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Research Article

Positive Solutions for Impulsive Equations of Third Order in Banach Space

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Using the fixed-point theorem, this paper is devoted to study the multiple and single positive solutions of third-order boundary value problems for impulsive differential equations in ordered Banach spaces. The arguments are based on a specially constructed cone. At last, an example is given to illustrate the main results.

1. Introduction

The purpose of this paper is to establish the existence of positive solutions for the following third-order three-point boundary value problems (BVP, for short) in Banach space E

$$\begin{gathered}
-x'''(t) = \lambda f_1(t, x(t), y(t)), \quad t \in [0,1] \setminus \{t_1, t_2, \ldots, t_m\}, \\
-y'''(t) = \mu f_2(t, x(t), y(t)), \quad t \in [0,1] \setminus \{t_1, t_2, \ldots, t_m\}, \\
\Delta x''(t_k) = -I_{1,k}(x(t_k)), \quad \Delta y''(t_k) = -I_{2,k}(y(t_k)), \quad k = 1, 2, \ldots, m, \\
x(0) = x'(0) = \theta, \quad x'(1) - \alpha x'(\eta) = \theta, \quad y(0) = y'(0) = \theta, \quad y'(1) - \alpha y'(\eta) = \theta,
\end{gathered} \tag{1.1}$$

where $f_i \in C([0,1] \times P \times P, P)$, $I_{i,k} \in C(P, P)$, $i = 1, 2$, $k = 1, 2, \ldots, m$. $\Delta x''(t_k) = x''(t_k^+) - x''(t_k^-)$, $\Delta y''(t_k) = y''(t_k^+) - y''(t_k^-)$, $\mu > 0$, $\lambda > 0$. θ is the zero element of E.

Recently, third-order boundary value problems (cf. [1–9]) have attracted many authors attention due to their wide range of applications in applied mathematics, physics, and engineering, especially in the bridge issue. To our knowledge, most papers in literature concern mainly about the existence of positive solutions for the cases in which the spaces are real and the equations have no parameters. And many authors consider nonlinear term have same linearity. In this paper, we consider the existence of solutions when the nonlinear terms have different properties, the space is abstract and the equations have two different parameters.

In [3], Guo et al. studied the following nonlinear three-point boundary value problem:

$$\begin{gathered} u'''(t) + a(t)f(u(t)) = 0, \\ u(0) = u'(0) = 0, \qquad u'(1) = \alpha u'(\eta), \end{gathered} \tag{1.2}$$

where $a \in C([0,1],[0,+\infty))$, $f \in C([0,+\infty),[0,+\infty))$. The authors obtained at least one positive solutions of BVP (1.2) by using fixed-point theorem when f is sublinear or suplinear.

In [8], Yao and Feng used the upper and lower solutions method proved some existence results for the following third-order two-point boundary value problem

$$\begin{gathered} u'''(t) + f(t,u(t)) = 0, \quad 0 \le t \le 1, \\ u(0) = u'(0) = u'(1) = 0. \end{gathered} \tag{1.3}$$

Inspired by the above work, the aim of this paper is to establish some simple criteria for the existence of nontrivial solutions for BVP (1.1) under some weaker conditions. The new features of this paper mainly include the following aspects. Firstly, we consider the system (1.1) in abstract space while [3, 8] talk about equations in real space ($E = R$). Secondly, we obtained the positive solutions when the two parameters have different ranges. Thirdly, f_1 and f_2 in system (1.1) may have different properties. Fourthly, f_i $(i = 1,2)$ in system (1.1) not only contains x, y but also t, which is much more complicated. Finally, the main technique used here is the fixed-point theory and a special cone is constructed to study the existence of nontrivial solutions.

We recall some basic facts about ordered Banach spaces E. The cone P in E induces a partial order on E, that is, $x \le y$ if and only if $y - x \in P$, P is said to be normal if there exists a positive constant N such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$, without loss of generality, suppose, in present paper, the normal constant $N = 1$. $\alpha(\cdot)$ denotes the measure of noncompactness (cf. [10]).

Some preliminaries and a number of lemmas to the derivation of the main results are given in Section 2, then the proofs of the theorems are given in Section 3, followed by an example, in Section 4, to demonstrate the validity of our main results.

2. Preliminaries and Lemmas

In this paper we will consider the Banach space $(E, \|\cdot\|)$, denote $J = [0,1]$ and $\mathrm{PC}^2(J,E) = \{x \mid x' \in C(J,E),\ x''$ is continuous at $t \neq t_k$ and x'' is left continuous at $t = t_k$, the right limit $x''(t_k^+)$ exists, $k = 1,2,\ldots,m\}$. For any $x \in \mathrm{PC}^2(J,E)$ we define $\|x\|_1 = \sup_{t\in J}\|x(t)\|$ and $\|(x,y)\|_2 = \|x\|_1 + \|y\|_1$ for $(x,y) \in \mathrm{PC}^2(J,E) \times \mathrm{PC}^2(J,E)$.

For convenience, let us list the following assumption.

(A) $f_i \in C([0,1] \times P \times P, P)$, $I_{i,k} \in C(P,P)$, $i = 1,2$, $k = 1,2,\ldots,m$. For any $t \in [0,1]$ and $r > 0$, $f(t, P_r, P_r) = \{f(t,u,v) : u,v \in P_r\}$ is relatively compact in E, where $P_r = \{x \in P \mid \|x\| \leq r\}$.

Lemma 2.1. *Assume that $\alpha\eta \neq 1$, then for any $y \in C[0,1]$, the following boundary value problem:*

$$\begin{gathered} -u'''(t) = y(t), \quad t \in [0,1] \setminus \{t_1, t_2, \ldots, t_m\}, \\ \Delta u''(t_k) = -I_k(u(t_k)), \quad k = 1,2,\ldots,m, \\ u(0) = u'(0) = \theta, \qquad u'(1) - \alpha u'(\eta) = \theta \end{gathered} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) y(s) ds + \sum_{k=1}^{m} G(t,t_k) I_k(u(t_k)), \tag{2.2}$$

where

$$G(t,s) = \frac{1}{2(1-\alpha\eta)} \begin{cases} (2ts - s^2)(1-\alpha\eta) + t^2 s(\alpha - 1), & s \leq \min\{\eta, t\}, \\ t^2(1-\alpha\eta) + t^2 s(\alpha - 1), & t \leq s \leq \eta, \\ (2ts - s^2)(1-\alpha\eta) + t^2(\alpha\eta - s), & \eta \leq s \leq t, \\ t^2(1-s), & \max\{\eta, t\} \leq s. \end{cases} \tag{2.3}$$

Proof. The proof is similar to Lemma 2.2 in [3], we omit it. □

Lemma 2.2 (see [3]). *Assume that $0 < \eta < 1$ and $1 < \alpha < 1/\eta$. Then $0 \leq G(t,s) \leq g(s)$ for any $(t,s) \in [0,1] \times [0,1]$, where $g(s) = ((1+\alpha)/(1-\alpha\eta)) s(1-s)$, $s \in [0,1]$.*

Lemma 2.3 (see [3]). *Let $0 < \eta < 1$ and $1 < \alpha < 1/\eta$, then for any $(t,s) \in [\eta/\alpha, \eta] \times [0,1]$, $G(t,s) \geq \sigma g(s)$, where*

$$0 < \sigma = \frac{\eta^2}{2\alpha^2(1+\alpha)} \min\{\alpha - 1, 1\} < 1. \tag{2.4}$$

In the paper, we define cone K as follows:

$$K = \left\{ x \in \mathrm{PC}^2(J,E) \mid x(t) \geq \theta,\ x(t) \geq \sigma x(s),\ t \in \left[\frac{\eta}{\alpha}, \eta\right],\ s \in [0,1] \right\}. \tag{2.5}$$

Lemma 2.4 (see [10]). *Let E be a Banach space and $K \subset E$ be a cone. Suppose Ω_1 and $\Omega_2 \in E$ are bounded open sets, $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow K$ is completely continuous such that* *either*

(i) $\|Au\| \leq \|u\|$ *for any* $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|$ *for any* $u \in K \cap \partial\Omega_2$ *or*

(ii) $\|Au\| \geq \|u\|$ *for any* $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|$ *for any* $u \in K \cap \partial\Omega_2$.

Then A *has a fixed-point in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.5. *The vector* $(x, y) \in PC^2(J, E) \times PC^2(J, E)$ *is a solution of differential systems* (1.1) *if and only if* $(x, y) \in PC^2(J, E)$ *is the solution of the following integral systems:*

$$\begin{aligned} x(t) &= \lambda \int_0^1 G(t,s) f_1(s, x(s), y(s)) ds + \sum_{k=1}^{m} G(t, t_k) I_{1,k}(x(t_k)), \\ y(t) &= \mu \int_0^1 G(t,s) f_2(s, x(s), y(s)) ds + \sum_{k=1}^{m} G(t, t_k) I_{2,k}(y(t_k)). \end{aligned} \tag{2.6}$$

Define operators $T_1 : K \to K$, $T_2 : K \to K$ *and* $T : K \times K \to K \times K$ *as follows:*

$$\begin{aligned} T_1(x, y) &= \lambda \int_0^1 G(t,s) f_1(s, x(s), y(s)) ds + \sum_{k=1}^{m} G(t, t_k) I_{1,k}(x(t_k)), \\ T_2(x, y) &= \mu \int_0^1 G(t,s) f_2(s, x(s), y(s)) ds + \sum_{k=1}^{m} G(t, t_k) I_{2,k}(y(t_k)), \\ T(x, y)(t) &= (T_1(x, y), T_2(x, y))(t). \end{aligned} \tag{2.7}$$

As we know, BVP (1.1) *has a positive solution* (x, y) *if and only if* $(x, y) \in K \times K$ *is the fixed-point of* T.

Lemma 2.6. $T : K \times K \to K \times K$ *is completely continuous.*

Proof. By condition (A) we get $T_1(x, y)(t) \geq \theta$, $T_2(x, y)(t) \geq \theta$, for all $x, y \in K$. For any $t \in [\eta/\alpha, \eta]$, we have

$$\begin{aligned} T_1(x, y)(t) &= \int_0^1 G(t,s) f_1(s, x(s), y(s)) ds + \sum_{k=1}^{m} G(t, t_k) I_{1,k}(x(t_k)) \\ &\geq \sigma \int_0^1 g(s) f_1(s, x(s), y(s)) ds + \sigma \sum_{k=1}^{m} g(t_k) I_{1,k}(x(t_k)) \\ &\geq \sigma \int_0^1 G(u,s) f_1(s, x(s), y(s)) ds + \sigma \sum_{k=1}^{m} G(u, t_k) I_{1,k}(x(t_k)) \\ &= \sigma T_1(x, y)(u), \quad u \in [0, 1]. \end{aligned} \tag{2.8}$$

Similarly

$$T_2(x,y)(t) \geq \sigma T_2(x,y)(u), \quad u \in [0,1]. \tag{2.9}$$

So $T : K \times K \to K \times K$.

Next, we prove $T : K \times K \to K \times K$ is completely continuous. We first prove that T_1 is continuous. Let $(x_n, y_n) \in K(n = 1,2,\ldots)$ and $(x_0, y_0) \in K$ such that $\|(x_n, y_n) - (x_0, y_0)\|_2 \to 0$ $(n \to \infty)$. Let $r = \sup_n \|(x_n, y_n)\|_2$, then

$$\left\|(x_0,y_0)\right\|_2 \leq r, \quad \|x_0\|_1 \leq r, \quad \left\|y_0\right\|_1 \leq r, \quad \|x_n\|_1 \leq r, \quad \left\|y_n\right\|_1 \leq r. \tag{2.10}$$

By (A), we obtain

$$\begin{gathered} f_i(t, x_n(t), y_n(t)) \longrightarrow f_i(t, x_0(t), y_0(t)), \quad (n \longrightarrow \infty), \quad \text{for any } t \in [0,1],\ i = 1,2, \\ I_{1,k}(x_n(t_k)) \longrightarrow I_{1,k}(x_0(t_k)), \quad (n \longrightarrow \infty), \quad k = 1,2,\ldots,m, \\ I_{2,k}(y_n(t_k)) \longrightarrow I_{2,k}(y_0(t_k)), \quad (n \longrightarrow \infty), \quad k = 1,2,\ldots,m. \end{gathered} \tag{2.11}$$

Hence

$$\begin{aligned} &\left\|T_1(x_n,y_n)(t) - T_1(x_0,y_0)(t)\right\| \\ &\quad = \left\| \int_0^1 G(t,s) f_1(s, x_n(s), y_n(s))\,ds + \sum_{k=1}^{m} G(t,t_k) I_{1,k}(x_n(t_k)) \right. \\ &\qquad \left. - \int_0^1 G(t,s) f_1(s, x_0(s), y_0(s))\,ds - \sum_{k=1}^{m} G(t,t_k) I_{1,k}(x_0(t_k)) \right\| \\ &\quad \leq \int_0^1 G(t,s) \left\| f_1(s, x_n(s), y_n(s)) - f_1(s, x_0(s), y_0(s)) \right\| ds \\ &\qquad + \sum_{k=1}^{m} G(t,t_k) \| I_{1,k}(x_n(t_k)) - I_{1,k}(x_0(t_k)) \| \\ &\quad \leq \int_0^1 g(s) \left\| f_1(s, x_n(s), y_n(s)) - f_1(s, x_0(s), y_0(s)) \right\| ds \\ &\qquad + \sum_{k=1}^{m} g(t_k) \| I_{1,k}(x_n(t_k)) - I_{1,k}(x_0(t_k)) \|. \end{aligned} \tag{2.12}$$

Since

$$\left\|T_1(x_n,y_n) - T_1(x_0,y_0)\right\|_1 = \sup_{t\in[0,1]} \left\|T_1(x_n,y_n)(t) - T_1(x_0,y_0)(t)\right\|. \tag{2.13}$$

By (2.11)–(2.13) and Lebesgue-dominated convergence theorem

$$\left\|T_1(x_n,y_n)-T_1(x_0,y_0)\right\|_1 \longrightarrow 0 \quad (n\longrightarrow\infty). \tag{2.14}$$

So T_1 is continuous. Similarly, T_2 is continuous. It follows that T is continuous.

Next we prove T is compact. Let $V=\{(x_n,y_n)\}\subset K\times K$ be bounded, $V_1=\{x_n\}$ and $V_2=\{y_n\}$. Let $\|(x_n,y_n)\|_2\le r$ for some $r>0$, then $\|x_n\|_1\le r$, $\|y_n\|_1\le r$. It is easy to see that $\{T_1(x_n,y_n)(t)\}$ is equicontinuous. By condition (A) we have

$$\begin{aligned}\alpha((T_1V)(t)) &= \alpha\left\{\int_0^1 G(t,s)f_1(s,x_n(s),y_n(s))ds+\sum_{k=1}^m G(t,t_k)I_{1,k}(x_n(t_k)) : x_n\in V_1,\ y_n\in V_2\right\}\\ &\le 2\int_0^1 \alpha\big(G(t,s)f_1(s,V_1(s),V_2(s))\big)+\sum_{k=1}^m \alpha(G(t,t_k))I_{1,k}(V_1(t_k))\\ &=0\end{aligned} \tag{2.15}$$

which implies that $\alpha(T_1V)=0$. So, $\alpha(TV)=0$, it follows that T is compact. The lemma is proved. □

In this paper, denote

$$\begin{gathered} f_i^\beta=\limsup_{\|x\|+\|y\|\to\beta}\max_{t\in[0,1]}\frac{\|f(t,x,y)\|}{\|x\|+\|y\|}, \qquad f_{i,\beta}=\liminf_{\|x\|+\|y\|\to\beta}\min_{t\in[\eta/\alpha,\eta]}\frac{\|f(t,x,y)\|}{\|x\|+\|y\|},\\ (\psi f_i)^\beta=\limsup_{\|x\|+\|y\|\to\beta}\max_{t\in[0,1]}\frac{\psi(f(t,x,y))}{\|x\|+\|y\|}, \qquad (\psi f_i)_\beta=\liminf_{\|x\|+\|y\|\to\beta}\min_{t\in[\eta/\alpha,\eta]}\frac{\psi(f(t,x,y))}{\|x\|+\|y\|}.\\ I_{i,\beta}(k)=\liminf_{\|x\|\to\beta}\frac{\|I_{i,k}(x)\|}{\|x\|},\quad I_i^\beta(k)=\limsup_{\|x\|\to\beta}\frac{\|I_{i,k}(x)\|}{\|x\|},\quad k=1,2,\dots,m.\end{gathered} \tag{2.16}$$

where $\beta=0$ or $\beta=+\infty$, $\psi\in P^*=\{\psi\in E^*:\psi(x)\ge\theta,\ \forall x\in P\}$ and $\|\psi\|=1$. P^* is a dual cone of P.

We list the assumptions:

(H$_1$) $(\psi f_1)_0>m_1$, $\psi(f_2)_\infty>m_2$, where $m_1,m_2\in(0,+\infty)$;

(H$_2$) $f_i^0<m_3$, $(\psi f_1)_\infty>m_4$, $I_{i,0}(k)=0$, $i=1,2$, where $m_3,m_4>0$ and $m_3\ll m_4$;

(H$_3$) $(\psi f_1)_0>m_5$, $f_i^\infty<m_6$, $I_i^\infty(k)=0$, $i=1,2$, where $m_5,m_6>0$ and $m_6\ll m_5$.

For convenience, denote

$$a_1 = \frac{1}{4}\left(m_3\int_0^1 g(s)ds\right)^{-1}, \qquad \alpha_2 = \left(m_4\sigma\int_{\alpha/\eta}^{\eta} G(\eta,s)ds\right)^{-1},$$
$$a_3 = \left(m_5\sigma\int_{\eta/\alpha}^{\eta} G(\eta,s)ds\right)^{-1}, \qquad \alpha_4 = \frac{1}{4}\left(m_6\int_0^1 g(s)ds\right)^{-1}. \tag{2.17}$$

3. Main Results

Theorem 3.1. *Assume that (A), (H_1) and the following condition* $(H)'$ *hold, then BVP* (1.1) *has at least two positive solution while* $\lambda \in (0, 1/(4M_1\int_0^1 g(s)ds))$ *and* $\mu \in (0, 1/(4M_2\int_0^1 g(s)ds))$.

$(H)'$: $m_1\lambda\sigma\int_{\eta/\alpha}^{\eta} G(\eta,s)ds \geq 1$; $m_2\mu\sigma\int_{\eta/\alpha}^{\eta} G(\eta,s)ds \geq 1$; $\sum_{i=1}^{2}\sum_{k=1}^{m} g(t_k)M_{i,k} < 1/2$, *where* $M_i = \max_{t\in[0,1],\, 0\leq\|u\|+\|v\|\leq 1}\|f_i(t,u,v)\| > 0$, $M_{i,k} = \max_{0\leq\|u\|\leq 1}\{\|I_{i,k}(u)\|\}$.

Proof. Let $\Omega_1 = \{(x,y) \in K\times K : \|(x,y)\|_2 < 1\}$, then for $(x,y) \in \partial\Omega_1$, we have

$$\begin{aligned}\|T_1(x,y)(t)\| &\leq \left\|\lambda\int_0^1 g(s)f_1(x(s),y(s))ds\right\| + \left\|\sum_{k=1}^{m} g(t_k)I_{1,k}(x(t_k))\right\| \\ &\leq \lambda M_1\int_0^1 g(s)ds + \sum_{k=1}^{m} g(t_k)M_{1,k},\end{aligned} \tag{3.1}$$

that is,

$$\|T_1(x,y)\|_1 \leq \lambda M_1\int_0^1 g(s)ds + \sum_{k=1}^{m} g(t_k)M_{1,k}, \tag{3.2}$$

Similarly

$$\|T_2(x,y)\|_1 \leq M_2\mu\int_0^1 g(s)ds + \sum_{k=1}^{m} g(t_k)M_{2,k}. \tag{3.3}$$

So

$$\begin{aligned}\|T(x,y)\|_2 &\leq (\lambda M_1 + \mu M_2)\int_0^1 g(s)ds + \sum_{i=1}^{2}\sum_{k=1}^{m} g(t_k)M_{i,k} \\ &< 1 = \|(x,y)\|_2.\end{aligned} \tag{3.4}$$

Hence

$$\|T(x,y)\|_2 < \|(x,y)\|_2, \quad \text{for any } (x,y) \in \partial\Omega_1. \tag{3.5}$$

Since $(\varphi f_1)_0 > m_1$, there exist $\varepsilon_1 > 0$ and $0 < R_1 < 1$ such that $\varphi(f_1(t,u,v)) \geq (m_1 + \varepsilon_1)(\|u\|+\|v\|)$ for $0 \leq \|u\|+\|v\| \leq R_1$ and $t \in [\eta/\alpha, \eta]$. Let $\Omega_2 = \{(x,y) \in K\times K : \|(x,y)\|_2 < R_1\}$. Then for any $(x,y) \in \partial\Omega_2$, by ($H_1$) and the definition of φ, we obtain

$$\begin{aligned}\|T_1(x,y)\|_1 \geq \varphi((T_1(x,y))(\eta)) &\geq \lambda \int_{\eta/\alpha}^{\eta} G(\eta,s)\varphi(f_1(t,x(s),y(s)))ds \\ &\geq (m_1+\varepsilon_1)\lambda\sigma \int_{\eta/\alpha}^{\eta} G(\eta,s)(\|x\|_1 + \|y\|_1)ds \\ &= R_1(m_1+\varepsilon_1)\lambda\sigma \int_{\eta/\alpha}^{\eta} G(\eta,s)ds.\end{aligned} \tag{3.6}$$

By (3.6) and (H)′

$$\begin{aligned}\|T(x,y)\|_2 \geq \|T_1(x,y)\|_1 &\geq R_1(m_1+\varepsilon_1)\lambda\sigma \int_{\eta/\alpha}^{\eta} G(\eta,s)ds \\ &> R_1 = \|(x,y)\|_2, \quad (x,y) \in \partial\Omega_2,\end{aligned} \tag{3.7}$$

Similarly, by $(\varphi f_2)_\infty > m_2$, there exist $\varepsilon_2 > 0$ and $R_2 > 1$ such that $\varphi(f_2(t,u,v)) \geq (m_2+\varepsilon_2)(\|u\|+\|v\|)$ for $t \in [\eta/\alpha, \eta]$ and $u, v \in P$ with $0 \leq \|u\|+\|v\| \leq R_2$. Let $\Omega_3 = \{(x,y) \in K \times K : \|(x,y)\|_2 < R_2\}$. Then for any $(x,y) \in \partial\Omega_3$,

$$\|T_2(x,y)\|_1 \geq R_2\mu(m_2+\varepsilon_2)\sigma \int_{\eta/\alpha}^{\eta} G(\eta,s)ds. \tag{3.8}$$

So we have by (3.8) and (H)′

$$\|T(x,y)\|_2 \geq \|T_2(x,y)\|_1 > R_2 = \|(x,y)\|_2, \quad \text{for any } (x,y) \in \partial\Omega_3. \tag{3.9}$$

By (3.5), (3.7), (3.9) and Lemma 2.4 we get that BVP (1.1) has at least two positive solutions with $\|(x_1,y_1)\|_2 < 1 < \|(x_2,y_2)\|_2$. □

Corollary 3.2. *Assume that (A) and the following condition hold, then the conclusion of Theorem 3.1 also holds.*

$$(\varphi f_1)_0 > m_1, \quad \varphi(f_2)_\infty > m_2, \quad \textit{where } m_1, m_2 \in (0,+\infty). \tag{3.10}$$

Theorem 3.3. *Assume that (A) and (H_2) hold, then BVP* (1.1) *has at least one positive solution when $\lambda \in [a_2, a_1]$ and $\mu \in (0, a_1]$.*

Proof. By Lemma 2.6, we see that $T : K \times K \to K \times K$ is completely continuous. By (H_2), there exists $r_1 > 0$, $\varepsilon_3 > 0$, $\varepsilon > 0$ such that for $i = 1, 2$,

$$\|f_i(t,x(t),y(t))\| \leq (m_3-\varepsilon_3)(\|x(t)\|+\|y(t)\|), \qquad \|I_{i,k}(x(t_k))\| \leq \varepsilon\|x(t_k)\|, \tag{3.11}$$

for any $x, y \in K$ with $0 \leq \|x\|_1 + \|y\|_1 \leq r_1$, where $m_3 - \varepsilon_3 > 0$, $\varepsilon > 0$ such that

$$\varepsilon \sum_{k=1}^{m} g(t_k) \leq \frac{1}{2}. \tag{3.12}$$

Let $\Omega_4 = \{(x, y) \in K \times K : \|(x, y)\|_2 < r_1\}$. Then for any $(x, y) \in \partial\Omega_4$, we obtain

$$\begin{aligned}
\|T_1(x, y)(t)\| &= \left\| \lambda \int_0^1 G(t, s) f_1(s, x(s), y(s)) ds + \sum_{k=1}^{m} G(t, t_k) I_{1,k}(x(t_k)) \right\| \\
&\leq \lambda \left\| \int_0^1 g(s) f_1(s, x(s), y(s)) ds \right\| + \left\| \sum_{k=1}^{m} g(t_k) I_{1,k}(x(t_k)) \right\| \\
&\leq \lambda(m_3 - \varepsilon_3) \int_0^1 g(s) ds \left(\|x(t)\| + \|y(t)\|\right) + \varepsilon \sum_{k=1}^{m} g(t_k) \|x(t_k)\| \\
&\leq \lambda(m_3 - \varepsilon_3) \int_0^1 g(s) ds \left(\|x\|_1 + \|y\|_1\right) + \varepsilon \sum_{k=1}^{m} g(t_k) \|x\|_1 \\
&= \lambda(m_3 - \varepsilon_3) r_1 \int_0^1 g(s) ds + \varepsilon \sum_{k=1}^{m} g(t_k) \|x\|_1 \\
&\leq \frac{1}{4} r_1 + \varepsilon \sum_{k=1}^{m} g(t_k) \|x\|_1,
\end{aligned} \tag{3.13}$$

Similarly

$$\|T_2(x, y)(t)\| \leq \mu(m_3 - \varepsilon_3) r_1 \int_0^1 g(s) ds + \varepsilon \sum_{k=1}^{m} g(t_k) \|y\|_1 \leq \frac{1}{4} r_1 + \varepsilon \sum_{k=1}^{m} g(t_k) \|y\|_1. \tag{3.14}$$

It follows that

$$\|T(x, y)\|_2 = \|T_1(x, y)\|_1 + \|T_2(x, y)\|_1 \leq r_1 = \|(x, y)\|_2, \tag{3.15}$$

which implies

$$\|T(x, y)\|_2 \leq \|(x, y)\|_2, \quad \text{for any } (x, y) \in \partial\Omega_4. \tag{3.16}$$

On the other hand, by $(\varphi f_1)_\infty > m_4$, there exists $R > 0$, $\varepsilon_4 > 0$ such that $\varphi(f_1(t, x(t), y(t))) \geq (m_4 + \varepsilon_4)(\|x(t)\| + \|y(t)\|)$ for $\|x\|_1 + \|y\|_1 > R$ and $t \in [\eta/\alpha, \eta]$. Let $R_1 = \max\{2r_1, R/\sigma\}$, $\Omega_5 = \{(x, y) \in K \times K : \|(x, y)\|_2 < R_1\}$. For any $(x, y) \in \partial\Omega_5$, we have

$$\begin{gathered}
x(t) \geq \sigma x(s), \quad y(t) \geq \sigma y(s), \quad \|x(t)\| \geq \sigma \|x(s)\|, \\
\|y(t)\| \geq \sigma \|y(s)\|, \quad t \in \left[\frac{\eta}{\alpha}, \eta\right], \ s \in [0, 1].
\end{gathered} \tag{3.17}$$

By the definition of T_1 we get

$$\begin{aligned}\|T_1(x,y)\|_1 \geq \psi((T_1(x,y))(\eta)) &\geq \lambda \int_{\eta/\alpha}^{\eta} G(\eta,s)\psi(f_1(t,x(s),y(s)))ds \\ &\geq \lambda(m_4+\varepsilon_4)\int_{\eta/\alpha}^{\eta} G(\eta,s)(\|x(s)\|+\|y(s)\|)ds \\ &\geq \lambda(m_4+\varepsilon_4)\sigma\int_{\eta/\alpha}^{\eta} G(\eta,s)(\|x(u)\|+\|y(u)\|)ds.\end{aligned} \tag{3.18}$$

So

$$\|T_1(x,y)\|_1 \geq \lambda(m_4+\varepsilon_4)\sigma\int_{\eta/\alpha}^{\eta} G(\eta,s)(\|x\|_1+\|y\|_1)ds = R_1\lambda(m_4+\varepsilon_4)\sigma\int_{\eta/\alpha}^{\eta} G(\eta,s)ds. \tag{3.19}$$

Hence

$$\|T(x,y)\|_2 \geq \|T_1(x,y)\|_1 \geq R_1\lambda(m_4+\varepsilon_4)\sigma\int_{\eta/\alpha}^{\eta} G(\eta,s)ds \geq R_1 = \|(x,y)\|_2. \tag{3.20}$$

Therefore

$$\|T(x,y)\|_2 \geq \|(x,y)\|_2, \quad \forall (x,y) \in \partial\Omega_5. \tag{3.21}$$

By (3.16), (3.21) and Lemma 2.4, it is easily seen that T has a fixed-point $(x^*,y^*) \in (\overline{\Omega}_5 \setminus \Omega_4)$. □

Corollary 3.4. *Let (A) and the following conditions hold, then BVP* (1.1) *has at least one positive solution while* $\mu \in [a_2,a_1]$ *and* $\lambda \in (0,a_1]$.

$$f_i^0 < m_3, \quad (\psi f_2)_\infty > m_4, \quad I_{i,0}(k) = 0, \quad i = 1,2. \tag{3.22}$$

Theorem 3.5. *Let (A) and (H$_3$) hold, then BVP* (1.1) *has at least one positive solution while* $\lambda \in [a_3,a_4]$ *and* $\mu \in (0,a_4]$.

Proof. Since $(\varphi f_1)_0 > m_5$, we choose $R_3 > 0, \varepsilon_5 > 0$ such that $\varphi(f_i(t,u,v)) \geq (m_5+\varepsilon_5)(\|u\|+\|v\|)$ for $0 \leq \|u\| + \|v\| \leq R_3$ and $t \in [\eta/\alpha, \eta]$. Let $\Omega_6 = \{(x,y) \in K \times K : \|(x,y)\|_2 < R_3\}$. Then for any $(x,y) \in \partial\Omega_6$,

$$\begin{aligned}\|T_1(x,y)\|_1 \geq \varphi((T_1(x,y))(\eta)) &\geq \lambda \int_{\eta/\alpha}^{\eta} G(\eta,s)\varphi(f_1(t,x(s),y(s)))ds \\ &\geq \lambda(m_5+\varepsilon_5)\sigma \int_{\eta/\alpha}^{\eta} G(\eta,s)(\|x\|_1 + \|y\|_1)ds \\ &= R_2\lambda(m_5+\varepsilon_5)\sigma \int_{\eta/\alpha}^{\eta} G(\eta,s)ds.\end{aligned} \tag{3.23}$$

So

$$\begin{aligned}\|T(x,y)\|_2 &= \|T_1(x,y)\|_1 + \|T_2(x,y)\|_1 \geq \|T_1(x,y)\|_1 \\ &\geq R_3\lambda(m_5+\varepsilon_5)\sigma \int_{\eta/\alpha}^{\eta} G(\eta,s)ds \\ &\geq R_3,\end{aligned} \tag{3.24}$$

which implies

$$\|T(x,y)\|_2 \geq \|(x,y)\|_2, \quad \forall (x,y) \in \partial\Omega_6. \tag{3.25}$$

On the other hand, by $f_i^{\infty} < m_6$ and $I_i^{\infty}(k) = 0$ $(i = 1,2)$, there exist $M > 0$, $\varepsilon_6 > 0$, $\varepsilon > 0$ such that $m_6 - \varepsilon_6 > 0$ and

$$\begin{gathered}\|f_i(t,x(t),y(t))\| \leq (m_6 - \varepsilon_6)(\|x(t)\| + \|y(t)\|), \quad \|I_{i,k}(x(t_k))\| \leq \varepsilon\|x(t_k)\|, \\ \text{for any } \|x\|_1 + \|y\|_1 = \|(x,y)\|_2 \geq M, \quad t \in [0,1],\end{gathered} \tag{3.26}$$

where ε satisfies

$$\varepsilon \sum_{k=1}^{m} g(t_k) \leq \frac{1}{2}. \tag{3.27}$$

Let $R_4 = \max\{M, 2R_3\}$ and $\Omega_7 = \{(x, y) \mid (x, y) \in K \times K : \|(x, y)\|_2 < R_4\}$. Then for any $(x, y) \in \partial\Omega_7$, we have

$$\begin{aligned}
\|T_1(x,y)(t)\| &= \left\|\lambda \int_0^1 G(t,s) f_1(s, x(s), y(s))\,ds + \sum_{k=1}^{m} G(t,t_k) I_{1,k}(x(t_k))\right\| \\
&\leq \lambda \left\|\int_0^1 g(s) f_1(s, x(s), y(s))\,ds\right\| + \left\|\sum_{k=1}^{m} g(t_k) I_{1,k}(x(t_k))\right\| \\
&\leq \lambda(m_6 - \varepsilon_6) \int_0^1 g(s)\,ds\big(\|x(t)\| + \|y(t)\|\big) + \varepsilon \sum_{k=1}^{m} g(t_k)\|x(t_k)\| \\
&\leq \lambda(m_6 - \varepsilon_6) \int_0^1 g(s)\,ds\big(\|x\|_1 + \|y\|_1\big) + \varepsilon \sum_{k=1}^{m} g(t_k)\|x\|_1 \\
&= \lambda(m_6 - \varepsilon_6) R_4 \int_0^1 g(s)\,ds + \varepsilon \sum_{k=1}^{m} g(t_k)\|x\|_1 \\
&\leq \frac{1}{4} R_4 + \varepsilon \sum_{k=1}^{m} g(t_k)\|x\|_1.
\end{aligned} \tag{3.28}$$

Similarly

$$\begin{aligned}
\|T_2(x,y)(t)\| &\leq \mu(m_6 - \varepsilon_6) R_4 \int_0^1 g(s)\,ds + \varepsilon \sum_{k=1}^{m} g(t_k)\|y\|_1 \\
&\leq \frac{1}{4} R_4 + \varepsilon \sum_{k=1}^{m} g(t_k)\|y\|_1.
\end{aligned} \tag{3.29}$$

Hence

$$\|T(x,y)\|_2 \leq \frac{1}{2} R_4 + \varepsilon R_4 \sum_{k=1}^{m} g(t_k) \leq R_4 = \|(x,y)\|_2. \tag{3.30}$$

So

$$\|T(x,y)\|_2 \leq \|(x,y)\|_2, \quad \forall (x,y) \in \partial\Omega_7. \tag{3.31}$$

By (3.25), (3.31), and Lemma 2.4, T has a fixed-point $(x^*, y^*) \in (\overline{\Omega}_7 \setminus \Omega_6)$. □

Corollary 3.6. *Assume that (A) and the following conditions hold, then BVP* (1.1) *has at least one positive solution while $\mu \in [a_3, a_4]$ and $\lambda \in (0, a_4]$.*

$$(\psi f_2)_0 > m_5, \quad f_i^{\infty} < m_6, \quad I_i^{\infty}(k) = 0, \quad i = 1, 2. \tag{3.32}$$

4. An Example

In this section, we construct an example to demonstrate the application of our main results obtained in Section 3. Consider the following third-order boundary value problem:

$$\begin{gathered}
-x_n'''(t) = \lambda t^2 e^{-t}\left(x_n(t) + y_n(t)\right)^2, \\
-y_n'''(t) = \mu t^2 e^{-t}\left(x_n(t) + y_n(t)\right)^2, \\
\Delta x_n''\left(\frac{1}{3}\right) = -x_n^4\left(\frac{1}{3}\right), \qquad \Delta y_n''\left(\frac{1}{3}\right) = -y_n^4\left(\frac{1}{3}\right), \\
\Delta x_n(0) = x_n'(0) = \theta, \qquad x_n'(1) - 2x_n'\left(\frac{2}{5}\right) = \theta, \\
y_n(0) = y_n'(0) = \theta, \qquad y_n'(1) - 2y_n'\left(\frac{2}{5}\right) = \theta.
\end{gathered} \tag{4.1}$$

Conclusion 1. BVP(4.1) has at least one positive solution.

Proof. $E = R^m = \{x = (x_1, x_2, \ldots, x_m),\ x_n \in R,\ n = 1, 2, \ldots, m\}$. Define $\|x\| = \max_{1\le n\le m}|x_n|$. $P = \{x \in E : x_i > 0,\ i = 1, 2, \ldots, m\}$. $x = (x_1, x_2, \ldots, x_m)$, $f = (f_1, f_2, \ldots, f_m)$. $g_n = f_n = t^2 e^{-t}(x_n(t) + y_n(t))^2$, we know that $P^* = P$, let $\varphi = (1, 1, \ldots, 1)$, then for any $x \in P$, $\varphi(f(t, x, y)) = \sum_{k=1}^{m} f_n(t, x, y)$. It is easy to see that (A) is satisfied. On the other hand,

$$\frac{\varphi(f(t,x,y))}{\|x\| + \|y\|} \ge \frac{\|f(t,x,y)\|}{\|x\| + \|y\|} = +\infty, \qquad f^0 = \limsup_{\|x\|+\|y\|\to 0} \max_{t\in[0,1]} \frac{\|f(t,x,y)\|}{\|x\| + \|y\|} = 0, \tag{4.2}$$

that is, $(\varphi f)_\infty = \infty$. Similarly, $g^0 = 0$, it is easy to see that $I_{i,0}(k) = 0$, where $k = 1$, $i = 1, 2$. In this example, $\alpha = 2$, $\eta = 2/5$, $\sigma = \eta^2/(2\alpha^2(1+\alpha)) = 25/96$ and

$$G(t,s) = \frac{5}{2}\begin{cases}
\dfrac{(2ts - s^2)}{5} + t^2 s, & s \le \left\{\dfrac{2}{5}, t\right\}, \\
\dfrac{(t^2 + t^2 s)}{5}, & t \le s \le \dfrac{2}{5}, \\
\dfrac{(2ts - s^2)}{5} + t^2\left(\dfrac{4}{5} - s\right), & \dfrac{2}{5} \le s \le t, \\
t^2(1-s), & \max\left\{\dfrac{2}{5}, t\right\} \le s,
\end{cases} \tag{4.3}$$

and $g(s) = (1+\alpha)s(1-s)/(1-\alpha\eta) = 15s(1-s)$.

Let $m_1 = 5, m_2 = 3000$. By computing, we get

$$a_1 = \frac{1}{4}\left(\int_0^1 5g(s)ds\right)^{-1} = \frac{8}{25}, \qquad a_2 = \frac{4}{3125}\left(\int_{1/5}^{2/5} G\left(\frac{2}{5}, s\right)ds\right)^{-1}. \tag{4.4}$$

Above all, the conditions of Theorem 3.3 are satisfied. Then for any $\lambda \in [a_2, +\infty)$ and $\mu \in (0, a_1]$, BVP (4.1) has at least one positive solution. □

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