Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 305018, 7 pages
doi:10.1155/2010/305018

Research Article

Some Identities of Bernoulli Numbers and Polynomials Associated with Bernstein Polynomials

Min-Soo Kim,[1] Taekyun Kim,[2] Byungje Lee,[3] and Cheon-Seoung Ryoo[4]

[1] *Department of Mathematics, KAIST, 373-1 Guseong-dong, Yuseong-gu, Daejeon 305-701, Republic of Korea*
[2] *Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea*
[3] *Department of Wireless Communications Engineering, Kwangwoon University, Seoul 139-701, Republic of Korea*
[4] *Department of Mathematics, Hannam University, Daejeon 306-791, Republic of Korea*

Correspondence should be addressed to Taekyun Kim, tkkim@kw.ac.kr

Received 30 August 2010; Accepted 27 October 2010

Academic Editor: Istvan Gyori

We investigate some interesting properties of the Bernstein polynomials related to the bosonic p-adic integrals on $\mathbb{Z}_p$.

1. Introduction

Let $C[0,1]$ be the set of continuous functions on $[0,1]$. Then the classical Bernstein polynomials of degree n for $f \in C[0,1]$ are defined by

$$\mathbb{B}_n(f) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x), \quad 0 \le x \le 1, \tag{1.1}$$

where $\mathbb{B}_n(f)$ is called the Bernstein operator and

$$B_{k,n}(x) = \binom{n}{k} x^k (x-1)^{n-k} \tag{1.2}$$

are called the Bernstein basis polynomials (or the Bernstein polynomials of degree n). Recently, Acikgoz and Araci have studied the generating function for Bernstein polynomials (see [1, 2]). Their generating function for $B_{k,n}(x)$ is given by

$$F_k(t,x) = \frac{t^k e^{(1-x)t} x^k}{k!} = \sum_{n=0}^{\infty} B_{k,n}(x)\frac{t^n}{n!}, \tag{1.3}$$

where $k = 0,1,\ldots$ and $x \in [0,1]$. Note that

$$B_{k,n}(x) = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k}, & \text{if } n \geq k, \\ 0, & \text{if } n < k \end{cases} \tag{1.4}$$

for $n = 0,1,\ldots$ (see [1, 2]). In [3], Simsek and Acikgoz defined generating function of the (q-)Bernstein-Type Polynomials, $Y_n(k,x,q)$ as follows:

$$F_{k,q}(t,x) = \frac{t^k e^{[1-x]_q t} [x]_q^k}{k!} = \sum_{n=k}^{\infty} Y_n(k,x,q)\frac{t^n}{n!}, \tag{1.5}$$

where $[x]_q = (1-q^x)/(1-q)$. Observe that

$$\lim_{q\to 1} Y_n(k,x,q) = B_{k,n}(x). \tag{1.6}$$

Hence by the above one can very easily see that

$$F_k(t,x) = \frac{t^k e^{(1-x)t} x^k}{k!} = \sum_{n=k}^{\infty} B_{k,n}(x)\frac{t^n}{n!}. \tag{1.7}$$

Thus, we have arrived at the generating function in [1, 2] and also in (1.3) as well.

The Bernstein polynomials can also be defined in many different ways. Thus, recently, many applications of these polynomials have been looked for by many authors. Some researchers have studied the Bernstein polynomials in the area of approximation theory (see [1–7]). In recent years, Acikgoz and Araci [1, 2] have introduced several type Bernstein polynomials.

In the present paper, we introduce the Bernstein polynomials on the ring of p-adic integers $\mathbb{Z}_p$. We also investigate some interesting properties of the Bernstein polynomials related to the bosonic p-adic integrals on the ring of p-adic integers $\mathbb{Z}_p$.

2. Bernstein Polynomials Related to the Bosonic p-Adic Integrals on $\mathbb{Z}_p$

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-1}$. For $N \geq 1$, the bosonic distribution μ_1 on $\mathbb{Z}_p$

$$\mu\left(a + p^N \mathbb{Z}_p\right) = \frac{1}{p^N} \tag{2.1}$$

is known as the p-adic Haar distribution μ_{Haar}, where $a + p^N\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x-a|_p \leq p^{-N}\}$ (cf. [8]). We will write $d\mu_1(x)$ to remind ourselves that x is the variable of integration. Let $\text{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. Then μ_1 yields the fermionic p-adic q-integral of a function $f \in \text{UD}(\mathbb{Z}_p)$

$$I_1(f) = \int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \tag{2.2}$$

(cf. [8]). Many interesting properties of (2.2) were studied by many authors (cf. [8, 9] and the references given there). For $n \in \mathbb{N}$, write $f_n(x) = f(x+n)$. We have

$$I_1(f_n) = I_1(f) + \sum_{l=0}^{n-1} f'(l). \tag{2.3}$$

This identity is to derives interesting relationships involving Bernoulli numbers and polynomials. Indeed, we note that

$$I_1((x+y)^n) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_1(y) = B_n(x), \tag{2.4}$$

where $B_n(x)$ are the Bernoulli polynomials (cf. [8]). From (1.2), we have

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(x)d\mu_1(x) &= \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{n-k-j}B_{n-j},\\
\int_{\mathbb{Z}_p} B_{k,n}(x)d\mu_1(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}(1-x)d\mu_1(x)\\
&= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\sum_{l=0}^{n-j}\binom{n-j}{l}(-1)^l B_l.
\end{aligned} \tag{2.5}$$

By (2.5), we obtain the following proposition.

Proposition 2.1. *For $n \geq k$,*

$$\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{n-k-j}B_{n-j} = \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\sum_{l=0}^{n-j}\binom{n-j}{l}(-1)^l B_l. \tag{2.6}$$

From (2.4), we note that

$$B_n(2) - n = (B(1) + 1)^n - n = (B + 1)^n = B_n, \quad n > 1 \tag{2.7}$$

with the usual convention of replacing B^n by B_n and $(B(1))^n$ by $B_n(1)$. Thus, we have

$$\begin{aligned}
\int_{\mathbb{Z}_p} x^n d\mu_1(x) &= \int_{\mathbb{Z}_p} (x+2)^n d\mu_1(x) - n \\
&= (-1)^n \int_{\mathbb{Z}_p} (x-1)^n d\mu_1(x) - n \\
&= \int_{\mathbb{Z}_p} (1-x)^n d\mu_1(x) - n
\end{aligned} \tag{2.8}$$

for $n > 1$, since $(-1)^n B_n(x) = B_n(1-x)$. Therefore we obtain the following theorem.

Theorem 2.2. *For $n > 1$,*

$$\int_{\mathbb{Z}_p} (1-x)^n d\mu_1(x) = \int_{\mathbb{Z}_p} x^n d\mu_1(x) + n. \tag{2.9}$$

Also we obtain

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{n-k,k}(x) d\mu_1(x) &= \int_{\mathbb{Z}_p} x^{n-k} (1-x)^k d\mu_1(x) \\
&= \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} (1-x)^{l+k} d\mu_1(x) \\
&= \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \left\{ \int_{\mathbb{Z}_p} x^{l+k} d\mu_1(x) + l + k \right\} \\
&= \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l (B_{l+k} + l + k).
\end{aligned} \tag{2.10}$$

Therefore we obtain the following result.

Corollary 2.3. *For $k > 1$,*

$$\int_{\mathbb{Z}_p} B_{n-k,k}(x) d\mu_1(x) = \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l (B_{l+k} + l + k). \tag{2.11}$$

From the property of the Bernstein polynomials of degree n, we easily see that

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)d\mu_1(x) &= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p} x^{2k}(1-x)^{n+m-2k}d\mu_1(x) \\
&= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{n+m-2k}\binom{n+m-2k}{l}(-1)^l B_{2k+l} \\
\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)B_{k,s}(x)d\mu_1(x) &= \binom{n}{k}\binom{m}{k}\binom{s}{k}\int_{\mathbb{Z}_p} x^{3k}(1-x)^{n+m-3k}d\mu_1(x) \\
&= \binom{n}{k}\binom{m}{k}\binom{s}{k}\sum_{l=0}^{n+m+s-3k}\binom{n+m+s-3k}{l}(-1)^l B_{3k+l}.
\end{aligned} \tag{2.12}$$

Continuing this process, we obtain the following theorem.

Theorem 2.4. *The multiplication of the sequence of Bernstein polynomials*

$$B_{k,n_1}(x), B_{k,n_2}(x), \dots, B_{k,n_s}(x) \tag{2.13}$$

for $s \in \mathbb{N}$ with different degree under p-adic integral on $\mathbb{Z}_p$, can be given as

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)\cdots B_{k,n_s}(x)d\mu_1(x) \\
&\quad = \binom{n_1}{k}\binom{n_2}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{n_1+n_2+\cdots+n_s-sk}\binom{n_1+n_2+\cdots+n_s-sk}{l}(-1)^l B_{sk+l}.
\end{aligned} \tag{2.14}$$

We put

$$B_{k,n}^m(x) = \underbrace{B_{k,n}(x)\times\cdots\times B_{k,n}(x)}_{m\text{-times}}. \tag{2.15}$$

Theorem 2.5. *The multiplication of*

$$B_{k,n_1}^{m_1}(x), B_{k,n_2}^{m_2}(x), \dots, B_{k,n_s}^{m_s}(x) \tag{2.16}$$

Bernstein polynomials with different degrees $n_1, n_2, \ldots, n_s$ under p-adic integral on $\mathbb{Z}_p$ can be given as

$$\int_{\mathbb{Z}_p} B_{k,n_1}^{m_1}(x) B_{k,n_2}^{m_2}(x) \cdots B_{k,n_s}^{m_s}(x) d\mu_1(x)$$
$$= \binom{n_1}{k}^{m_1} \binom{n_2}{k}^{m_2} \cdots \binom{n_s}{k}^{m_s} \sum_{l=0}^{n_1m_1+n_2m_2+\cdots+n_sm_s-(m_1+\cdots+m_s)k} (-1)^l \tag{2.17}$$
$$\times \binom{n_1m_1+n_2m_2+\cdots+n_sm_s-(m_1+\cdots+m_s)k}{l} B_{(m_1+\cdots+m_s)k+l}.$$

Theorem 2.6. *The multiplication of*

$$B_{k_1,n_1}^{m_1}(x), B_{k_2,n_2}^{m_2}(x), \ldots, B_{k_s,n_s}^{m_s}(x) \tag{2.18}$$

Bernstein polynomials with different degrees $n_1, n_2, \ldots, n_s$ with different powers $m_1, m_2, \ldots, m_s$ under p-adic integral on $\mathbb{Z}_p$ can be given as

$$\int_{\mathbb{Z}_p} B_{k_1,n_1}^{m_1}(x) B_{k_2,n_2}^{m_2}(x) \cdots B_{k_s,n_s}^{m_s}(x) d\mu_1(x)$$
$$= \binom{n_1}{k_1}^{m_1} \binom{n_2}{k_2}^{m_2} \cdots \binom{n_s}{k_s}^{m_s} \sum_{l=0}^{n_1m_1+n_2m_2+\cdots+n_sm_s-(k_1m_1+\cdots+k_sm_s)} (-1)^l \tag{2.19}$$
$$\times \binom{n_1m_1+n_2m_2+\cdots+n_sm_s-(k_1m_1+\cdots+k_sm_s)}{l} B_{k_1m_1+\cdots+k_sm_s+l}.$$

Problem. Find the Witt's formula for the Bernstein polynomials in p-adic number field.

Acknowledgments

The first author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science, and Technology (2010-0001654). The second author was supported by the research grant of Kwangwoon University in 2010.

References

[1] M. Acikgoz and S. Araci, "A study on the integral of the product of several type Bernstein polynomials," *IST Transaction of Applied Mathematics-Modelling and Simulation*. In press.

[2] M. Acikgoz and S. Araci, "On the generating function of the Bernstein polynomials," in *Proceedings of the 8th International Conference of Numerical Analysis and Applied Mathematics (ICNAAM '10)*, AIP, Rhodes, Greece, March 2010.

[3] Y. Simsek and M. Acikgoz, "A new generating function of (q-) Bernstein-type polynomials and their interpolation function," *Abstract and Applied Analysis*, vol. 2010, Article ID 769095, 12 pages, 2010.

[4] S. Bernstein, "Demonstration du theoreme de Weierstrass, fondee sur le calcul des probabilities," *Communications of the Kharkov Mathematical Society*, vol. 13, pp. 1–2, 1913.

[5] L.-C. Jang, W.-J. Kim, and Y. Simsek, "A study on the p-adic integral representation on $\mathbb{Z}_p$ associated with Bernstein and Bernoulli polynomials," *Advances in Difference Equations*, vol. 2010, Article ID 163217, 6 pages, 2010.

[6] T. Kim, L. -C. Jang, and H. Yi, "A note on the modified q-bernstein polynomials," *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 706483, 12 pages, 2010.

[7] G. M. Phillips, "Bernstein polynomials based on the q-integers," *Annals of Numerical Mathematics*, vol. 4, no. 1–4, pp. 511–518, 1997.

[8] T. Kim, "On a q-analogue of the p-adic log gamma functions and related integrals," *Journal of Number Theory*, vol. 76, no. 2, pp. 320–329, 1999.

[9] T. Kim, J. Choi, and Y.-H. Kim, "Some identities on the q-Bernstein polynomials, q-Stirling numbers and q-Bernoulli numbers," *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 3, pp. 335–341, 2010.