$

(3.43)

On the other hand, it follows from (3.29) and (2.14) that, not indicating μ_k explicitly,

$$
\begin{aligned}
\varphi'_N\psi'_{N-1} - \varphi'_{N-1}\psi'_N &= \sum_{j=0}^{N-1} w_j\varphi_j(\varphi_N\psi_j - \varphi_j\psi_N)\sum_{j=0}^{N-2} w_j\psi_j(\varphi_{N-1}\psi_j - \varphi_j\psi_{N-1}) \\
&\quad - \sum_{j=0}^{N-2} w_j\varphi_j(\varphi_{N-1}\psi_j - \varphi_j\psi_{N-1})\sum_{j=0}^{N-1} w_j\psi_j(\varphi_N\psi_j - \varphi_j\psi_N) \\
&= \left(\sum_{j=0}^{N-1} w_j\varphi_j\psi_j\right)^2 - \sum_{j=0}^{N-1} w_j\varphi_j^2\sum_{j=0}^{N-1} w_j\psi_j^2.
\end{aligned}
$$

(3.44)

Since φ_n and ψ_n are linearly independent on $[-1, N]$, the above relation implies that $f''(\mu_k) < 0$ by Hölder's inequality, which proves the first conclusion.

The second conclusion can be shown similarly. Hence, the proof is complete. □

Finally, we turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. By Propositions 3.3–3.8, and the intermediate value theorem, one can obtain the graph of f (see Figure 1), which implies the results of Theorem 3.1. We now give its detailed proof.

By Propositions 3.3–3.6, $f(\mu_0) \leq -2$, $f'(\lambda) < 0$ for all $\lambda < \mu_0$ with $-2 \leq f(\lambda) \leq 2$, and there exists $\nu_0 < \mu_0$ such that $f(\nu_0) \geq 2$. Therefore, by the continuity of $f(\lambda)$ and the intermediate value theorem, (1.1) and (1.2) with $\alpha = 0$ has only one eigenvalue $\lambda_0(K) < \mu_0$, (1.1) and (1.2) with $\alpha = \pi$ has only one eigenvalue $\lambda_0(-K) \leq \mu_0$, and (1.1) and (1.2) with $\alpha \neq 0$, $-\pi < \alpha < \pi$ has only one eigenvalue $\lambda_0(K) < \lambda_0(e^{i\alpha}K) < \lambda_0(-K)$, and they satisfy

$$\nu_0 \leq \lambda_0(K) < \lambda_0\left(e^{i\alpha}K\right) < \lambda_0(-K) \leq \mu_0.$$

(3.45)

Similarly, by Propositions 3.3–3.6, the continuity of $f(\lambda)$, and the intermediate value theorem, $f(\lambda)$ reaches -2, $2\cos\alpha$ ($\alpha \neq 0$, $-\pi < \alpha < \pi$), and 2 exactly one time, respectively, between
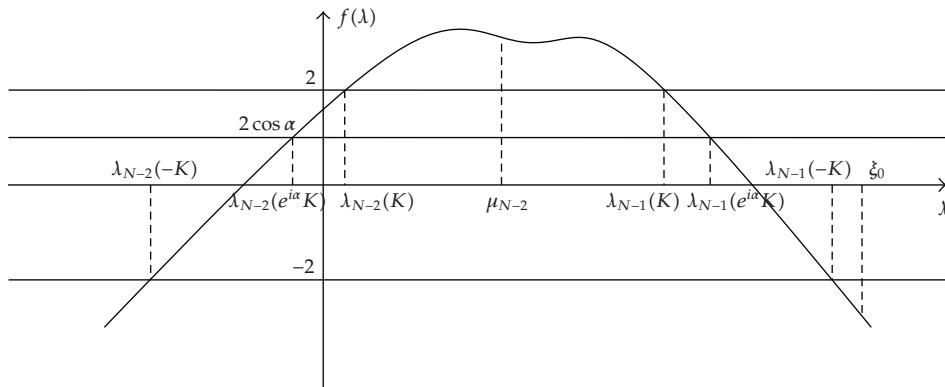
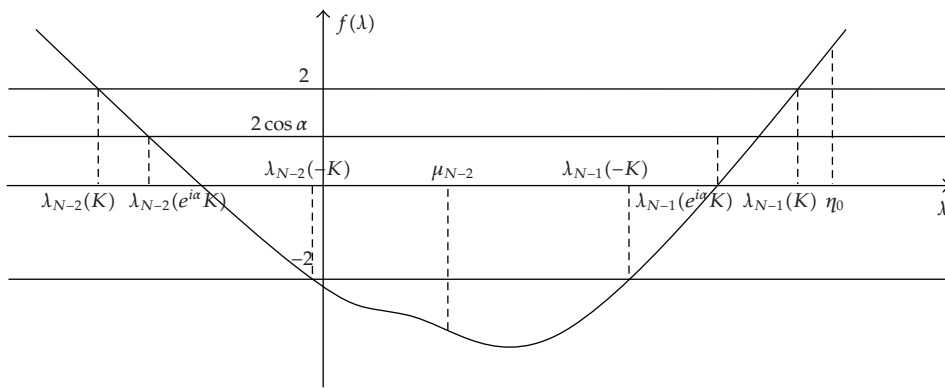**Figure 2:** The graph of $f(\lambda)$ in the case that N is odd.

Figure 3: The graph of $f(\lambda)$ in the case that N is even.

any two consecutive eigenvalues of the separated boundary value problem (1.1) with (2.7). Hence, (1.1) and (1.2) with $\alpha = 0$; $\alpha \neq 0$, $-\pi < \alpha < \pi$; $\alpha = \pi$ has only one eigenvalue between any two consecutive eigenvalues of (1.1) with (2.7), respectively. In addition, by Proposition 3.6, if $f(\mu_k) = 2$ or -2 and $f\prime(\mu_k) = 0$, then μ_k is not only an eigenvalue of (1.1) with (2.7) but also a multiple eigenvalue of (1.1) and (1.2) with $\alpha = 0$ and $\alpha = \pi$.

By Proposition 3.5 (i), if N is odd, $f(\mu_{N-2}) \geq 2$ and if N is even, $f(\mu_{N-2}) \leq -2$. It follows (3.22) that if N is odd, then $f(\lambda) \rightarrow -\infty$ as $\lambda \rightarrow +\infty$, and if N is even, then $f(\lambda) \rightarrow +\infty$ as $\lambda \rightarrow +\infty$. Hence, if N is odd, then there exists a constant $\xi_0 > \mu_{N-2}$ such that $f(\xi_0) \leq -2$, which, together with Proposition 3.6, implies that (1.1) and (1.2) with $\alpha = 0$; $\alpha \neq 0$, $-\pi < \alpha < \pi$; $\alpha = \pi$, has only one eigenvalue $\lambda_{N-1}(K)$, $\lambda_{N-1}(e^{i\alpha}K)$, and $\lambda_{N-1}(-K)$, satisfying

$$\mu_{N-2} \leq \lambda_{N-1}(K) < \lambda_{N-1}\left(e^{i\alpha}K\right) < \lambda_{N-1}(-K) \leq \xi_0 \qquad (3.46)$$

(see Figure 2). Similarly, in the other case that N is even, there exists a constant $\eta_0 > \mu_{N-2}$ such that $f(\eta_0) \geq 2$, which, together with Proposition 3.6, implies that (1.1) and (1.2) with

$\alpha = 0$; $\alpha \neq 0$, $-\pi < \alpha < \pi$; $\alpha = \pi$ has only one eigenvalue $\lambda_{N-1}(K)$, $\lambda_{N-1}(e^{i\alpha}K)$, and $\lambda_{N-1}(-K)$, satisfying

$$\mu_{N-2} \leq \lambda_{N-1}(-K) < \lambda_{N-1}\left(e^{i\alpha}K\right) < \lambda_{N-1}(K) \leq \eta_0 \tag{3.47}$$

(see Figure 3). Therefore, we get that (1.1) and (1.2) with $\alpha \neq 0$, $-\pi < \alpha < \pi$, has N eigenvalues and it is real and satisfies

$$\begin{aligned}
&\nu_0 \leq \lambda_0(K) < \lambda_0\left(e^{i\alpha}K\right) < \lambda_0(-K) \leq \mu_0 \leq \lambda_1(-K) < \lambda_1\left(e^{i\alpha}K\right) < \lambda_1(K) \leq \mu_1 \\
&\quad \leq \lambda_2(K) < \lambda_2\left(e^{i\alpha}K\right) < \lambda_2(-K) \leq \mu_2 \leq \lambda_3(-K) < \lambda_3\left(e^{i\alpha}K\right) < \lambda_3(K) \leq \mu_3 \\
&\quad \leq \cdots \leq \mu_{N-3} \leq \lambda_{N-2}(-K) < \lambda_{N-2}\left(e^{i\alpha}K\right) < \lambda_{N-2}(K) \leq \mu_{N-2} \leq \lambda_{N-1}(K) \\
&\quad < \lambda_{N-1}\left(e^{i\alpha}K\right) < \lambda_{N-1}(-K) \leq \xi_0, \quad \text{if } N \text{ is odd,} \\
&\nu_0 \leq \lambda_0(K) < \lambda_0\left(e^{i\alpha}K\right) < \lambda_0(-K) \leq \mu_0 \leq \lambda_1(-K) < \lambda_1\left(e^{i\alpha}K\right) < \lambda_1(K) \leq \mu_1 \\
&\quad \leq \lambda_2(K) < \lambda_2\left(e^{i\alpha}K\right) < \lambda_2(-K) \leq \mu_2 \leq \lambda_3(-K) < \lambda_3\left(e^{i\alpha}K\right) < \lambda_3(K) \leq \mu_3 \\
&\quad \leq \cdots \leq \mu_{N-3} \leq \lambda_{N-2}(K) < \lambda_{N-2}\left(e^{i\alpha}K\right) < \lambda_{N-2}(-K) \leq \mu_{N-2} \leq \lambda_{N-1}(-K) \\
&\quad < \lambda_{N-1}\left(e^{i\alpha}K\right) < \lambda_{N-1}(K) \leq \eta_0, \quad \text{if } N \text{ is even.}
\end{aligned} \tag{3.48}$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.9. Let $K = I$, that is, $k_{11} = k_{22} = 1$, $k_{12} = k_{21} = 0$. Then $f(\lambda) = \varphi_{N-1}(\lambda) + \psi_N(\lambda)$. In this case, Propositions 3.5 and 3.8 are the same as those mentioned in [4, Propositions 3.1, 3.3–3.5], respectively, and most of the results of Proposition 3.6 are the same as the results of [4, Proposition 3.2].

Acknowledgments

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