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Research Article

Existence and Uniqueness of Mild Solution for Fractional Integrodifferential Equations

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We study the existence and uniqueness of mild solution of a class of nonlinear fractional integrodifferential equations $d^q u(t)/dt^q + Au(t) = f(t, u(t)) + \int_0^t a(t-s)g(s, u(s))ds$, $t \in [0, T]$, $u(0) = u_0$, in a Banach space X, where $0 < q < 1$. New results are obtained by fixed point theorem. An application of the abstract results is also given.

1. Introduction

An integrodifferential equation is an equation which involves both integrals and derivatives of an unknown function. It arises in many fields like electronic, fluid dynamics, biological models, and chemical kinetics. A well-known example is the equations of basic electric circuit analysis. In recent years, the theory of various integrodifferential equations in Banach spaces has been studied deeply due to their important values in sciences and technologies, and many significant results have been established (see, e.g., [1–11] and references therein).

On the other hand, many phenomena in Engineering, Physics, Economy, Chemistry, Aerodynamics, and Electrodynamics of complex medium can be modeled by fractional differential equations. During the past decades, such problem attracted many researchers (see [1, 12–21] and references therein).

However, among the previous researches on the fractional differential equations, few are concerned with mild solutions of the fractional integrodifferential equations as follows:

$$\frac{d^q u(t)}{dt^q} + Au(t) = f(t, u(t)) + \int_0^t a(t-s)g(s, u(s))ds, \quad t \in [0, T], \qquad u(0) = u_0, \tag{1.1}$$

where $0 < q < 1$, and the fractional derivative is understood in the Caputo sense.

In this paper, motivated by [1–27] (especially the estimating approaches given in [4, 6, 10, 24, 27]), we investigate the existence and uniqueness of mild solution of (1.1) in a Banach space X: $-A$ generates a compact semigroup $S(\cdot)$ of uniformly bounded linear operators on a Banach space X. The function $a(\cdot)$ is real valued and locally integrable on $[0,\infty)$, and the nonlinear maps f and g are defined on $[0,T]\times X$ into X. New existence and uniqueness results are given. An example is given to show an application of the abstract results.

2. Preliminaries

In this paper, we set $I=[0,T]$, a compact interval in $\mathbb{R}$. We denote by X a Banach space with norm $\|\cdot\|$. Let $-A: D(A)\to X$ be the infinitesimal generator of a compact semigroup $S(\cdot)$ of uniformly bounded linear operators. Then there exists $M\geq 1$ such that $\|S(t)\|\leq M$ for $t\geq 0$.

According to [22, 23], a mild solution of (1.1) can be defined as follows.

Definition 2.1. A continuous function $u: I\to X$ satisfying the equation

$$u(t)=Q(t)u_0+\int_0^t R(t-s)\left[f(s,u(s))+K(u)(s)\right]ds \tag{2.1}$$

for $t\in I$ is called a mild solution of (1.1), where

$$\begin{aligned} Q(t)&=\int_0^\infty \xi_q(\sigma)S(t^q\sigma)d\sigma,\\ R(t)&=q\int_0^\infty \sigma t^{q-1}\xi_q(\sigma)S(t^q\sigma)d\sigma,\\ K(u)(t)&=\int_0^t a(t-s)g(s,u(s))ds, \end{aligned} \tag{2.2}$$

and ξ_q is a probability density function defined on $(0,\infty)$ such that its Laplace transform is given by

$$\int_0^\infty e^{-\sigma x}\xi_q(\sigma)d\sigma=\sum_{j=0}^{\infty}\frac{(-x)^j}{\Gamma(1+qj)},\quad 0<q\leq 1,\ x>0. \tag{2.3}$$

Remark 2.2. Noting that $\int_0^\infty \sigma\xi_q(\sigma)d\sigma=1$ (cf., [23]), we can see that

$$\|R(t)\|\leq qMt^{q-1},\quad t>0. \tag{2.4}$$

In this paper, we use $\|f\|_p$ to denote the L^p norm of f whenever $f\in L^p(0,T)$ for some p with $1\leq p<\infty$. $C([0,T],X)$ denotes the Banach space of all continuous functions $[0,T]\to X$ endowed with the sup-norm given by $\|u\|_\infty:=\sup_{t\in I}\|u\|$ for $u\in C([0,T],X)$. Set $a_T:=\int_0^T|a(s)|ds$.

The following well-known theorem will be used later.

Theorem 2.3 (Krasnosel'skii). *Let Ω be a closed convex and nonempty subset of a Banach space X. Let A, B be two operators such that*

(i) *$Ax + By \in \Omega$ whenever $x, y \in \Omega$,*

(ii) *A is compact and continuous,*

(iii) *B is a contraction mapping.*

Then there exists $z \in \Omega$ such that $z = Az + Bz$.

3. Main Results

We will require the following assumptions.

(H1) The function $f : [0,T] \times X \to X$ is continuous, and there exists $L > 0$ such that

$$\|f(t,u) - f(t,v)\| \le L\|u - v\|, \quad u, v \in C([0,T], X). \tag{3.1}$$

(H2) The function $L_q : I \to \mathbb{R}^+$, $0 < q < 1$, satisfies

$$L_q(t) = Mt^q \cdot (L + L\ a_T) \le \omega < 1, \quad t \in [0,T]. \tag{3.2}$$

Theorem 3.1. *Let $-A$ be the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\ge 0}$ with $\|S(t)\| \le M$, $t \ge 0$. If the maps f and g satisfy (H1), $L_q(t)$ satisfies (H2), and*

$$L \le \gamma\left[M \cdot T^q \cdot (1 + a_T)\right]^{-1}, \quad 0 < \gamma < 1, \tag{3.3}$$

then (1.1) *has a unique mild solution for every $u_0 \in X$.*

Proof. Define the mapping $\mathcal{F} : C([0,T], X) \to C([0,T], X)$ by

$$(\mathcal{F}u)(t) = Q(t)u_0 + \int_0^t R(t-s)\left[f(s, u(s)) + K(u)(s)\right]ds. \tag{3.4}$$

Set $\sup_{t\in[0,T]}\|f(t,0)\| = M_1$, $\sup_{t\in[0,T]}\|g(t,0)\| = M_2$.

Choose r such that

$$r \ge \frac{M}{1-\gamma}\left[T^q(M_1 + M_2\ a_T) + \|u_0\|\right]. \tag{3.5}$$

Let B_r be the nonempty closed and convex set given by

$$B_r = \{u \in C([0,T], X) \mid \|u\|_\infty \le r\}. \tag{3.6}$$

Then for $u \in B_r$, we have

$$\begin{aligned}
\|(\mathcal{F}u)(t)\| &\leq \|Q(t)u_0\| + \int_0^t \|R(t-s)\| \cdot \|f(s,u(s)) + K(u)(s)\| ds \\
&\leq M\|u_0\| + qM\int_0^t (t-s)^{q-1}\left[\|f(s,u(s))\| + \|K(u)(s)\|\right] ds \\
&\leq M\|u_0\| + qM\int_0^t (t-s)^{q-1}\left[\|f(s,u(s)) - f(s,0)\| + \|f(s,0)\|\right] ds \\
&\quad + qM\int_0^t (t-s)^{q-1}\|K(u)(s)\| ds.
\end{aligned} \tag{3.7}$$

Noting that

$$\begin{aligned}
\|K(u)(s)\| &= \left\|\int_0^s a(s-\tau)g(\tau,u(\tau))d\tau\right\| \\
&\leq \int_0^s |a(s-\tau)| \cdot \left[\|g(\tau,u(\tau)) - g(\tau,0)\| + \|g(\tau,0)\|\right] d\tau \\
&\leq (Lr + M_2)a_T,
\end{aligned} \tag{3.8}$$

we obtain

$$\|(\mathcal{F}u)(t)\| \leq M\|u_0\| + MT^q[(Lr + M_1) + (Lr + M_2)a_T] \leq r, \tag{3.9}$$

for $t \in [0,T]$. Hence $\mathcal{F} : B_r \to B_r$.

Let u and v be two elements in $C([0,T],X)$. Then

$$\begin{aligned}
&\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| \\
&\quad\leq qM\int_0^t (t-s)^{q-1}\big\|f(s,u(s)) - f(s,v(s)) + K(u)(s) - K(v)(s)\big\| ds \\
&\quad\leq qM\int_0^t (t-s)^{q-1}\left[\|f(s,u(s)) - f(s,v(s))\| + \int_0^s |a(s-\tau)|\|g(\tau,u(\tau)) - g(\tau,v(\tau))\|d\tau\right] ds \\
&\quad\leq Mt^q \cdot (L + La_T)\|u-v\| \\
&\quad= L_q(t)\|u-v\|.
\end{aligned} \tag{3.10}$$

So

$$\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\|_\infty \leq L_q(T)\|u-v\|_\infty. \tag{3.11}$$

The conclusion follows by the contraction mapping principle. □

We assume the following.

(H3) The function $f : I \times X \to X$ is continuous, and there exists a positive function $\mu(\cdot) \in L^p_{\mathrm{loc}}(I, \mathbb{R}^+)$ $(p > 1/q > 1)$ such that

$$\|f(t,u(t))\| \leq \mu(t), \quad \text{the function } s \longmapsto \frac{\mu(s)}{(t-s)^{1-q}} \text{ belongs to } L^1([0,t],\mathbb{R}^+), \tag{3.12}$$

and set $T_{p,q} := \max\{T^{q-1/p}, T^q\}$.

Let $-A$ be the infinitesimal generator of a compact semigroup $S(\cdot)$ of uniformly bounded linear operators. Then there exists a constant $M \geq 1$ such that $\|S(t)\| \leq M$ for $t \geq 0$.

Theorem 3.2. *If the maps g and f satisfy (H1), (H3), respectively, and*

$$L \leq \lambda \left(M \cdot T_{p,q} \cdot a_T\right)^{-1}, \quad 0 < \lambda < 1, \tag{3.13}$$

then (1.1) *has a mild solution for every $u_0 \in X$.*

Proof. Define

$$\begin{gathered}
(\Phi u)(t) := \int_0^t R(t-s) f(s,u(s))ds, \\
(\Psi u)(t) := Q(t)u_0 + \int_0^t R(t-s)K(u)(s)ds.
\end{gathered} \tag{3.14}$$

Choose r such that

$$r \geq \frac{M}{1-\lambda}\left[T_{p,q}\left(q \cdot M_{p,q}\|\mu\|_{L^p_{\mathrm{loc}}(I,\mathbb{R}^+)} + a_T M_2\right) + \|u_0\|\right], \tag{3.15}$$

where $M_{p,q} := ((p-1)/(pq-1))^{(p-1)/p}$.

Let $B_r = \{u \in C([0,T],X) \mid \|u\|_\infty \leq r\}$ be the closed convex and nonempty subset of the space $C([0,T],X)$.

Letting $u, v \in B_r$, we have

$$\begin{aligned}
\|(\Phi v)(t) + (\Psi u)(t)\| &\leq \int_0^t \|R(t-s)f(s,v(s))\|ds + \|Q(t)u_0\| \\
&\quad + \int_0^t \|R(t-s)K(u)(s)\|ds \\
&\leq M\|u_0\| + qM\int_0^t (t-s)^{q-1}\|f(s,v(s))\|ds \\
&\quad + qM\int_0^t (t-s)^{q-1}\|K(u)(s)\|ds.
\end{aligned} \tag{3.16}$$

Set $\sup_{t\in[0,T]}\|g(t,0)\| = M_2$.

According to the Hölder inequality, (H1), and (3.8), for $t \in [0,T]$, we have

$$
\begin{aligned}
\|(\Phi v)(t) + (\Psi u)(t)\| &\le M\|u_0\| + qM\int_0^t (t-s)^{q-1}\|f(s,v(s))\|ds \\
&\quad + qM\int_0^t (t-s)^{q-1}\|K(u)(s)\|ds \\
&\le M\|u_0\| + MT_{p,q}\Big[qM_{p,q}\|\mu\|_{L^p_{\mathrm{loc}}(I,\mathbb{R}^+)} + (Lr+M_2)a_T\Big] \\
&\le r.
\end{aligned} \tag{3.17}
$$

Thus, $(\Phi v) + (\Psi u) \in B_r$.

For $u, v \in C([0,T],X)$ and $t \in [0,T]$, using (H1), we obtain

$$
\begin{aligned}
\|(\Psi u)(t) - (\Psi v)(t)\| &\le qM\int_0^t (t-s)^{q-1}\|K(u)(s) - K(v)(s)\|ds \\
&\le qM\int_0^t (t-s)^{q-1}\cdot\left\|\int_0^s a(s-\tau)\big[g(\tau,u(\tau)) - g(\tau,v(\tau))\big]d\tau\right\|ds \\
&\le MT^q\cdot a_T\cdot L\|u-v\|_\infty \\
&\le \lambda\|u-v\|_\infty.
\end{aligned} \tag{3.18}
$$

So, we know that Ψ is a contraction mapping.

Set $U(t) = \{(\Phi u)(t) \mid u \in B_r\}$.

Fix $t \in (0,T]$. For $0 < \varepsilon < t$, set

$$
\begin{aligned}
(\Phi_\varepsilon u)(t) &= \int_0^{t-\varepsilon} R(t-s)f(s,u(s))ds \\
&= qS(\varepsilon^q\sigma)\int_0^{t-\varepsilon}(t-s)^{q-1}f(s,u(s))\int_0^\infty \sigma\xi_q(\sigma)S\big((t-s)^q\sigma - \varepsilon^q\sigma\big)d\sigma\, ds.
\end{aligned} \tag{3.19}
$$

Since $S(t)$ is compact for each $t \in (0,T]$, the sets $U_\varepsilon(t) = \{(\Phi_\varepsilon u)(t) \mid u \in B_r\}$ are relatively compact in X for each ε, $0 < \varepsilon < t$. Furthermore,

$$
\begin{aligned}
\|(\Phi u)(t) - (\Phi_\varepsilon u)(t)\| &\le qM\int_{t-\varepsilon}^t (t-s)^{q-1}\|f(s,u(s))\|ds \\
&\le qM\cdot M_{p,q}\cdot\|\mu\|_{L^p_{\mathrm{loc}}(I,\mathbb{R}^+)}\cdot\varepsilon^{q-1/p},
\end{aligned} \tag{3.20}
$$

which implies that $U(t)$ is relatively compact in X.

Next, we prove that $(\Phi u)(t)$ is equicontinuous.

For $0 < t_2 < t_1 < T$, we have

$$
\begin{aligned}
&\|(\Phi u)(t_1) - (\Phi u)(t_2)\| \\
&\quad = \left\| \int_0^{t_1} R(t_1 - s) f(s, u(s)) ds - \int_0^{t_2} R(t_2 - s) f(s, u(s)) ds \right\| \\
&\quad = \left\| \int_0^{t_2} [R(t_1 - s) - R(t_2 - s)] f(s, u(s)) ds + \int_{t_2}^{t_1} R(t_1 - s) f(s, u(s)) ds \right\| \\
&\quad \le q \left\| \int_0^{t_2} \int_0^{\infty} \sigma \left[(t_1 - s)^{q-1} - (t_2 - s)^{q-1} \right] \xi_q(\sigma) S\big((t_1 - s)^q \sigma\big) f(s, u(s)) d\sigma\, ds \right\| \\
&\qquad + \int_{t_2}^{t_1} \|R(t_1 - s)\| \|f(s, u(s))\| ds \\
&\qquad + q \left\| \int_0^{t_2} \int_0^{\infty} \sigma (t_2 - s)^{q-1} \xi_q(\sigma) \left[S\big((t_1 - s)^q \sigma\big) - S\big((t_2 - s)^q \sigma\big) \right] f(s, u(s)) d\sigma\, ds \right\| \\
&\quad = I_1 + I_2 + I_3.
\end{aligned}
\tag{3.21}
$$

By (H3), we get

$$
\begin{aligned}
I_1 &\le qM \int_0^{t_2} \left| (t_1 - s)^{q-1} - (t_2 - s)^{q-1} \right| \|f(s, u(s))\| ds \\
&\le qM \int_0^{t_2} \left| (t_1 - s)^{q-1} - (t_2 - s)^{q-1} \right| \mu(s) ds.
\end{aligned}
\tag{3.22}
$$

In view of the assumption of $\mu(s)$, we see that I_1 tends to 0 as $t_2 \to t_1$, and one

$$
I_2 \le qM \int_{t_2}^{t_1} (t_1 - s)^{q-1} \|f(s, u(s))\| ds \le qM \int_{t_2}^{t_1} (t_1 - s)^{q-1} \mu(s) ds. \tag{3.23}
$$

Clearly, the last term tends to 0 as $t_2 \to t_1$. Hence $I_2 \to 0$ as $t_2 \to t_1$, and

$$
\begin{aligned}
I_3 &= q \left\| \int_0^{t_2} \int_0^{\infty} \sigma (t_2 - s)^{q-1} \xi_q(\sigma) \left[S\big((t_1 - s)^q \sigma\big) - S\big((t_2 - s)^q \sigma\big) \right] f(s, u(s)) d\sigma\, ds \right\| \\
&\le q \int_0^{t_2} (t_2 - s)^{q-1} \mu(s) \int_0^{\infty} \sigma \xi_q(\sigma) \left\| S\big((t_1 - s)^q \sigma\big) - S\big((t_2 - s)^q \sigma\big) \right\| d\sigma\, ds.
\end{aligned}
\tag{3.24}
$$

The right-hand side of (3.24) tends to 0 as $t_2 \to t_1$ as a consequence of the continuity of $S(t)$ in the uniform operator topology for $t > 0$ by the compactness of $S(t)$. So $I_3 \to 0$ as $t_2 \to t_1$. Thus, $\|(\Phi u)(t_1) - (\Phi u)(t_2)\| \to 0$, as $t_2 \to t_1$, which is independent of u. Therefore Φ is compact by the Arzela-Ascoli theorem.

Next we show that Φ is continuous.

Let $\{u_n\}$ be a sequence of B_r such that $u_n \to u$ in B_r. By the continuity of f on $I \times X$, we have

$$f(s, u_n(s)) \longrightarrow f(s, u(s)), \quad n \longrightarrow \infty. \tag{3.25}$$

Noting the continuity of f, we get

$$\begin{aligned} \|(\Phi u_n)(t) - (\Phi u)(t)\| &= \left\| \int_0^t R(t-s)\left[f(s, u_n(s)) - f(s, u(s))\right] ds \right\| \\ &\le qM \int_0^t (t-s)^{q-1} \|f(s, u_n(s)) - f(s, u(s))\| ds \\ &\le MT^q \|f(\cdot, u_n(\cdot)) - f(\cdot, u(\cdot))\|_\infty \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.26}$$

Thus, we have

$$\lim_{n \to \infty} \|\Phi u_n - \Phi u\|_\infty = 0. \tag{3.27}$$

So Φ is continuous.

By Krasnosel'skii's theorem, we have the conclusion of the theorem. □

Remark 3.3. In Theorem 3.2, if we furthermore suppose that the hypothesis

(H4)

$$\|f(t, u(t)) - f(t, v(t))\| \le L' \|u - v\|, \quad L' > 0, \tag{3.28}$$

holds, then we can obtain the uniqueness of the mild solution in Theorem 3.2.

Actually, from what we have just proved, (1.1) has a mild solution $u(t)$ and

$$u(t) = Q(t)u_0 + \int_0^t R(t-s)\left[f(s, u(s)) + K(u)(s)\right] ds. \tag{3.29}$$

Let $v(t)$ be another mild solution of (1.1). Then

$$\begin{aligned} \|u(t) - v(t)\| &\le \int_0^t \|R(t-s)\| \left(\|f(s, u(s)) - f(s, v(s))\| + \|K(u)(s) - K(v)(s)\|\right) ds \\ &\le qM \int_0^t (t-s)^{q-1} \left(La_T + L'\right) \|u(s) - v(s)\| ds, \end{aligned} \tag{3.30}$$

which implies by Gronwall's inequality that (1.1) has a unique mild solution $u(t)$.

Example 3.4. Let $X = L^2([0,1], \|\cdot\|_2)$. Define

$$D(A) = H^2(0,1) \cap H_0^1(0,1),$$
$$Au = -u''. \tag{3.31}$$

Then $-A$ generates a compact, analytic semigroup $S(\cdot)$ of uniformly bounded linear operators.

Let $(t,s) \in [0,T] \times [0,1]$, $\xi \in X$, and let C, r_0 be positive constants. We set

$$g(t,\xi)(s) = C \sin|\xi(s)|,$$
$$f(t,\xi)(s) = \frac{1}{\sqrt{t+r_0}} \frac{|\xi(s)|}{1+|\xi(s)|}, \tag{3.32}$$
$$a(t) = t,$$

$q = 1/2$, and $p = 3$.

It is not hard to see that g and f satisfy (H1), (H3), respectively, and if

$$\frac{C \cdot T_{p,q} \cdot T^2}{2} \le \lambda < 1, \tag{3.33}$$

then (1.1) has a unique mild solution by Theorem 3.2 and Remark 3.3.

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