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Research Article

Stability of an Additive-Cubic-Quartic Functional Equation

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In this paper, we consider the additive-cubic-quartic functional equation $11[f(x+2y)+f(x-2y)] = 44[f(x+y)+f(x-y)]+12f(3y)-48f(2y)+60f(y)-66f(x)$ and prove the generalized Hyers-Ulam stability of the additive-cubic-quartic functional equation in Banach spaces.

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1. Introduction

The stability problem of functional equations is originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach (see [2, 5–13]).

Jun and Kim [14] introduced and investigate the following functional equation:

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x), \tag{1.1}$$

and prove the generalized Hyers-Ulam stability for the functional equation (1.1). Obviously, the function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a cubic functional equation. Every solution of the cubic functional equation is said to be a *cubic mapping*. Jun and Kim proved that a mapping f between two real vector spaces X and Y is a solution of (1.1) if and only if there exists a unique mapping $C : X \times X \times X \to Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$; moreover, C is symmetric for each fixed one variable and is additive for fixed two variables.

In [15], Park and Bae considered the following quartic functional equation:

$$f(2x + y) + f(2x - y) = 4(f(x + y) + f(x - y)) + 24f(x) - 6f(y). \tag{1.2}$$

In fact, they proved that a mapping f between two real vector spaces X and Y is a solution of (1.2) if and only if there exists a unique symmetric multi-additive mapping $B : X\times X\times X\times X \to Y$ such that $f(x) = B(x, x, x, x)$ for all x (see [7, 11]). It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.2), which is called a quartic functional equation. Every solution of the quartic functional equation is said to be a *quartic mapping*.

In this paper, we aim to deal with the next functional equation derived from additive, cubic, and quadric mappings,

$$\begin{aligned} &11[f(x + 2y) + f(x - 2y)] \\ &\quad = 44[f(x + y) + f(x - y)] + 12f(3y) - 48f(2y) + 60f(y) - 66f(x). \end{aligned} \tag{1.3}$$

It is easy to show that the function $f(x) = ax + bx^3 + cx^4$ satisfies the functional equation (1.3). We establish the general solution and prove the generalized Hyers-Ulam stability for the functional equation (1.3).

2. An Additive-Cubic-Quartic Functional Equation

Throughout this section, X and Y will be real vector spaces. Before proceeding the proof of Theorem 2.4 which is the main result in this section, we shall need the following two lemmas.

Lemma 2.1. *If an even mapping $f : X \to Y$ satisfies* (1.3), *then f is quartic.*

Proof. Putting $x = y = 0$ in (1.3), we get $f(0) = 0$. Setting $x = 0$ in (1.3), by the evenness of f, we obtain

$$6f(3y) = 35f(2y) - 74f(y) \tag{2.1}$$

for all $y \in X$. Hence (1.3) can be written as

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] + 2f(2y) - 8f(y) - 6f(x) \tag{2.2}$$

for all $x, y \in X$. Replacing x by y in (1.3), we obtain

$$f(3y) = 4f(2y) + 17f(y) \tag{2.3}$$

for all $y \in X$. By (2.1) and (2.3), we obtain

$$f(2y) = 16f(y) \tag{2.4}$$

for all $y \in X$. According to (2.4), (2.2) can be written as

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) + 24f(y) - 6f(x) \tag{2.5}$$

for all $x, y \in X$. This shows that f is quartic, which completes the proof of the lemma. □

Lemma 2.2. *If an odd mapping $f : X \to Y$ satisfies* (1.3), *then f is cubic-additive.*

Proof. We show that the mappings $g : X \to Y$ and $h : X \to Y$, respectively, defined by $g(x) := f(2x) - 8f(x)$ and $h(x) := f(2x) - 2f(x)$, are additive and cubic, respectively.

Since f is odd, $f(0) = 0$. Letting $x = 0$ in (1.3), we obtain

$$f(3y) = 4f(2y) - 5f(y) \tag{2.6}$$

for all $y \in X$. Hence (1.3) can be written as

$$f(x+2y) + f(x-2y) = 4[f(x+y) + f(x-y)] - 6f(x) \tag{2.7}$$

for all $x, y \in X$. Replacing x, y by $x+y$ and $x-y$ in (2.7), respectively, we get

$$f(3x-y) - f(x-3y) = -6f(x+y) + 4f(2x) + 4f(2y) \tag{2.8}$$

for all $x, y \in X$. Replacing x by $x+y$ in (2.7), we obtain

$$f(x+3y) + f(x-y) = 4f(x+2y) - 6f(x+y) + 4f(x) \tag{2.9}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.9), we get

$$f(x-3y) + f(x+y) = 4f(x-2y) - 6f(x-y) + 4f(x) \tag{2.10}$$

for all $x, y \in X$. Replacing x by y and y by x in (2.9), we get

$$f(3x+y) - f(x-y) = 4f(2x+y) - 6f(x+y) + 4f(y) \tag{2.11}$$

for all $x, y \in X$. Replacing $-y$ by y in (2.11), we get

$$f(3x-y) - f(x+y) = 4f(2x-y) - 6f(x-y) - 4f(y) \tag{2.12}$$

for all $x, y \in X$.

Subtracting (2.12) from (2.10), we obtain

$$f(3x-y)-f(x-3y)=4f(2x-y)-4f(x-2y)+2f(x+y)-4f(x)-4f(y) \tag{2.13}$$

for all $x,y\in X$. By (2.8) and (2.13), we obtain

$$f(x-2y)=f(2x-y)+2f(x+y)-f(2x)-f(2y)-f(x)-f(y) \tag{2.14}$$

for all $x,y\in X$.

Replacing y by $-y$ in (2.14), we get

$$f(x+2y)=f(2x+y)+2f(x-y)-f(2x)+f(2y)-f(x)+f(y) \tag{2.15}$$

for all $x,y\in X$.

By (2.14) and (2.15), we obtain

$$\begin{aligned} &f(x+2y)+f(x-2y)\\ &\quad=f(2x+y)+f(2x-y)+2f(x+y)+2f(x-y)-2f(2x)-2f(x) \end{aligned} \tag{2.16}$$

for all $x,y\in X$.

By (2.7) and (2.16), we have

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x) \tag{2.17}$$

for all $x,y\in X$. Replacing y by $x+y$ in (2.17), we get

$$f(3x+y)+f(x-y)=2f(2x+y)-2f(y)+2f(2x)-4f(x) \tag{2.18}$$

for all $x,y\in X$. Replacing x, y by y, x in (2.18), respectively, we get

$$f(x+3y)-f(x-y)=2f(x+2y)-2f(x)+2f(2y)-4f(y) \tag{2.19}$$

for all $x,y\in X$.

By (2.18) and (2.19), we obtain

$$\begin{aligned} &f(3x+y)+f(x+3y)\\ &\quad=2f(2x+y)+2f(x+2y)+2f(2x)+2f(2y)-6f(x)-6f(y) \end{aligned} \tag{2.20}$$

for all $x,y\in X$. Replacing x, y by $x+y$, $x-y$ in (2.17), respectively, we get

$$f(3x+y)+f(x+3y)=2f(2x+2y)-4f(x+y)+2f(2x)+2f(2y) \tag{2.21}$$

for all $x, y \in X$. Thus it follows from (2.20) and (2.21) that

$$f(2x+y) + f(x+2y) = f(2x+2y) - 2f(x+y) + 3f(x) + 3f(y) \tag{2.22}$$

for all $x, y \in X$. Replacing x by $x - y$ in (2.22), we obtain

$$f(2x-y) + f(x+y) = 3f(x-y) + f(2x) - 2f(x) + 3f(y) \tag{2.23}$$

for all $x, y \in X$. Replacing x, y by y, x in (2.23), respectively, we get

$$f(2y-x) + f(x+y) = 3f(y-x) + f(2y) - 2f(y) + 3f(x) \tag{2.24}$$

for all $x, y \in X$. By (2.23) and (2.24), we obtain

$$f(2x-y) + f(2y-x) = -2f(x+y) + f(x) + f(y) + f(2x) + f(2y) \tag{2.25}$$

for all $x, y \in X$. Adding (2.22) to (2.25) and using (2.17), we get

$$f(2x+2y) - 8f(x+y) = [f(2x) - 8f(x)] + [f(2y) - 8f(y)] \tag{2.26}$$

for all $x, y \in X$. The last equality means that

$$g(x+y) = g(x) + g(y) \tag{2.27}$$

for all $x, y \in X$. Thus the mapping $g : X \to Y$ is additive.

Replacing x, y by $2x$, $2y$ in (2.17), respectively, we get

$$f(4x+2y) + f(4x-2y) = 2f(2x+2y) + 2f(2x-2y) + 2f(4x) - 4f(2x) \tag{2.28}$$

for all $x, y \in X$. Since $g(2x) = 2g(x)$ for all $x \in X$,

$$f(4x) = 10f(2x) - 16f(x) \tag{2.29}$$

for all $x, y \in X$. Hence it follows from (2.17) and (2.28) that

$$\begin{aligned} h(2x+y) + h(2x-y) &= [f(2(2x+y)) - 2f(2x+y)] + [f(2(2x-y)) - 2f(2x-y)] \\ &= 2[f(2(x+y)) - 2f(x+y)] \\ &\quad + 2[f(2(x-y)) - 2f(x-y)] + 12[f(2x) - 2f(x)] \\ &= 2h(x+y) + 2h(x-y) + 12h(x) \end{aligned} \tag{2.30}$$

for all $x, y \in X$. Thus the mapping $h : X \to Y$ is cubic.

On the other hand, we have $f(x) = (1/6)h(x) - (1/6)g(x)$ for all $x \in X$. This means that f is cubic-additive. This completes the proof of the lemma. □

The following is suggested by an anonymous referee.

Remark 2.3. The functional equation (1.3) is equivalent to the functional equation

$$\begin{aligned} &11f(x+2y) + 11f(x-2y) - 44f(x+y) - 44f(x-y) + 66f(x) \\ &\quad = 12f(3y) - 48f(2y) + 60f(y). \end{aligned} \tag{2.31}$$

The left hand side is even with respect to y and the right hand side is odd by the assumption of Lemma 2.2. Thus

$$11f(x+2y) + 11f(x-2y) - 44f(x+y) - 44f(x-y) + 66f(x) = 0. \tag{2.32}$$

So we conclude that $f(x) = A(x) + C(x,x,x)$, as desired.

Theorem 2.4. *If a mapping $f : X \to Y$ satisfies* (1.3) *for all $x, y \in X$, then there exist a unique additive mapping $A : X \to Y$, a unique mapping $C : X \times X \times X \to Y$, and a unique symmetric multi-additive mapping $Q : X \times X \times X \times X \to Y$ such that $f(x) = A(x) + C(x,x,x) + Q(x,x,x,x)$ for all $x \in X$, and that C is symmetric for each fixed one variable and is additive for fixed two variables.*

Proof. Let f satisfy (1.3). We decompose f into the even part and the odd part by setting

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)), \qquad f_o(x) = \frac{1}{2}(f(x) - f(-x)), \tag{2.33}$$

for all $x \in X$. By (1.3), we have

$$\begin{aligned} &11[f_e(x+2y) + f_e(x-2y)] \\ &\quad = \frac{1}{2}[11f(x+2y) + 11f(-x-2y) + 11f(x-2y) + 11f(-x+2y)] \\ &\quad = \frac{1}{2}[11f(x+2y) + 11f(x-2y)] + \frac{1}{2}[11f(-x+(-2y)) + 11f(-x-(-2y))] \\ &\quad = \frac{1}{2}[44(f(x+y) + f(x-y)) + 12f(3y) - 48f(2y) + 60f(y) - 66f(x)] \\ &\qquad + \frac{1}{2}[44(f(-x-y) + f(-x-(-y))) + 12f(-3y) - 48f(-2y) + 60f(-y) - 66f(-x)] \end{aligned}$$

$$
\begin{aligned}
&= 44\left[\frac{1}{2}(f(x+y)+f(-x-y))+\frac{1}{2}(f(-x+y)+f(x-y))\right] \\
&\quad +12\left[\frac{1}{2}(f(3y)+f(-3y))\right]-48\left[\frac{1}{2}(f(2y)+f(-2y))\right] \\
&\quad +60\left[\frac{1}{2}(f(y)+f(-y))\right]-66\left[\frac{1}{2}(f(x)+f(-x))\right] \\
&= 44[f_e(x+y)+f_e(x-y)]+12f_e(3y)-48f_e(2y)+60f_e(y)-66f_e(x)
\end{aligned}
\tag{2.34}
$$

for all $x, y \in X$. This means that f_e satisfies (1.3). Similarly we can show that f_o satisfies (1.3). By Lemmas 2.1 and 2.2, f_e and f_o are quartic and cubic-additive, respectively. Thus there exist a unique additive mapping $A : X \to Y$, a unique mapping $C : X \times X \times X \to Y$, and a unique symmetric multi-additive mapping $Q : X \times X \times X \times X \to Y$ such that $f_e(x) = Q(x,x,x,x)$ and that $f_o(x) = A(x) + C(x,x,x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables. Thus $f(x) = A(x) + C(x,x,x) + Q(x,x,x,x)$ for all $x \in X$, as desired. □

3. Stability of an Additive-Cubic-Quartic Functional Equation

We now investigate the generalized Hyers-Ulam stability problem of the functional equation (1.3). From now on, let X be a real vector space and let Y be a Banach space. Now before taking up the main subject, given $f : X \to Y$, we define the difference operator $D_f : X \times X \to Y$ by

$$
\begin{aligned}
D_f(x,y) = 11[f(x+2y)+f(x-2y)] - 44[f(x+y)+f(x-y)] \\
-12f(3y)+48f(2y)-60f(y)+66f(x)
\end{aligned}
\tag{3.1}
$$

for all $x, y \in X$. We consider the following functional inequality:

$$
\|D_f(x,y)\| \le \phi(x,y), \tag{3.2}
$$

for an upper bound $\phi : X \times X \to [0,\infty)$.

Theorem 3.1. *Let $s \in \{1,-1\}$ be fixed. Suppose that an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$
\|D_f(x,y)\| \le \phi(x,y) \tag{3.3}
$$

for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0,\infty)$ is a function such that

$$
\sum_{i=0}^{\infty} 4^{si}\left[\phi\left(2^{-si}x, 2^{-si}y\right)+\frac{1}{2}\phi\left(0, 2^{-si}x\right)\right] < \infty \tag{3.4}
$$

and that $\lim_{n\to\infty} 16^{sn}\phi(2^{-sn}x, 2^{-sn}x) = 0$ *for all* $x, y \in X$*, then the limit*

$$Q(x) := \lim_{n\to\infty} 16^{sn} f\left(2^{-sn}x\right) \tag{3.5}$$

exists for all $x \in X$*, and* $Q : X \to Y$ *is a unique quartic mapping satisfying* (1.3) *and*

$$\|f(x) - Q(x)\| \leq \sum_{i=(s+1)/2}^{\infty} 16^{si-1}\left[\frac{6}{11}\phi\left(2^{-si}x, 2^{-si}x\right) + \phi\left(0, 2^{-si}x\right)\right] \tag{3.6}$$

for all $x \in X$.

Proof. Putting $x = 0$ in (3.3), we obtain

$$\|-12f(3y) + 70f(2y) - 148f(y)\| \leq \phi(0, y) \tag{3.7}$$

for all $y \in X$. On the other hand, replacing y by x in (3.3), we get

$$\|-f(3y) + 4f(2y) + 17f(y)\| \leq \phi(y, y) \tag{3.8}$$

for all $y \in X$. By (3.7) and (3.8), we get

$$\|f(2y) - 16f(y)\| \leq \frac{6}{11}\phi(y, y) + \phi(0, y) \tag{3.9}$$

for all $y \in X$. Replacing y by $x/2$ in (3.9), we get

$$\left\|f(x) - 16f\left(\frac{x}{2}\right)\right\| \leq \frac{6}{11}\phi\left(\frac{x}{2}, \frac{x}{2}\right) + \phi\left(0, \frac{x}{2}\right) \tag{3.10}$$

for all $x \in X$. It follows from (3.10) that

$$\left\|f(x) - 16^n f\left(\frac{x}{2^n}\right)\right\| \leq \sum_{i=0}^{n-1} 16^i\left[\frac{6}{11}\phi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right) + \phi\left(0, \frac{x}{2^{i+1}}\right)\right] \tag{3.11}$$

for all $x \in X$. It follows from (3.11) that

$$\begin{aligned}\left\|16^m f\left(\frac{x}{2^m}\right) - 16^{m+n} f\left(\frac{x}{2^{m+n}}\right)\right\| &\leq \sum_{i=m}^{n-1} 16^{m+i}\left[\frac{6}{11}\phi\left(\frac{x}{2^{m+i+1}}, \frac{x}{2^{m+i+1}}\right) + \phi\left(0, \frac{x}{2^{m+i+1}}\right)\right] \\ &= \sum_{i=m}^{m+n-1} 16^i\left[\frac{6}{11}\phi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right) + \phi\left(0, \frac{x}{2^{i+1}}\right)\right]\end{aligned} \tag{3.12}$$

for all $x \in X$.

This shows that $\{16^n f(x/2^n)\}$ is a Cauchy sequence in Y. Since Y is complete, the sequence $\{16^n f(x/2^n)\}$ converges. We now define $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right) \tag{3.13}$$

for all $x \in X$. It is clear that (3.6) holds, and $Q(-x) = Q(x)$ for all $x \in X$. By (3.3), we have

$$\|D_Q(x,y)\| = \lim_{n\to\infty} 16^n \left\|D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \le \lim_{n\to\infty} 16^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{3.14}$$

for all $x, y \in X$. Hence by Lemma 2.1, Q is quartic.

It remains to show that Q is unique. Suppose that there exists a quartic mapping $Q' : X \to Y$ which satisfies (1.3) and (3.6). Since $Q(2^n x) = 16^n Q(x)$ and $Q'(2^n x) = 16^n Q'(x)$ for all $x \in X$, we conclude that

$$\begin{aligned}\|Q(x) - Q'(x)\| &= 16^n \left\|Q\left(\frac{x}{2^n}\right) - Q'\left(\frac{x}{2^n}\right)\right\| \\ &\le 16^n \left\|Q\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\| + 16^n \left\|Q'\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\| \\ &\le 2\sum_{i=0}^{\infty} 16^{n+i} \left[\frac{6}{11}\phi\left(\frac{x}{2^{n+i+1}}, \frac{x}{2^{n+i+1}}\right) + \phi\left(0, \frac{x}{2^{n+i+1}}\right)\right]\end{aligned} \tag{3.15}$$

for all $x \in X$. By taking $n \to \infty$ in this inequality, we have $Q(x) = Q'(x)$ for all $x \in X$, which gives the conclusion for the case $s = 1$. Let $s = -1$. Then by (3.9), we have

$$\left\|\frac{f(2x)}{16} - f(x)\right\| \le \frac{1}{16}\left(\frac{6}{11}\phi(x,x) + \phi(0,x)\right) \tag{3.16}$$

for all $x \in X$. Replacing x by $2x$ in (3.16) and dividing by 16, we get

$$\left\|\frac{f(4x)}{16^2} - \frac{f(2x)}{16}\right\| \le \frac{1}{16^2}\left(\frac{6}{11}\phi(2x,2x) + \phi(0,2x)\right) \tag{3.17}$$

for all $x \in X$. By (3.16) and (3.17), we obtain

$$\left\|f(x) - \frac{f(4x)}{16^2}\right\| \le \frac{1}{16}\left[\frac{6}{11}\phi(x,x) + \left(\frac{1}{16} \times \frac{6}{11}\right)\phi(2x,2x) + \phi(0,x) + \frac{1}{16}\phi(0,2x)\right] \tag{3.18}$$

for all $x \in X$. It follows from (3.18) that

$$\left\|f(x) - \frac{f(2^n x)}{16^n}\right\| \le \frac{1}{16}\left(\sum_{i=0}^{n-1} 16^{-i}\left[\frac{6}{11}\phi\left(2^i x, 2^i x\right) + \phi\left(0, 2^i x\right)\right]\right) \tag{3.19}$$

for all $x \in X$. Dividing both sides of (3.19) by 16^m and then replacing x by $2^m x$, we get

$$\begin{aligned} \left\| \frac{f(2^m x)}{16^m} - \frac{f(2^{m+n} x)}{16^{m+n}} \right\| &\le \frac{1}{6}\sum_{i=0}^{n-1} 16^{-m-i} \left[\frac{6}{11}\phi\left(2^{m+i}x, 2^{m+i}x\right) + \phi\left(0, 2^{m+i}x\right) \right] \\ &= \frac{1}{16}\sum_{i=m}^{m+n-1} 16^{-i} \left[\frac{6}{11}\phi\left(2^{i}x, 2^{i}x\right) + \phi\left(0, 2^{i}x\right) \right] \end{aligned} \tag{3.20}$$

for all $x \in X$. By taking $m \to \infty$ in (3.20), $\{16^{-n} f(2^n x)\}$ is a Cauchy sequence in Y. Then $Q(x) := \lim_{n\to\infty} 16^{-n} f(2^n x)$ exists for all $x \in X$. It is easy to see that (3.6) holds for $s = -1$.

The rest of the proof is similar to the case $s = 1$. □

Theorem 3.2. *Suppose that an odd mapping $f : X \to Y$ satisfies*

$$\left\| D_f(x, y) \right\| \le \phi(x, y) \tag{3.21}$$

for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0, \infty)$ is a function such that

$$\sum_{i=1}^{\infty} 2^i \left[\phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \phi\left(0, \frac{x}{2^{i+1}}\right) \right] < \infty \tag{3.22}$$

and that $\lim_{n\to\infty} 2^n \phi(x/2^n, y/2^n) = 0$ for all $x, y \in X$, then the limit

$$A(x) := \lim_{n\to\infty} 2^n \left[f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right) \right] \tag{3.23}$$

exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping satisfying (1.3) and

$$\left\| f(2x) - 8f(x) - A(x) \right\| \le \frac{1}{11}\sum_{i=0}^{\infty} 2^i \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{14}{33}\sum_{i=0}^{\infty} 2^i \phi\left(0, \frac{x}{2^{i+1}}\right) \tag{3.24}$$

for all $x \in X$.

Proof. Set $x = 0$ in (3.21). Then by the oddness of f, we have

$$\left\| 12f(3y) - 48f(2y) + 60f(y) \right\| \le \phi(0, y) \tag{3.25}$$

for all $y \in X$. Replacing x by $2y$ in (3.21), we obtain

$$\left\| 11f(4y) - 56f(3y) + 114f(2y) - 104f(y) \right\| \le \phi(2y, y) \tag{3.26}$$

for all $y \in X$. Combining (3.25) and (3.26) yields that

$$\left\| f(4y) - 10f(2y) + 16f(y) \right\| \le \frac{1}{11}\left[\phi(2y, y) + \frac{14}{3}\phi(0, y) \right] \tag{3.27}$$

for all $y \in X$. Putting $y := x/2$ and $g(x) := f(2x) - 8f(x)$ for all $x \in X$, we get

$$\left\| g(x) - 2g\left(\frac{x}{2}\right) \right\| \le \frac{1}{11}\phi\left(x, \frac{x}{2}\right) + \frac{14}{33}\phi\left(0, \frac{x}{2}\right) \tag{3.28}$$

for all $x \in X$. It follows from (3.28) that

$$\left\| 2^n g\left(\frac{x}{2^n}\right) - g(x) \right\| \le \frac{1}{11}\sum_{i=0}^{n-1} 2^i \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{14}{33}\sum_{i=0}^{n-1} 2^i \phi\left(0, \frac{x}{2^{i+1}}\right) \tag{3.29}$$

for all $x \in X$. Multiplying both sides of (3.29) by 2^m and then replacing x by $2^{-m}x$, we get

$$\begin{aligned} \left\| 2^m g\left(\frac{x}{2^m}\right) - 2^{m+n} g\left(\frac{x}{2^{m+n}}\right) \right\| &\le \frac{1}{11}\sum_{i=0}^{n-1} 2^{i+m} \phi\left(\frac{x}{2^{i+m}}, \frac{x}{2^{m+i+1}}\right) + \frac{14}{33}\sum_{i=0}^{n-1} 2^{m+i} \phi\left(0, \frac{x}{2^{m+i+1}}\right) \\ &= \frac{1}{11}\sum_{i=m}^{m+n-1} 2^i \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{14}{33}\sum_{i=0}^{m+n-1} 2^i \phi\left(0, \frac{x}{2^{i+1}}\right) \end{aligned} \tag{3.30}$$

for all $x \in X$. So $\{2^n g(x/2^n)\}$ is a Cauchy sequence in Y. Put $A(x) := \lim_{n\to\infty} 2^n g(x/2^n)$ for all $x \in X$. Then we have

$$\begin{aligned} \|A(2x) - 2A(x)\| &= \lim_{n\to\infty} \left\| 2^n g\left(\frac{x}{2^{n-1}}\right) - 2^{n+1} g\left(\frac{x}{2^n}\right) \right\| \\ &= \lim_{n\to\infty} 2 \left\| 2^{n-1} g\left(\frac{x}{2^{n-1}}\right) - 2^n g\left(\frac{x}{2^n}\right) \right\| = 0 \end{aligned} \tag{3.31}$$

for all $x \in X$. On the other hand, it is easy to show that

$$D_g(x, y) = D_f(2x, 2y) - 8D_f(x, y) \tag{3.32}$$

for all $x, y \in X$. Hence it follows that

$$\begin{aligned} \|D_A(x, y)\| &= \lim_{n\to\infty} \left\| 2^n D_g\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| = \lim_{n\to\infty} \left\| \left[2^n D_f\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right) - 2^{n+3} D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right] \right\| \\ &\le 2\lim_{n\to\infty} \left[2^{n-1}\phi\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right) \right] + 8\lim_{n\to\infty} 2^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \end{aligned} \tag{3.33}$$

for all $x, y \in X$. This means that A satisfies (1.3). Then by Lemma 2.2, $x \mapsto A(2x) - 8A(x)$ is additive. Thus (3.31) implies that A is additive.

To prove the uniqueness of A, suppose that $A' : X \to Y$ is an additive mapping satisfying (3.24). Then for every $x \in X$, we have $A(2^{-n}x) = 2^{-n}A(x)$, and $A'(2^{-n}x) = 2^{-n}A'(x)$. Hence it follows that

$$\begin{aligned}\|A(x) - A'(x)\| &= \lim_{n\to\infty} 2^n \left\|A\left(\frac{x}{2^n}\right) - A'\left(\frac{x}{2^n}\right)\right\| \le \lim_{n\to\infty} 2^n \left\|A\left(\frac{x}{2^n}\right) - g\left(\frac{x}{2^n}\right)\right\| \\ &\quad + \lim_{n\to\infty} 2^n \left\|A'\left(\frac{x}{2^n}\right) - g\left(\frac{x}{2^n}\right)\right\| \le \frac{1}{11}\sum_{i=0}^{\infty} 2^{n+i}\phi\left(\frac{x}{2^{n+i}}, \frac{x}{2^{n+i+1}}\right) \\ &\quad + \frac{14}{33}\sum_{i=0}^{\infty} 2^{n+i}\phi\left(0, \frac{x}{2^{n+i+1}}\right) = \frac{1}{11}\sum_{i=n}^{\infty} 2^{i}\phi\left(\frac{x}{2^{i}}, \frac{x}{2^{i+1}}\right) + \frac{14}{33}\sum_{i=n}^{\infty} 2^{i}\phi\left(0, \frac{x}{2^{i+1}}\right)\end{aligned} \tag{3.34}$$

for all $x \in X$. This shows that $A(x) = A'(x)$ for all $x \in X$. □

Theorem 3.3. *Suppose that an odd mapping $f : X \to Y$ satisfies*

$$\|D_f(x, y)\| \le \phi(x, y) \tag{3.35}$$

for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0, \infty)$ is a function such that

$$\sum_{i=1}^{\infty} 8^i \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \sum_{i=1}^{\infty} 8^i \phi\left(0, \frac{x}{2^{i+1}}\right) < \infty \tag{3.36}$$

and that

$$\lim_{n\to\infty} 8^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{3.37}$$

for all $x, y \in X$, then the limit

$$C(x) := \lim_{n\to\infty} 8^n \left[f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right] \tag{3.38}$$

exists for all $x \in X$, and $C : X \to Y$ is a unique cubic mapping satisfying (1.3)*, and*

$$\|f(2x) - 2f(x) - C(x)\| \le \sum_{i=0}^{\infty} 8^i \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + 2\sum_{i=0}^{\infty} 8^i \phi\left(0, \frac{x}{2^{i+1}}\right) \tag{3.39}$$

for all $x \in X$.

Proof. It is easy to show that f satisfies (3.27). Setting $h(x) := f(2x) - 2f(x)$ and then putting $y := x/2$ in (3.27), we obtain

$$\left\|h(x) - 8h\left(\frac{x}{2}\right)\right\| \le \frac{1}{11}\phi\left(x, \frac{x}{2}\right) + \frac{14}{33}\phi\left(0, \frac{x}{2}\right) \tag{3.40}$$

for all $x \in X$. It follows from (3.40) that

$$\left\| 8^n h\left(\frac{x}{2^n}\right) - h(x) \right\| \le \frac{1}{11} \sum_{i=0}^{n-1} 8^i \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{14}{33} \sum_{i=0}^{n-1} 8^i \phi\left(0, \frac{x}{2^{i+1}}\right) \tag{3.41}$$

for all $x \in X$. Replacing x by $x/2^m$ in (3.41) and then multiplying both sides of (3.41) by 8^m, we get

$$\begin{aligned} \left\| 8^{n+m} h\left(\frac{x}{2^{n+m}}\right) - 8^m h\left(\frac{x}{2^m}\right) \right\| &\le \frac{1}{11} \sum_{i=0}^{n-1} 8^{m+i} \phi\left(\frac{x}{2^{i+m}}, \frac{x}{2^{i+m+1}}\right) + \frac{14}{33} \sum_{i=0}^{n-1} 8^{m+i} \phi\left(0, \frac{x}{2^{i+m+1}}\right) \\ &= \frac{1}{11} \sum_{i=m}^{m+n-1} 8^i \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{14}{33} \sum_{i=m}^{m+n-1} 8^i \phi\left(0, \frac{x}{2^{i+1}}\right) \end{aligned} \tag{3.42}$$

for all $x \in X$. Since the right hand side of the inequality (3.42) tends to 0 as $m \to \infty$, the sequence $\{8^n h(x/2^n)\}$ is Cauchy. Now we define

$$C(x) := \lim_{n\to\infty} 8^n h\left(\frac{x}{2^n}\right) \tag{3.43}$$

for all $x \in X$. Then we have

$$\|C(2x) - 8C(x)\| = \lim_{n\to\infty} \left\| 8^n h\left(\frac{x}{2^{n-1}}\right) - 8^{n+1} h\left(\frac{x}{2^n}\right) \right\| = 0 \tag{3.44}$$

for all $x \in X$. Let

$$D_h(x, y) = D_f(2x, 2y) - 2D_f(x, y) \tag{3.45}$$

for all $x, y \in X$. Then we have

$$\begin{aligned} \|D_C(x, y)\| &= \lim_{n\to\infty} \left\| 8^n D_h\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| = \lim_{n\to\infty} 8^n \left[\left\| D_f\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right) - 2D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \right] \\ &= \lim_{n\to\infty} 8 \left\| 8^{n-1} D_f\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right) \right\| + 2 \left\| 8^n D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \\ &\le \lim_{n\to\infty} 8\left(8^{n-1} \phi\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right)\right) + \lim_{n\to\infty} 2\left(8^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right) = 0 \end{aligned} \tag{3.46}$$

for all $x, y \in X$. Since C is an odd mapping, C satisfies (2.6). By (3.44), we conclude that $C(3x) = 27C(x)$ for all $x \in X$. Then C is cubic.

We have to show that C is unique. Suppose that there exists another cubic mapping $C' : X \to Y$ which satisfies (1.3) and (3.39). Since $C(2^n x) = 8^n C(x)$ and $C'(2^n x) = 8^n C'(x)$ for all $x \in X$, we have

$$\begin{aligned}\|C(x) - C'(x)\| &= \lim_{n\to\infty} 8^n \left\|C\left(\frac{x}{2^n}\right) - C'\left(\frac{x}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty} 8^n \left\|C\left(\frac{x}{2^n}\right) - h\left(\frac{x}{2^n}\right)\right\| + \lim_{n\to\infty} 8^n \left\|h\left(\frac{x}{2^n}\right) - C'\left(\frac{x}{2^n}\right)\right\| \\ &\le \frac{1}{11}\sum_{i=0}^{\infty} 8^{n+i}\phi\left(\frac{x}{2^{n+i}}, \frac{x}{2^{n+i+1}}\right) + \frac{14}{33}\sum_{i=0}^{\infty} 8^{n+i}\phi\left(0, \frac{x}{2^{n+i+1}}\right) \\ &= \frac{1}{11}\sum_{i=n}^{\infty} 8^{i}\phi\left(\frac{x}{2^{i}}, \frac{x}{2^{i+1}}\right) + \frac{14}{33}\sum_{i=n}^{\infty} 8^{i}\phi\left(0, \frac{x}{2^{i+1}}\right)\end{aligned} \tag{3.47}$$

for all $x \in X$. By letting $n \to \infty$ in the above inequality, we get $C(x) = C'(x)$ for all $x \in X$, which gives the conclusion. □

Theorem 3.4. *Suppose that an odd mapping $f : X \to Y$ satisfies*

$$\|D_f(x, y)\| \le \phi(x, y) \tag{3.48}$$

for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0, \infty)$ is a function such that

$$\sum_{i=1}^{\infty} 8^{i}\phi\left(\frac{x}{2^{i}}, \frac{x}{2^{i+1}}\right) + \sum_{i=1}^{\infty} 8^{i}\phi\left(0, \frac{x}{2^{i+1}}\right) < \infty \tag{3.49}$$

and that $\lim_{n\to\infty} 8^n \phi(x/2^n, y/2^n) = 0$ for all $x, y \in X$, then there exist a unique cubic mapping $C : X \to Y$, and a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - C(x) - A(x)\| \le \frac{1}{66}\sum_{i=0}^{\infty}\left(2^i + 8^i\right)\phi\left(\frac{x}{2^{i}}, \frac{x}{2^{i+1}}\right) + \frac{7}{99}\sum_{i=0}^{\infty}\left(2^i + 8^i\right)\phi\left(0, \frac{x}{2^{i+1}}\right) \tag{3.50}$$

for all $x \in X$.

Proof. By Theorems 3.2 and 3.3, there exist an additive mapping $A_o : X \to Y$ and a cubic mapping $C_o : X \to Y$ such that

$$\begin{aligned}\|f(2x) - 8f(x) - A_o(x)\| &\le \frac{1}{11}\sum_{i=0}^{\infty} 2^{i}\phi\left(\frac{x}{2^{i}}, \frac{x}{2^{i+1}}\right) + \frac{14}{33}\sum_{i=0}^{\infty} 2^{i}\phi\left(0, \frac{x}{2^{i+1}}\right), \\ \|f(2x) - 2f(x) - C_o(x)\| &\le \frac{1}{11}\sum_{i=0}^{\infty} 8^{i}\phi\left(\frac{x}{2^{i}}, \frac{x}{2^{i+1}}\right) + \frac{14}{33}\sum_{i=0}^{\infty} 8^{i}\phi\left(0, \frac{x}{2^{i+1}}\right)\end{aligned} \tag{3.51}$$

for all $x \in X$. Combining two equations in (3.51) yields that

$$\left\| f(x) - \frac{1}{6}C_o(x) + \frac{1}{6}A_o(x) \right\| \leq \frac{1}{66}\sum_{i=0}^{\infty}\left(2^i + 8^i\right)\phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{7}{99}\sum_{i=0}^{\infty}\left(2^i + 8^i\right)\phi\left(0, \frac{x}{2^{i+1}}\right) \tag{3.52}$$

for all $x \in X$. So we get (3.50) by letting $A(x) = -(1/6)A_o(x)$ and $C(x) = (1/6)C_o(x)$ for all $x \in X$.

To prove the uniqueness of A and C, let $A_1, C_1 : X \to Y$ be other additive and cubic mappings satisfying (3.50). Let $A' = A - A_1$, $C' = C - C_1$. Then

$$\begin{aligned} \left\| A'(x) - C'(x) \right\| &\leq \left\| f(x) - A(x) - C(x) \right\| + \left\| f(x) - A_1(x) - C_1(x) \right\| \\ &\leq 2\left[\frac{1}{66}\sum_{i=0}^{\infty}\left(2^i + 8^i\right)\phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{7}{99}\sum_{i=0}^{\infty}\left(2^i + 8^i\right)\phi\left(0, \frac{x}{2^{i+1}}\right)\right] \end{aligned} \tag{3.53}$$

for all $x \in X$. Since

$$\begin{aligned} &\lim_{n\to\infty}\left\{\sum_{i=1}^{\infty}8^{i+n}\phi\left(\frac{x}{2^{i+n}}, \frac{x}{2^{i+n+1}}\right) + \sum_{i=1}^{\infty}8^{i+n}\phi\left(0, \frac{x}{2^{i+n+1}}\right)\right\} = 0, \\ &\lim_{n\to\infty}\left\{\sum_{i=1}^{\infty}2^{i+n}\phi\left(\frac{x}{2^{i+n}}, \frac{x}{2^{i+n+1}}\right) + \sum_{i=1}^{\infty}2^{i+n}\phi\left(0, \frac{x}{2^{i+n+1}}\right)\right\} = 0 \end{aligned} \tag{3.54}$$

for all $x \in X$. Hence (3.53) implies that

$$\lim_{n\to\infty}8^n\left\| A'\left(\frac{x}{2^n}\right) - C'\left(\frac{x}{2^n}\right) \right\| = 0 \tag{3.55}$$

for all $x \in X$. Since $C'(x/2^n) = (1/8^n)C'(x)$, by (3.55), we obtain that $A'(x) = 0$ for all $x \in X$. Again by (3.55), we have $C'(x) = 0$ for all $x \in X$. □

Now we prove the generalized Hyers-Ulam stability of the functional equation (1.3).

Theorem 3.5. *Suppose that a mapping $f : X \to Y$ satisfies $f(0) = 0$ and $\|D_f(x,y)\| \leq \phi(x,y)$ for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0,\infty)$ is a function such that*

$$\sum_{i=0}^{\infty}\left\{8^i\left[\phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \phi\left(0, \frac{x}{2^{i+1}}\right)\right] + 16^i\phi\left(\frac{x}{2^i}, \frac{x}{2^i}\right)\right\} < \infty \tag{3.56}$$

and that $\lim_{n\to\infty} 8^n \phi(x/2^n, y/2^n) = 0$ *for all* $x, y \in X$, *then there exist a unique additive mapping* $A : X \to Y$, *a unique cubic mapping* $C : X \to Y$, *and a unique quartic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - A(x) - C(x) - Q(x)\| \\ \leq \frac{1}{11}\sum_{i=0}^{\infty}\left(2^i + 8^i\right)\left[\frac{1}{6}\phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{7}{9}\phi\left(0, \frac{x}{2^{i+1}}\right)\right] + \frac{1}{8}\sum_{i=1}^{\infty}16^i\left[\frac{6}{11}\phi\left(\frac{x}{2^i}, \frac{x}{2^i}\right) + \phi\left(0, \frac{x}{2^i}\right)\right] \tag{3.57}$$

for all $x \in X$.

Proof. Let $f_e(x) = (1/2)(f(x) + f(-x))$ for all $x \in X$. Then $f_e(0) = 0$, $f_e(-x) = f_e(x)$ and

$$\|D_{f_e}(x, y)\| \leq \frac{1}{2}[\phi(x, y) + \phi(-x, -y)] \tag{3.58}$$

for all $x, y \in X$. Hence in view of Theorem 3.1, there exists a unique quartic mapping $Q : X \to Y$ satisfying (3.6). Let $f_o(x) = (1/2)(f(x) - f(-x))$ for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$, and $\|D_{f_o}(x, y)\| \leq (1/2)[\phi(x, y) + \phi(-x, -y)]$ for all $x, y \in X$. From Theorem 3.4, it follows that there exist a unique cubic mapping $C : X \to Y$ and a unique additive mapping $A : X \to Y$ satisfying (3.44). Now it is obvious that (3.57) holds for all $x \in X$ and the proof of the theorem is complete. □

Corollary 3.6. *Let* $p > 4$ *and let* θ *be a positive real number. Suppose that a mapping* $f : X \to Y$ *satisfies* $f(0) = 0$ *and*

$$\|D_f(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{3.59}$$

for all $x, y \in X$. *Then there exist a unique additive mapping* $A : X \to Y$, *a unique cubic mapping* $C : X \to Y$, *and a unique quartic mapping* $Q : X \to Y$ *satisfying*

$$\|f(x) - A(x) - C(x) - Q(x)\| \\ \leq \left\{\frac{1}{11}\left[\frac{1}{6}\left(1 + \frac{1}{2^p}\right) + \frac{7}{9 \times 2^p}, \frac{1}{1 - 2^{1-p}} + \frac{1}{1 - 2^{3-p}}\right] + \frac{23}{88}\left(\frac{1}{1 - 2^{4-p}} - 1\right)\right\}\theta\|x\|^p \tag{3.60}$$

for all $x \in X$.

Proof. It follows from Theorem 3.5 by taking $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. □

Theorem 3.7. *Suppose that an odd mapping* $f : X \to Y$ *satisfies*

$$\|D_f(x, y)\| \leq \phi(x, y) \tag{3.61}$$

for all $x, y \in X$. *If the upper bound* $\phi : X \times X \to [0, \infty)$ *is a function such that*

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \left[\phi\left(2^i x, 2^{i-1} x\right) + \phi\left(0, 2^{i-1} x\right) \right] < \infty \tag{3.62}$$

and that $\lim_{n\to\infty}(1/2^n)\phi(2^n x, 2^n y) = 0$ *for all* $x, y \in X$, *then the limit*

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} \left[f\left(2^{n+1} x\right) - 8 f(2^n x) \right] \tag{3.63}$$

exists for all $x \in X$, *and* $A : X \to Y$ *is a unique additive mapping satisfying* (1.3) *and*

$$\| f(2x) - 8f(x) - A(x) \| \le \frac{1}{11} \sum_{i=1}^{\infty} \frac{1}{2^i} \phi\left(2^i x, 2^{i-1} x\right) + \frac{14}{33} \sum_{i=1}^{\infty} \frac{1}{2^i} \phi\left(0, 2^{i-1} x\right) \tag{3.64}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 3.2. □

Employing a similar way to the proof of Theorem 3.3, we get the following theorem.

Theorem 3.8. *Suppose that an odd mapping* $f : X \to Y$ *satisfies*

$$\| D_f(x, y) \| \le \phi(x, y) \tag{3.65}$$

for all $x, y \in X$. *If the upper bound* $\phi : X \times X \to [0, \infty)$ *is a function such that*

$$\sum_{i=1}^{\infty} \frac{1}{8^i} \phi\left(2^i x, 2^{i-1} x\right) + \sum_{i=1}^{\infty} \frac{1}{8^i} \phi\left(0, 2^{i-1} x\right) < \infty \tag{3.66}$$

and that $\lim_{n\to\infty}(1/8^n)\phi(2^n x, 2^n y) = 0$ *for all* $x, y \in X$, *then the limit*

$$C(x) := \lim_{n\to\infty} \frac{1}{8^n} \left[f\left(2^{n+1} x\right) - 2 f(2^n x) \right] \tag{3.67}$$

exists for all $x \in X$, *and* $C : X \to Y$ *is a unique cubic mapping satisfying* (1.3), *and*

$$\| f(2x) - 2f(x) - C(x) \| \le \sum_{i=1}^{\infty} \frac{1}{8^i} \phi\left(2^i x, 2^{i-1} x\right) + 2 \sum_{i=1}^{\infty} \frac{1}{8^i} \phi\left(0, 2^{i-1} x\right) \tag{3.68}$$

for all $x \in X$.

Theorem 3.9. *Suppose that an odd mapping* $f : X \to Y$ *satisfies*

$$\| D_f(x, y) \| \le \phi(x, y) \tag{3.69}$$

for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0, \infty)$ is a function such that

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \phi\left(2^i x, 2^{i-1} x\right) + \sum_{i=1}^{\infty} 2^i \phi\left(0, 2^{i-1} x\right) < \infty \tag{3.70}$$

and that $\lim_{n\to\infty}(1/2^n)\phi(2^n x, 2^n y) = 0$ for all $x, y \in X$, then there exist a unique additive mapping $A : X \to Y$, and a unique cubic mapping $C : X \to Y$ such that

$$\|f(x) - A(x) - C(x)\| \leq \frac{1}{330} \sum_{i=1}^{\infty} \left(\frac{1}{2^i} + \frac{1}{8^i}\right) \left(\phi\left(2^i x, 2^{i-1} x\right)\right) + \frac{14}{495} \sum_{i=1}^{\infty} \left(\frac{1}{2^i} + \frac{1}{8^i}\right) \left(\phi\left(0, 2^{i-1} x\right)\right) \tag{3.71}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 3.4. □

Theorem 3.10. *Suppose that $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\|D_f(x, y)\| \leq \phi(x, y) \tag{3.72}$$

for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0, \infty)$ is a function such that

$$\sum_{i=1}^{\infty} \left\{ \frac{1}{2^i} \left[\phi\left(2^i x, 2^{i-1} x\right) + \phi\left(0, 2^{i-1} x\right)\right] + \frac{1}{16^i} \phi\left(2^i x, 2^i x\right) \right\} < \infty \tag{3.73}$$

and that $\lim_{n\to\infty}(1/2^n)\phi(2^n x, 2^n y) = 0$ for all $x, y \in X$, then there exist a unique additive mapping $A : X \to Y$, a unique cubic mapping $C : X \to Y$, and a unique quartic mapping $Q : X \to Y$ such that

$$\begin{aligned} \|f(x) - A(x) - C(x) - Q(x)\| \leq{}& \frac{1}{66} \left[\sum_{i=1}^{\infty} \left(\frac{1}{2^i} + \frac{1}{8^i}\right) \left(\phi\left(2^i x, 2^{i-1} x\right) + \frac{14}{3} \phi\left(0, 2^{i-1} x\right)\right) \right] \\ &+ \frac{1}{8} \sum_{i=0}^{\infty} \frac{1}{16^i} \left[\frac{6}{11} \phi\left(2^i x, 2^i x\right) + \phi\left(0, 2^i x\right)\right] \end{aligned} \tag{3.74}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 3.5. □

Corollary 3.11. *Let $0 < p < 1$ and let θ be a positive real number. Suppose that $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\|D_f(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{3.75}$$

for all $x, y \in X$. *Then there exist a unique additive mapping* $A : X \to Y$, *a unique cubic mapping* $C : X \to Y$, *and a unique quartic mapping* $Q : X \to Y$ *satisfying*

$$\|f(x) - A(x) - C(x) - Q(x)\| \\ \leq \frac{\theta\|x\|^p}{22}\left\{\frac{1}{3}\left(1 + \frac{17}{3 \times 2^p}\right)\left(\frac{1}{1 - 2^{p-1}} + \frac{1}{1 - 2^{p-3}} - 2\right) + \frac{23}{4(1 - 2^{p-4})}\right\} \tag{3.76}$$

for all $x \in X$.

Corollary 3.12. *Let* ϵ *be a positive real number. Suppose that a mapping* $f : X \to Y$ *satisfies* $f(0) = 0$ *and* $\|D_f(x, y)\| \leq \epsilon$ *for all* $x, y \in X$. *Then there exist a unique additive mapping* $A : X \to Y$, *a unique cubic mapping* $C : X \to Y$, *and a unique quartic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - A(x) - C(x) - Q(x)\| \leq \frac{34782}{114345}\epsilon \tag{3.77}$$

for all $x \in X$.

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