

Research Article

Existence and Uniqueness of Periodic Solutions for a Class of Nonlinear Equations with p -Laplacian-Like Operators

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We investigate the following nonlinear equations with p -Laplacian-like operators $(\varphi(x'(t)))' + f(x(t))x'(t) + g(x(t)) = e(t)$: some criteria to guarantee the existence and uniqueness of periodic solutions of the above equation are given by using Mawhin's continuation theorem. Our results are new and extend some recent results due to Liu (B. Liu, Existence and uniqueness of periodic solutions for a kind of Lienard type p -Laplacian equation, *Nonlinear Analysis TMA*, 69, 724–729, 2008).

1. Introduction

In this paper, we deal with the existence and uniqueness of periodic solutions for the following nonlinear equations with p -Laplacian-like operators:

$$(\varphi(x'(t)))' + f(x(t))x'(t) + g(x(t)) = e(t), \quad (1.1)$$

where f, g are continuous functions on \mathbb{R} , and e is a continuous function on \mathbb{R} with period $T > 0$; moreover, $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying the following:

(H1) for any $x_1, x_2 \in \mathbb{R}, x_1 \neq x_2, [\varphi(x_1) - \varphi(x_2)] \cdot (x_1 - x_2) > 0$ and $\varphi(0) = 0$;

(H2) there exists a function $\alpha : [0, +\infty) \rightarrow [0, +\infty)$ such that $\lim_{s \rightarrow +\infty} \alpha(s) = +\infty$ and

$$\varphi(x) \cdot x \geq \alpha(|x|)|x|, \quad \forall x \in \mathbb{R}. \quad (1.2)$$

It is obvious that under these two conditions, φ is an homeomorphism from \mathbb{R} onto \mathbb{R} and is increasing on \mathbb{R} .

Recall that p -Laplacian equations have been of great interest for many mathematicians. Especially, there is a large literature (see, e.g., [1–7] and references therein) about the existence of periodic solutions to the following p -Laplacian equation:

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g(x(t)) = e(t), \quad (1.3)$$

and its variants, where $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$. Obviously, (1.3) is a special case of (1.1).

However, there are seldom results about the existence of periodic solutions to (1.1). The main difficulty lies in the p -Laplacian-like operator φ of (1.1), which is more complicated than φ_p in (1.3). Since there is no concrete form for the p -Laplacian-like operator φ of (1.1), it is more difficult to prove the existence of periodic solutions to (1.1).

Therefore, in this paper, we will devote ourselves to investigate the existence of periodic solutions to (1.1). As one will see, our theorem generalizes some recent results even for the case of $\varphi(s) = \varphi_p(s)$ (see Remark 2.2).

Next, let us recall some notations and basic results. For convenience, we denote

$$C_T^1 := \left\{ x \in C^1(\mathbb{R}, \mathbb{R}) : x \text{ is } T\text{-periodic} \right\}, \quad (1.4)$$

which is a Banach space endowed with the norm $\|x\| = \max\{|x|_\infty, |x'|_\infty\}$, where

$$|x|_\infty = \max_{t \in [0, T]} |x(t)|, \quad |x'|_\infty = \max_{t \in [0, T]} |x'(t)|. \quad (1.5)$$

In the proof of our main results, we will need the following classical Mawhin's continuation theorem.

Lemma 1.1 ([8]). *Let (H1), (H2) hold and \tilde{f} is Carathéodory. Assume that Ω is an open bounded set in C_T^1 such that the following conditions hold.*

(S1) *For each $\lambda \in (0, 1)$, the problem*

$$(\varphi(x'(t)))' = \lambda \tilde{f}(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T) \quad (1.6)$$

has no solution on $\partial\Omega$.

(S2) *The equation*

$$F(a) := \frac{1}{T} \int_0^T \tilde{f}(t, a, 0) dt = 0 \quad (1.7)$$

has no solution on $\partial\Omega \cap \mathbb{R}$.

(S3) *The Brouwer degree*

$$\deg(F, \Omega \cap \mathbb{R}, 0) \neq 0. \quad (1.8)$$

Then the periodic boundary value problem

$$(\varphi(x'(t)))' = \tilde{f}(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T) \tag{1.9}$$

has at least one T -periodic solution on $\overline{\Omega}$.

2. Main Results

In this section, we prove an existence and uniqueness theorem for (1.1).

Theorem 2.1. *Suppose the following assumptions hold:*

- (A1) $g \in C^1(\mathbb{R}, \mathbb{R})$ and $g'(x) < 0$ for all $x \in \mathbb{R}$;
- (A2) there exist a constant $r \geq 0$ and a function $\varepsilon(t) \in C(\mathbb{R}, \mathbb{R})$ such that for all $t \in \mathbb{R}$ and $|x| > r$,

$$\begin{aligned} x[g(x) - \varepsilon(t)] &< 0, \\ \int_0^T [\varepsilon(t) - e(t)] dt &\leq 0, \quad \int_0^T |\varepsilon(t) - e(t)| dt < 2. \end{aligned} \tag{2.1}$$

Then (1.1) has a unique T -periodic solution.

Proof

Existence. For the proof of existence, we use Lemma 1.1. First, let us consider the homotopic equation of (1.1):

$$(\varphi(x'(t)))' + \lambda f(x(t))x'(t) + \lambda g(x(t)) = \lambda e(t), \quad \lambda \in (0, 1). \tag{2.2}$$

Let $x(t) \in C_T^1$ be an arbitrary solution of (2.2). By integrating the two sides of (2.2) over $[0, T]$, and noticing that $x'(0) = x'(T)$ and $x(0) = x(T)$, we have

$$\int_0^T [g(x(t)) - e(t)] dt = 0, \tag{2.3}$$

that is,

$$\frac{1}{T} \int_0^T g(x(t)) dt = \bar{e} := \frac{1}{T} \int_0^T e(t) dt. \tag{2.4}$$

Since $g(x(\cdot))$ is continuous, there exists $t_0 \in [0, T]$ such that

$$g(x(t_0)) \geq \bar{e}. \tag{2.5}$$

In view of (A1), we obtain

$$x(t_0) \leq \tilde{e}, \quad (2.6)$$

where $\tilde{e} = g^{-1}(\bar{e})$. Then, for each $t \in [t_0, t_0 + T]$, we have

$$\begin{aligned} 2x(t) &= x(t) + x(t - T) \\ &= x(t_0) + \int_{t_0}^t x'(s) ds + x(t_0) - \int_{t-T}^{t_0} x'(s) ds \\ &\leq 2x(t_0) + \int_{t_0}^t |x'(s)| ds + \int_{t-T}^{t_0} |x'(s)| ds \\ &\leq 2\tilde{e} + \int_0^T |x'(s)| ds, \end{aligned} \quad (2.7)$$

which gives that

$$|x|_\infty \leq \tilde{e} + \frac{1}{2} \int_0^T |x'(s)| ds. \quad (2.8)$$

Thus,

$$|x|_\infty \leq |\tilde{e}| + \frac{1}{2} \int_0^T |x'(s)| ds. \quad (2.9)$$

Since $\lim_{s \rightarrow +\infty} \alpha(s) = +\infty$, there is a constant $M > 0$ such that

$$\alpha(s) \geq 1, \quad \forall s \geq M. \quad (2.10)$$

Set

$$\begin{aligned} E_1 &= \{t : t \in [0, T], |x'(t)| > M\}, & E_2 &= \{t : t \in [0, T], |x'(t)| \leq M\}, \\ F_1 &= \{t : t \in [0, T], |x(t)| > r\}, & F_2 &= \{t : t \in [0, T], |x(t)| \leq r\}. \end{aligned} \quad (2.11)$$

In view of (A2) and

$$\varphi(x) \cdot x \geq \alpha(|x|)|x|, \quad \forall x \in \mathbb{R}, \quad (2.12)$$

we get

$$\begin{aligned}
 \int_0^T |x'(t)| dt &= \int_{E_1} |x'(t)| dt + \int_{E_2} |x'(t)| dt \\
 &\leq \int_{E_1} |x'(t)| dt + MT \\
 &\leq \int_{E_1} \frac{\varphi(x'(t))x'(t)}{\alpha(|x'(t)|)} dt + MT \\
 &\leq \int_{E_1} \varphi(x'(t))x'(t) dt + MT \\
 &\leq \int_0^T \varphi(x'(t))x'(t) dt + MT \\
 &= \int_0^T \varphi(x'(t)) dx(t) + MT \\
 &= - \int_0^T (\varphi(x'(t)))' x(t) dt + MT \\
 &= \lambda \int_0^T [g(x(t)) - e(t)] x(t) dt + \lambda \int_0^T f(x(t))x'(t)x(t) dt + MT \\
 &= \lambda \int_0^T [g(x(t)) - e(t)] x(t) dt + MT \\
 &= \lambda \int_{F_1} [g(x(t)) - \varepsilon(t)] x(t) dt + \lambda \int_{F_2} [g(x(t)) - \varepsilon(t)] x(t) dt \\
 &\quad + \lambda \int_0^T [\varepsilon(t) - e(t)] x(t) dt + MT \\
 &\leq \int_{F_2} |g(x(t)) - \varepsilon(t)| \cdot |x(t)| dt + MT + \int_0^T |\varepsilon(t) - e(t)| dt \cdot |x|_\infty \\
 &\leq M'T + MT + \int_0^T |\varepsilon(t) - e(t)| dt \cdot |x|_\infty,
 \end{aligned} \tag{2.13}$$

where

$$M' = \left[\max_{|x| \leq r} |g(x)| + \max_{t \in [0, T]} |\varepsilon(t)| \right] \cdot r. \tag{2.14}$$

By (2.9), we have

$$|x|_\infty \leq |\tilde{e}| + \frac{(M' + M)T}{2} + \frac{1}{2} \int_0^T |\varepsilon(t) - e(t)| dt \cdot |x|_\infty. \tag{2.15}$$

Noticing that

$$\int_0^T |\varepsilon(t) - e(t)| dt < 2, \quad (2.16)$$

there exists a constant $M'' > |\tilde{e}|$ such that

$$|x|_\infty \leq M''. \quad (2.17)$$

On the other hand, it follows from

$$|\varphi(x)| \cdot |x| = \varphi(x) \cdot x \geq \alpha(|x|)|x|, \quad \forall x \in \mathbb{R}, \quad (2.18)$$

that $\alpha(|x|) \leq |\varphi(x)|$ for $x \neq 0$. In addition, since $x(0) = x(T)$, there exists $t_1 \in [0, T]$ such that $x'(t_1) = 0$. Thus $\varphi(x'(t_1)) = 0$.

Then, for all $t \in E := \{t \in [0, T] : x'(t) \neq 0\}$, we have

$$\begin{aligned} \alpha(|x'(t)|) &\leq |\varphi(x'(t))| \\ &= \left| \int_{t_1}^t (\varphi(x'(s)))' ds \right| \\ &\leq \left| \int_{t_1}^t f(x(s)) \cdot x'(s) ds \right| + \int_0^T |g(x(s))| ds + \int_0^T |e(s)| ds \\ &\leq \left| \int_{x(t_1)}^{x(t)} f(u) du \right| + \left[\max_{|x| \leq M''} |g(x)| + \max_{t \in [0, T]} |e(t)| \right] \cdot T \\ &\leq \int_{-M''}^{M''} |f(u)| du + \left[\max_{|x| \leq M''} |g(x)| + \max_{t \in [0, T]} |e(t)| \right] \cdot T \\ &\leq \max_{|x| \leq M''} |f(x)| \cdot 2M'' + \left[\max_{|x| \leq M''} |g(x)| + \max_{t \in [0, T]} |e(t)| \right] \cdot T := M'''. \end{aligned} \quad (2.19)$$

For the above M''' , it follows from $\lim_{s \rightarrow +\infty} \alpha(s) = +\infty$ that there exists $G > M''$ such that

$$\alpha(s) > M''', \quad s \geq G. \quad (2.20)$$

Combining this with $\alpha(|x'(t)|) \leq M'''$, we get

$$|x'(t)| < G, \quad t \in E, \quad (2.21)$$

which yields that $|x'|_\infty < G$.

Now, we have proved that any solution $x(t) \in C_T^1$ of (2.2) satisfies

$$|x|_\infty < G, \quad |x'|_\infty < G. \quad (2.22)$$

Since $G > |\tilde{e}|$, we have

$$G > \tilde{e} = g^{-1}(\bar{e}), \quad -G < \tilde{e} = g^{-1}(\bar{e}). \quad (2.23)$$

In view of g being strictly decreasing, we get

$$g(G) < \bar{e}, \quad g(-G) > \bar{e} \quad (2.24)$$

Set

$$\Omega = \left\{ x \in C_T^1 : |x|_\infty \leq G, |x'|_\infty \leq G \right\}. \quad (2.25)$$

Then, we know that (2.2) has no solution on $\partial\Omega$ for each $\lambda \in (0, 1)$, that is, the assumption (S1) of Lemma 1.1 holds. In addition, it follows from (2.24) that

$$\begin{aligned} -\frac{1}{T} \int_0^T [g(G) - e(t)] dt &= \bar{e} - g(G) > 0, \\ -\frac{1}{T} \int_0^T [g(-G) - e(t)] dt &= \bar{e} - g(-G) < 0. \end{aligned} \quad (2.26)$$

So the assumption (S2) of Lemma 1.1 holds. Let

$$H(x, \mu) = \mu x - (1 - \mu) \frac{1}{T} \int_0^T [g(x) - e(t)] dt. \quad (2.27)$$

For $x \in \partial\Omega \cap \mathbb{R}$ and $\mu \in [0, 1]$, by (2.24), we have

$$\begin{aligned} xH(x, \mu) &= \mu x^2 - (1 - \mu) x \frac{1}{T} \int_0^T [g(x) - e(t)] dt \\ &= \mu x^2 + (1 - \mu) x \frac{1}{T} \int_0^T [\bar{e} - g(x)] dt > 0. \end{aligned} \quad (2.28)$$

Thus, $H(x, \mu)$ is a homotopic transformation. So

$$\begin{aligned} \deg(F, \Omega \cap \mathbb{R}, 0) &= \deg(H(x, 0), \Omega \cap \mathbb{R}, 0) \\ &= \deg(H(x, 1), \Omega \cap \mathbb{R}, 0) \\ &= \deg(I, \Omega \cap \mathbb{R}, 0) \neq 0, \end{aligned} \quad (2.29)$$

that is, the assumption (S3) of Lemma 1.1 holds. By applying Lemma 1.1, there exists at least one solution with period T to (1.1).

Uniqueness. Let

$$\varphi(x) = \int_0^x f(u)du, \quad y(t) = \varphi(x'(t)) + \varphi(x(t)). \quad (2.30)$$

Then (1.1) is transformed into

$$\begin{aligned} x'(t) &= \varphi^{-1}[y(t) - \varphi(x(t))], \\ y'(t) &= -g(x(t)) + e(t). \end{aligned} \quad (2.31)$$

Let $x_1(t)$ and $x_2(t)$ being two T -periodic solutions of (1.1); and

$$y_i(t) = \varphi(x'_i(t)) + \varphi(x_i(t)), \quad i = 1, 2. \quad (2.32)$$

Then we obtain

$$\begin{aligned} x'_i(t) &= \varphi^{-1}[y_i(t) - \varphi(x_i(t))], \\ y'_i(t) &= -g(x_i(t)) + e(t). \end{aligned} \quad i = 1, 2. \quad (2.33)$$

Setting

$$v(t) = x_1(t) - x_2(t), \quad u(t) = y_1(t) - y_2(t), \quad (2.34)$$

it follows from (2.33) that

$$\begin{aligned} v'(t) &= \varphi^{-1}[y_1(t) - \varphi(x_1(t))] - \varphi^{-1}[y_2(t) - \varphi(x_2(t))], \\ u'(t) &= -[g(x_1(t)) - g(x_2(t))]. \end{aligned} \quad (2.35)$$

Now, we claim that

$$u(t) \leq 0, \quad \forall t \in \mathbb{R}. \quad (2.36)$$

If this is not true, we consider the following two cases.

Case 1. There exists $t_2 \in (0, T)$ such that

$$u(t_2) = \max_{t \in [0, T]} u(t) = \max_{t \in \mathbb{R}} u(t) > 0, \quad (2.37)$$

which implies that

$$\begin{aligned} u'(t_2) &= -[g(x_1(t_2)) - g(x_2(t_2))] = 0, \\ u''(t_2) &= -[g(x_1(t)) - g(x_2(t))]'|_{t=t_2} = -[g'(x_1(t_2))x'_1(t_2) - g'(x_2(t_2))x'_2(t_2)] \leq 0. \end{aligned} \quad (2.38)$$

By (A1), $g'(x) < 0$. So it follows from $g(x_1(t_2)) - g(x_2(t_2)) = 0$ that $x_1(t_2) = x_2(t_2)$. Thus, in view of

$$-g'(x_1(t_2)) > 0, \quad u(t_2) = y_1(t_2) - y_2(t_2) > 0, \quad (2.39)$$

and (H1), we obtain

$$\begin{aligned} u''(t_2) &= -g'(x_1(t_2)) [x'_1(t_2) - x'_2(t_2)] \\ &= -g'(x_1(t_2)) \left\{ \varphi^{-1}[y_1(t_2) - \varphi(x_1(t_2))] - \varphi^{-1}[y_2(t_2) - \varphi(x_2(t_2))] \right\} \\ &= -g'(x_1(t_2)) \left\{ \varphi^{-1}[y_1(t_2) - \varphi(x_1(t_2))] - \varphi^{-1}[y_2(t_2) - \varphi(x_1(t_2))] \right\} > 0, \end{aligned} \quad (2.40)$$

which contradicts with $u''(t_2) \leq 0$.

Case 2.

$$u(0) = \max_{t \in [0, T]} u(t) = \max_{t \in \mathbb{R}} u(t) > 0. \quad (2.41)$$

Also, we have $u'(0) = 0$ and $u''(0) \leq 0$. Then, similar to the proof of Case 1, one can get a contradiction.

Now, we have proved that

$$u(t) \leq 0, \quad \forall t \in \mathbb{R}. \quad (2.42)$$

Analogously, one can show that

$$u(t) \geq 0, \quad \forall t \in \mathbb{R}. \quad (2.43)$$

So we have $u(t) \equiv 0$. Then, it follows from (2.35) that

$$g(x_1(t)) - g(x_2(t)) \equiv 0, \quad \forall t \in \mathbb{R}, \quad (2.44)$$

which implies that

$$x_2(t) \equiv x_1(t), \quad \forall t \in \mathbb{R}. \quad (2.45)$$

Hence, (1.1) has a unique T -periodic solution. The proof of Theorem 2.1 is now completed.

Remark 2.2. In Theorem 2.1, setting $\varepsilon(t) \equiv e(t)$, then (A2) becomes as follows:

(A2') there exists a constant $r \geq 0$ such that for all $t \in \mathbb{R}$ and $|x| > r$,

$$x[g(x) - e(t)] < 0. \quad (2.46)$$

In the case $\varphi(s) = \varphi_p(s)$, Liu [7, Theorem 1] proved that (1.1) has a unique T -periodic solution under the assumptions (A1) and (A2'). Thus, even for the case of $\varphi(s) = \varphi_p(s)$, Theorem 2.1 is a generalization of [7, Theorem 1].

In addition, we have the following interesting corollary.

Corollary 2.3. *Suppose (A1) and*

(A2'') there exist a constant $\alpha \geq 0$ such that

$$\int_0^T [g(\alpha) - e(t)] dt \leq 0, \quad \int_0^T |g(\alpha) - e(t)| dt < 2 \quad (2.47)$$

hold. Then (1.1) has a unique T -periodic solution.

Proof. Let $\varepsilon(t) \equiv g(\alpha)$. Noticing that

$$x[g(x) - g(\alpha)] < 0, \quad |x| > \alpha, \quad (2.48)$$

we know that (A2) holds with $r = \alpha$. This completes the proof. \square

At last, we give two examples to illustrate our results.

Example 2.4. Consider the following nonlinear equation:

$$(\varphi(x'(t)))' + f(x(t))x'(t) + g(x(t)) = e(t), \quad (2.49)$$

where $\varphi(x) = 2xe^{x^2} - 2x$, $f(x) = e^{-x}$, $g(x) = -(x^3 + x)$, and $e(t) = \sin t$. One can easily check that φ satisfy (H1) and (H2). Obviously, (A1) holds. Moreover, since

$$\lim_{x \rightarrow +\infty} g(x) = -\infty, \quad \lim_{x \rightarrow -\infty} g(x) = +\infty, \quad (2.50)$$

it is easy to verify that (A2) holds. By Theorem 2.1, (2.49) has a unique 2π -periodic solution.

Example 2.5. Consider the following p -Laplacian equation:

$$(\varphi_p(x'(t)))' + g(x(t)) = e(t), \quad (2.51)$$

where $g(x) = -(1/2) \arctan x$, and $e(t) = \sin^2 t$. Obviously, (A1) holds. Moreover, we have

$$\int_0^\pi [g(0) - e(t)] dt \leq 0, \quad \int_0^\pi |g(0) - e(t)| dt = \int_0^\pi \sin^2 t = \frac{\pi}{2} < 2. \quad (2.52)$$

So (A2'') holds. Then, by Corollary 2.3, (2.51) has a unique π -periodic solution.

Remark 2.6. In Example 2.5, $\forall r > 0$, we have

$$x\left(g(x) - e\left(\frac{\pi}{2}\right)\right) > r\left(1 - \frac{\pi}{4}\right) > 0, \quad \forall x < -r. \quad (2.53)$$

Thus, (A2') does not hold. So [7, Theorem 1] cannot be applied to Example 2.5. This means that our results generalize [7, Theorem 1] in essence even for the case of $\varphi(s) = \varphi_p(s)$.

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