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Research Article

Gevrey Regularity of Invariant Curves of Analytic Reversible Mappings

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We prove the existence of a Gevrey family of invariant curves for analytic reversible mappings under weaker nondegeneracy condition. The index of the Gevrey smoothness of the family could be any number $\mu > \tau + 2$, where $\tau > m - 1$ is the exponent in the small divisors condition and m is the order of degeneracy of the reversible mappings. Moreover, we obtain a Gevrey normal form of the reversible mappings in a neighborhood of the union of the invariant curves.

1. Introduction and Main Results

In this paper we consider the following reversible mapping A:

$$\begin{aligned} x_1 &= x + h(y) + f(x, y), \\ y_1 &= y + g(x, y), \end{aligned} \tag{1.1}$$

where the rotation $h(y)$ is real analytic and satisfies the weaker non-degeneracy condition

$$h^{(j)}(0) = 0, \quad 0 < j < m, \quad h^{(m)}(0) \neq 0, \tag{1.2}$$

where $f(x, y)$ and $g(x, y)$ are real analytic and 2π periodic in x, the variable y ranges in an open interval of the real line $\mathbb{R}$. We suppose that the mapping A is reversible with respect to the involution $R : (x, y) \to (-x, y)$, that is, $ARA = R$. When some nonresonance and non-degeneracy conditions are satisfied and f, g are sufficiently small, the existence of invariant curve of reversible mapping (1.1) has been proved in [1–3]. For related works, we refer the readers to [4–6] and the references there.

It is well known that reversible mappings have many similarities as Hamiltonian systems. Since many KAM theorems are proved for Hamiltonian systems, some mathematicians turn to study the regular property of KAM tori with respect to parameters. One of the earliest results is due to Pöschel [7], who proved that the KAM tori of nearly integrable analytic Hamiltonian systems form a Cantor family depending on parameters only in C^∞-way. Because the notorious small divisors can result in loss of smoothness with respect to parameters involving in small divisors in KAM steps, we can only expect Gevrey smoothness of KAM tori even for analytic systems. Gevrey smoothness is a notion intermediate between C^∞-smoothness and analyticity (see definition below). Popov [8] obtained Gevrey smoothness of invariant tori for analytic Hamiltonian systems. In [9], Wagener used the inverse approximation lemma to prove a more general conclusion. Recently, the preceding result has been generalized to Rüssmann's non-degeneracy condition [10–12]. Gevrey smoothness of the family of KAM tori is important for constructing Gevrey normal form near KAM tori, which can lead to the effective stability [8, 13].

For reversible mappings, if $h'(y) \neq 0$, the existence of a C^∞-family of invariant curves has been proved in [1, 2]. But in the case of weaker non-degeneracy condition (1.2), there is no result about Gevrey smoothness. In this paper, we are concerned with Gevrey smoothness of invariant curve of reversible mapping (1.1). The Gevrey smoothness is expressed by Gevrey index. In the following, we specifically obtain the Gevrey index of invariant curve which is related to smoothness of reversible mapping (1.1) and the exponent of the small divisors condition. Moreover, we obtain a Gevrey normal form of the reversible mappings in a neighborhood of the union of the invariant curves.

As in [7, 14, 15], we introduce some parameters, so that the existence of invariant curve of reversible mapping (1.1) can be reduced to that of a family of reversible mappings with some parameters. We write $y = p + z$, and expand $h(y)$ around p, so that $h(y) = h(p) + \int_0^1 h'(y_t)z\,dt$, where $y_t = p + tz$, $0 \le t \le 1$, z varies in a neighborhood of origin of the real line $\mathbb{R}$. We put $\omega(p) = h(p)$, $f(x, z; p) = \int_0^1 h'(y_t)z\,dt + f(x, p+z)$, $g(x, z; p) = g(x, p+z)$ and obtain the family of reversible mappings

$$\begin{aligned} x_1 &= x + \omega(p) + f(x, z; p), \\ z_1 &= z + g(x, z; p). \end{aligned} \tag{1.3}$$

Now, we turn to consider this family of reversible mappings with parameters $p \in \Pi$, where $\Pi \subset \mathbb{R}$ is a bounded interval.

Before stating our theorem, we first give some definitions and notations. Usually, denote by $\mathbb{Z}$ and $\mathbb{Z}_+$ the set of integers and positive integers.

Definition 1.1. Let D be a domain of $\mathbb{R}^n$. A function $F : D \to \mathbb{R}$ is said to belong to the Gevrey-class $G^\mu(D)$ of index $\mu (\mu \ge 1)$ if F is $C^\infty(D)$-smooth and there exists a constant M such that for all $p \in D$,

$$\left|\partial_p^\beta F(p)\right| \le cM^{|\beta|+1}\beta!^\mu, \tag{1.4}$$

where $|\beta| = \beta_1 + \cdots + \beta_n$ and $\beta! = \beta_1! \cdots \beta_n!$ for $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$.

Remark 1.2. By definition, it is easy to see that the Gevrey-smooth functions class G^1 coincides with the class of analytic functions. Moreover, we have

$$G^1 \subset G^{\mu_1} \subset G^{\mu_2} \subset C^{\infty}, \tag{1.5}$$

for $1 < \mu_1 < \mu_2 < \infty$.

In this paper, we will prove Gevrey smoothness of function in a closed set, so we give the following definition.

Definition 1.3. A function F is Gevrey of index μ on a compact set Π_* if it can be extended as a Gevrey function of the same index in a neighborhood of Π_*.

Define

$$D(s,r) = \{(x,z) \in \mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C} \mid |\mathrm{Im}\, x| \le s, |z| \le r\}, \tag{1.6}$$

and denote a complex neighborhood of Π by

$$\Pi_h = \{p \in \mathbb{C} \mid \mathrm{dist}(p,\Pi) \le h\}. \tag{1.7}$$

Now the function $f(x,z;p)$ is real analytic on $D(s,r) \times \Pi_h$. We expand $f(x,z;p)$ as Fourier series with respect to x

$$f(x,z;p) = \sum_{k\in\mathbb{Z}} f_k(z;p)e^{ikx}, \tag{1.8}$$

then define

$$\|f\|_{D(s,r)\times\Pi_h} = \sum_{k\in\mathbb{Z}} \|f_k\|_{r,h} e^{s|k|}, \tag{1.9}$$

where

$$\|f_k\|_{r,h} = \sup_{|z|\le r, p\in\Pi_h} |f_k(z;p)|. \tag{1.10}$$

We write $F(x,z;p) \in G^{1,\mu}(D(s,r) \times \Pi_*)$ if $F(x,z;p)$ is analytic with respect to (x,z) on $D(s,r)$ and G^{μ}-smooth in p on Π_*.

Denote $T = \max_{p\in\Pi_h} |\omega'(p)|$. Fix $\delta \in (0,1)$ and $\tau > m-1$, and let $\mu = \tau + 2 + \delta$ and $\sigma = (2/3)^{\delta/(\tau+1+\delta)}$. Let $W_0 = \mathrm{diag}(\rho_0^{-1}, r_0^{-1})$.

Theorem 1.4. *We consider the mapping A defined in* (1.3)*, which is reversible with respect to the involution $R : (x, z) \to (-x, z)$, that is, $ARA = R$. Suppose that $\omega(p)$ satisfies the non-degeneracy condition: $\omega^{(j)}(0) = 0$, $0 < j < m$, $\omega^{(m)}(0) \neq 0$. Suppose that $f(x, z; p)$ and $g(x, z; p)$ are real analytic on $D(s, r) \times \Pi_h$. Then, there exists $\gamma > 0$ such that for any $0 < \alpha < 1$, if*

$$\|f\|_{D(s,r)\times\Pi_h} + \frac{1}{r}\|g\|_{D(s,r)\times\Pi_h} = \epsilon \leq \gamma\alpha s^{\tau+2}, \tag{1.11}$$

there is a nonempty Cantor set $\Pi_* \subset \Pi$, and a family of transformations $V_*(\cdot, \cdot; p) : D(s/2, r/2) \to D(s, r)$, $\forall p \in \Pi_*$,

$$\begin{gathered} x = \xi + p_*(\xi; p), \\ z = \eta + q_*(\xi, \eta; p), \end{gathered} \tag{1.12}$$

satisfying $V_*(x, z; p) \in G^{1,\mu}(D(s/2, r/2) \times \Pi_*)$, and for any $\beta \in \mathbb{Z}_+$,

$$\left\| W_0 \partial_p^\beta (V_* - id) \right\|_{D(s/2,r/2)\times\Pi_*} \leq cM^\beta \beta!^{\tau+2+\delta} \gamma^{1/2}, \tag{1.13}$$

where $M = 2^{\tau+2+\delta}(T + 1)\ (\tau + 1 + \delta)^{\tau+1+\delta}/\pi\alpha$, the constant c depends on n, τ, and δ. Under these transformations, the mapping (1.3) *is transformed to*

$$\begin{gathered} \xi_1 = \xi + \omega_*(p) + f_*(\xi, \eta; p), \\ \eta_1 = \eta + g_*(\xi, \eta; p), \end{gathered} \tag{1.14}$$

where $f_* = O(\eta)$, $g_* = O(\eta^2)$ at $\eta = 0$. Thus, for any $p \in \Pi_*$, the mapping (1.3) *has an invariant curve Γ such that the induced mapping on this curve is the translation $\xi_1 = \xi + \omega_*(p)$, whose frequency $\omega_*(p)$ satisfies that*

$$\left| \partial_p^\beta (\omega_*(p) - \omega(p)) \right| \leq c\alpha M^\beta \beta!^{\tau+2+\delta} \gamma^{1/2} s^{\tau+2}, \quad \forall \beta \in \mathbb{Z}_+, \tag{1.15}$$

$$\left| \frac{k\omega_*(p)}{2\pi} - l \right| \geq \frac{\alpha}{2|k|^\tau}, \quad \forall (k, l) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0, 0\}. \tag{1.16}$$

Moreover, one has meas $(\Pi \setminus \Pi_*) \leq c\alpha^{1/m}$.

Remark 1.5. From Theorem 1.4, we can see that for any $\mu > \tau + 2$, if ϵ is sufficiently small, the family of invariant curves is G^μ-smooth in the parameters. The Gevrey index $\mu = \tau + 2 + \delta$ should be optimal.

Remark 1.6. The derivatives in (1.13) and (1.15) should be understood in the sense of Whitney [16]. In fact, the estimates (1.13) and (1.15) also hold in a neighborhood of Π_* with the same Gevrey index.

2. Proof of the Main Results

In this section, we will prove our Theorem 1.4. But in the case of weaker non-degeneracy condition, the previous methods in [1, 2] are not valid and the difficulty is how to control the parameters in small divisors. We use an improved KAM iteration carrying some parameters to obtain the existence and Gevrey regularity of invariant curves of analytic reversible mappings. This method is outlined in the paper [7] by Pöschel and adapted to Gevrey classes in [13] by Popov. We also extend the method of Liu [1, 2].

KAM step

The KAM step can be summarized in the following lemma.

Lemma 2.1. *Consider the following real analytic mapping A:*

$$\begin{aligned} x_1 &= x + \omega(p) + f(x, z; p), \\ z_1 &= z + g(x, z; p), \end{aligned} \tag{2.1}$$

on $D(s, r) \times \Pi_h$. Suppose the mapping is reversible with respect to the involution $R : (x, z) \to (-x, z)$, that is, $ARA = R$. Let $0 < E < 1$, $0 < \rho < s/5$, and $K > 0$ such that $e^{-K\rho} = E$. Suppose $\forall p \in \Pi$, the following small divisors condition holds:

$$\left| \frac{k\omega(p)}{2\pi} - l \right| \geq \frac{\alpha}{|k|^{\tau}}, \quad \forall (k, l) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0, 0\},\ 0 < |k| \leq K. \tag{2.2}$$

Let

$$\max_{p \in \Pi_h} \left| \omega'(p) \right| \leq T, \qquad h = \frac{\pi\alpha}{TK^{\tau+1}}. \tag{2.3}$$

Suppose that

$$\|f\|_{s,r;h} + \frac{1}{r}\|g\|_{s,r;h} \leq \epsilon = \alpha\rho^{\tau+2}E, \tag{2.4}$$

where the norm $\|\cdot\|_{s,r;h}$ indicates $\|\cdot\|_{D(s,r)\times\Pi_h}$ for simplicity. Then, for any $p \in \Pi_h$, there exists a transformation U:

$$\begin{aligned} x &= \xi + u(\xi; p), \\ z &= \eta + v(\xi, \eta; p), \end{aligned} \tag{2.5}$$

which is affine in η, such that the mapping A is transformed to $A_+ = U^{-1}AU$:

$$\begin{aligned} \xi_1 &= \xi + \omega_+(p) + f_+(\xi, \eta; p), \\ \eta_1 &= \eta + g_+(\xi, \eta; p), \end{aligned} \tag{2.6}$$

where the new perturbation satisfies

$$\|f_+\|_{s_+,r_+;h} + \frac{1}{r_+}\|g_+\|_{s_+,r_+;h} \le \epsilon_+ = \alpha_+ \rho_+^{\tau+2} E_+, \tag{2.7}$$

with

$$s_+ = s - 5\rho, \quad \rho_+ = \sigma\rho, \quad \mu = \sqrt{E}, \quad r_+ = \mu r, \quad E_+ = cE^{3/2}, \quad \frac{\alpha}{2} \le \alpha_+ \le \alpha, \tag{2.8}$$

where σ is defined in Theorem 1.4. Moreover, one has

$$\left|\omega_+(p) - \omega(p)\right| \le \epsilon, \quad \forall p \in \Pi_h. \tag{2.9}$$

Let $\alpha_+ = \alpha - (\epsilon/2\pi)K^{\tau+1}$, and denote

$$R_k^+ = \left\{ p \in \Pi \mid \left| \frac{k\omega_+(p)}{2\pi} - l \right| < \frac{\alpha_+}{|k|^\tau}, \forall K < |k| \le K_+ \right\} \tag{2.10}$$

and $\Pi_+ = \Pi \setminus R_k^+$. Then, $\forall p \in \Pi_+$, it follows that

$$\left| \frac{k\omega_+(p)}{2\pi} - l \right| \ge \frac{\alpha_+}{|k|^\tau}, \quad \forall (k,l) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0,0\},\ 0 < |k| \le K_+, \tag{2.11}$$

where $K_+ > 0$ such that $e^{-K_+\rho_+} = E_+$. Let

$$T_+ = T + \frac{3\epsilon}{h}, \qquad h_+ = \frac{\pi\alpha_+}{T_+ K_+^{\tau+1}}. \tag{2.12}$$

If $h_+ \le 2h/3$, then $\max_{p\in\Pi_{h_+}} |\omega_+'(p)| \le T_+$. Moreover, one has

$$\|f_+\|_{s_+,r_+;h_+} + \frac{1}{r_+}\|g_+\|_{s_+,r_+;h_+} \le \epsilon_+. \tag{2.13}$$

Thus, the above result also holds for A_+ in place of A.

Proof of Lemma 2.1. The above lemma is actually one KAM step. We divide the KAM step into several pats.

(A) Truncation

Let $Q_f = f(x,0;p)$, $Q_g = g(x,0;p) + g_z(x,0;p)z$. It follows that $\|Q_f\|_{s,r;h} \le \epsilon$, $\|Q_g\|_{s,r;h} \le 2r\epsilon$. Write $Q_f = \sum_{k\in\mathbb{Z}} Q_{fk}(p)e^{ikx}$, $Q_g = \sum_{k\in\mathbb{Z}} Q_{gk}(z;p)e^{ikx}$, and let

$$R_f = \sum_{|k|\le K} Q_{fk}(p)e^{ikx}, \qquad R_g = \sum_{|k|\le K} Q_{gk}(z;p)e^{ikx}. \tag{2.14}$$

By the definition of norm, we have

$$\|Q_f - R_f\|_{s-\rho,r;h} \le ce^{-K\rho}\epsilon, \qquad \|Q_g - R_g\|_{s-\rho,r;h} \le ce^{-K\rho} r\epsilon. \tag{2.15}$$

(B) Construction of the Transformation

As in [1–3], for a reversible mapping, if the change of variables commutes with the involution R, then the transformed mapping is also reversible with respect to the same involution R. If the change of variables $U : (\xi, \eta) \to (x, z)$ is of the form

$$\begin{aligned} x &= \xi + u(\xi), \\ z &= \eta + v(\xi, \eta), \end{aligned} \tag{2.16}$$

then from the equality $RU = UR$, it follows that

$$\begin{aligned} u(-\xi) &= -u(\xi), \\ v(-\xi, \eta) &= v(\xi, \eta). \end{aligned} \tag{2.17}$$

In this case, the transformed mapping $U^{-1}AU$ of A is also reversible with respect to the involution $R : (\xi, \eta) \to (-\xi, \eta)$.

In the following, we will determine the unknown functions u and v to satisfy the condition (2.17) in order to guarantee that the transformed mapping $U^{-1}AU$ is also reversible.

We may solve u and v from the following equations:

$$\begin{aligned} u(\xi + \omega(p)) - u(\xi) &= R_f(\xi) - [R_f(\xi)], \\ v(\xi + \omega(p), \eta) - v(\xi, \eta) &= R_g(\xi, \eta) - [R_g(\xi, \eta)], \end{aligned} \tag{2.18}$$

where $[\cdot]$ denotes the mean value of a function over the angular variable ξ. Indeed, we can solve these functions from the above equations. But the problem is that such functions u and v do not, in general, satisfy the condition (2.17), that is, the transformed mapping $U^{-1}AU$ is no longer a reversible mapping with respect to R. Therefore, we cannot use the above equations to determine the functions u and v.

Instead of solving the above equations (2.18), we may find these functions u and v from the following modified equations:

$$\begin{aligned} u(\xi + \omega(p)) - u(\xi) &= \tilde{f}(\xi), \\ v(\xi + \omega(p), \eta) - v(\xi, \eta) &= \tilde{g}(\xi, \eta), \end{aligned} \tag{2.19}$$

with

$$\begin{aligned}\widetilde{f}(\xi) &= \frac{1}{2}\left(R_f(\xi) - [R_f(\xi)] + R_f(-\xi-\omega(p)) - [R_f(-\xi-\omega(p))]\right),\\ \widetilde{g}(\xi,\eta) &= \frac{1}{2}\left(R_g(\xi,\eta) - R_g(-\xi-\omega(p),\eta)\right),\end{aligned} \tag{2.20}$$

where $[\cdot]$ denotes the mean value of a function over the angular variable ξ.

It is easy to verify that $\widetilde{f}(-\xi-\omega(p)) = \widetilde{f}(\xi)$ and $\widetilde{g}(-\xi-\omega(p),\eta) = -\widetilde{g}(\xi,\eta)$. So, by Lemma A.1, the functions u and v meet the condition (2.17). In this case, the transformed mapping $U^{-1}AU$ is also reversible with respect to the involution $R:(\xi,\eta)\to(-\xi,\eta)$.

Because the right hand sides of (2.19) have the mean value zero, we can solve u, v from (2.19). But the difference equations introduce small divisors. By the definition of Π_h, it follows that $\forall p\in\Pi_h$,

$$\left|\frac{k\omega(p)}{2\pi} - l\right| \geq \frac{\alpha}{2|k|^{\tau}}, \quad \forall(k,l)\in\mathbb{Z}\times\mathbb{Z}\setminus\{0,0\},\ 0<|k|\leq K. \tag{2.21}$$

Let $\widetilde{f}_k$, $\widetilde{g}_k$ be Fourier coefficients of $\widetilde{f}$ and $\widetilde{g}$. Then, we have

$$u_k = \frac{\widetilde{f}_k}{e^{ik\omega(p)}-1}, \quad v_k = \frac{\widetilde{g}_k}{e^{ik\omega(p)}-1}, \quad 0<|k|\leq K, \tag{2.22}$$

and $u_k=0$, $v_k=0$ for $k=0$ or $|k|>K$. Moreover, v is affine in η, u is independent of η.

(C) Estimates of the Transformation

By the definition of norm, we have

$$\left\|\widetilde{f}\right\|_{s-\rho,r;h} \leq c\epsilon, \quad \|\widetilde{g}\|_{s-\rho,r;h} \leq cr\epsilon. \tag{2.23}$$

By Lemma A.1, it follows that

$$\|u\|_{s-2\rho,r;h} \leq \frac{c\epsilon}{\alpha\rho^{\tau+1}}, \quad \|v\|_{s-2\rho,r;h} \leq \frac{cr\epsilon}{\alpha\rho^{\tau+1}}. \tag{2.24}$$

Using Cauchy's estimate on the derivatives of u, v, we obtain

$$\begin{aligned}&\|u_\xi\|_{s-3\rho,r/2;h} \leq \frac{c\epsilon}{\alpha\rho^{\tau+2}},\\ &\|v_\xi\|_{s-3\rho,r/2;h} < \frac{cr\epsilon}{\alpha\rho^{\tau+2}}, \quad \|v_\eta\|_{s-3\rho,r/2;h} < \frac{c\epsilon}{\alpha\rho^{\tau+1}}.\end{aligned} \tag{2.25}$$

In the same way as in [1, 2, 4], we can verify that $U^{-1}AU$ is well defined in $D(s-5\rho, \mu r)$, $0<\mu\leq 1/8$. Moreover, according to (2.24)–(2.25), we have

$$\|W_0(U-id)\|_{D_{(s-5\rho,\mu r)}\times\Pi_h} \leq \frac{c\epsilon}{\alpha\rho^{\tau+2}},$$
$$\left\|W_0(DU-Id)W_0^{-1}\right\|_{D_{(s-5\rho,\mu r)}\times\Pi_h} \leq \frac{c\epsilon}{\alpha\rho^{\tau+2}}, \tag{2.26}$$

where $\|\cdot\|$ denotes the maximum of the absolute value of the elements of a matrix, $W_0 = \mathrm{diag}(\rho_0^{-1}, r_0^{-1})$, DU denotes the Jacobian matrix with respect to (ξ,η).

(D) Estimates of the New Perturbation

Let $\alpha_+ = \alpha - (\epsilon/2\pi)K^{\tau+1}$. We have $|k\omega_+(p)/2\pi - l| \geq \alpha_+/|k|^\tau$, $\forall p\in\Pi$, $\forall 0<|k|\leq K$. Then, by the definition of R_k^+, it follows that (2.11) holds. Thus, the small divisors condition for the next step holds.

Let $R_f(p) = [R_f(\xi;p)]$, then we have $\|R_f(p)\| \leq \epsilon = \alpha\rho^{\tau+2}E$. Due to $U^{-1}AU = A_+$, we have

$$f_+(\xi,\eta) = u(\xi) - u(\xi_1) - R_f(p) + f(\xi+u, \eta+v). \tag{2.27}$$

By the first difference equation of (2.19), we have

$$f_+ = u(\xi+\omega(p)) - u(\xi_1) + f(\xi+u,\eta+v) - \tilde{f}(\xi) - R_f(p). \tag{2.28}$$

From the reversibility of A, it follows that

$$f(-x-\omega(p)-f, z+g) - f(x,z) = 0,$$
$$g(-x-\omega(p)-f, z+g) + g(x,z) = 0. \tag{2.29}$$

Hence, we have

$$\begin{aligned} f(\xi,\eta) - \tilde{f}(\xi) - R_f(p) &= \frac{1}{2}\left(f(\xi,\eta) - R_f(\xi) + f(\xi,\eta) - R_f(-\xi-\omega(p))\right) \\ &= \frac{1}{2}\left(f(\xi,\eta) - R_f(\xi) + f(-\xi-\omega(p),\eta) - R_f(-\xi-\omega(p))\right. \\ &\quad \left. - f(-\xi-\omega(p),\eta) + f(-\xi-\omega(p)-f,\eta+g)\right), \end{aligned} \tag{2.30}$$

which yields that

$$\left\|f(\xi,\eta) - \tilde{f}(\xi) - R_f(p)\right\| \leq c\mu\epsilon + ce^{-K\rho}\epsilon + \frac{2\epsilon^2}{\rho}. \tag{2.31}$$

By (2.15) and (2.24)–(2.25), the following estimate of f_+ holds:

$$\begin{aligned} \|f_+\| &\le \|u_\xi\|\cdot\|R_f(p)+f_+\| + \|f_\xi\|\cdot\|u\| + \|f_\eta\|\cdot\|v\| + c\mu\epsilon + ce^{-K\rho}\epsilon + \frac{2\epsilon^2}{\rho} \\ &\le \frac{c\epsilon}{\alpha\rho^{\tau+2}}\|f_+\| + \frac{c\epsilon^2}{\alpha\rho^{\tau+2}} + c\mu\epsilon + ce^{-K\rho}\epsilon. \end{aligned} \tag{2.32}$$

Similarly, for g_+, we get

$$g_+ = v(\xi+\omega(p),\eta) - v(\xi_1,\eta_1) + g(\xi+u,\eta+v) - \widetilde{g}(\xi,\eta), \tag{2.33}$$

$$\frac{1}{r_+}\|g_+\| \le \frac{c\epsilon}{\alpha\mu\rho^{\tau+2}}\|f_+\| + \frac{c\epsilon}{\alpha\rho^{\tau+1}}\frac{\|g_+\|}{r_+} + \frac{c\epsilon^2}{\alpha\mu\rho^{\tau+2}} + c\mu\epsilon + \frac{ce^{-K\rho}\epsilon}{\mu}. \tag{2.34}$$

If ϵ is sufficiently small such that

$$\frac{c\epsilon}{\alpha\mu\rho^{\tau+2}} < \frac{1}{2}, \tag{2.35}$$

then combing with (2.32) and (2.34), we have

$$\|f_+\|_{s_+,r_+;h} + \frac{1}{r_+}\|g_+\|_{s_+,r_+;h} \le \frac{c\epsilon^2}{\alpha\mu\rho^{\tau+2}} + c\mu\epsilon + \frac{ce^{-K\rho}\epsilon}{\mu}. \tag{2.36}$$

Suppose $h_+ \le (2/3)h$. Then, by Cauchy's estimates, we have

$$\left|\omega'_+(p) - \omega'(p)\right| \le \frac{3\epsilon}{h}, \quad \forall p \in \Pi_{h_+}. \tag{2.37}$$

Let $T_+ = T + 3\epsilon/h$. Then, $\max_{p\in\Pi_{h_+}}|\omega'_+(p)| \le T_+$.

Moreover, by the definition of ρ_+ and E_+, we have

$$\|f_+\|_{s_+,r_+;h_+} + \frac{1}{r_+}\|g_+\|_{s_+,r_+;h_+} \le \frac{c\epsilon^2}{\alpha\mu\rho^{\tau+2}} \le c\alpha_+\rho_+^{\tau+2}E^{3/2} = \alpha_+\rho_+^{\tau+2}E_+. \tag{2.38}$$

Thus, this ends the proof of Lemma 2.1. □

Setting the Parameters and Iteration

Now, we choose some suitable parameters so that the above iteration can go on infinitely.

At the initial step, let $\rho_0 = (1-\sigma)s/10$, $r_0 = r$, $\epsilon_0 = \alpha_0\rho_0^{\tau+2}E_0$. Let K_0 satisfy $e^{-K_0\rho_0} = E_0$, $\alpha_0 = \alpha > 0$, $\omega_0 = \omega$, $T_0 = T = \max_{p\in\Pi_h}|\omega'(p)|$. Denote

$$\Pi_0 = \left\{ p \in \Pi \mid \left|\frac{k\omega(p)}{2\pi} - l\right| \ge \frac{\alpha}{|k|^\tau}, \forall 0 < |k| \le K_0 \right\}. \tag{2.39}$$

Choose $h = \alpha^{1/m}$. Note that this choice for h is only for measure estimate for parameters and has no conflict with the assumption in Theorem 1.4, since we can use a smaller h.

Let $h_0 = \pi\alpha_0/T_0 K_0^{\tau+1} \leq h$ and $\mu_0 = E_0^{1/2}$. Assume the above parameters are all well defined for j. Then, we define $\rho_{j+1} = \sigma\rho_j$, $r_{j+1} = \mu_j r_j$ and $E_{j+1} = cE_j^{3/2}$, $\alpha_{j+1} = \alpha_j - (\epsilon_j/2\pi)K_j^{\tau+1}$. Define ϵ_{j+1}, μ_{j+1}, K_{j+1}, and h_{j+1} in the same way as the previous step.

Let

$$\Pi_j = \left\{ p \in \Pi_{j-1} \mid \left| \frac{k\omega_j(p)}{2\pi} - l \right| \geq \frac{\alpha_j}{|k|^\tau},\ \forall K_{j-1} < |k| \leq K_j \right\}. \tag{2.40}$$

Denote $\Pi_{h_j} = \{\xi \in \mathbb{C} \mid \operatorname{dist}(\xi, \Pi_j) \leq h_j\}$ and $D_j = D(s_j, r_j)$ for simplicity. By the iteration lemma, we have a sequence of transformations U_j:

$$\begin{aligned} x &= \xi + u_j(\xi; p), \\ z &= \eta + v_j(\xi, \eta; p), \end{aligned} \tag{2.41}$$

such that for any $p \in \Pi_{h_j}$, $U_j : D_j \to D_{j-1}$, satisfying

$$\begin{aligned} \left\| W_j(U_j - id) \right\|_{D_j \times \Pi_{h_j}} &\leq \frac{c\epsilon_j}{\alpha_j \rho_j^{\tau+2}}, \\ \left\| W_j(DU_j - Id)W_j^{-1} \right\|_{D_j \times \Pi_{h_j}} &\leq \frac{c\epsilon_j}{\alpha_j \rho_j^{\tau+2}}, \end{aligned} \tag{2.42}$$

where $W_j = \operatorname{diag}(\rho_j^{-1}, r_j^{-1})$, DU_j denotes the Jacobian matrix with respect to (ξ, η).

Thus, the transformation $V_j = U_0 \circ U_1 \cdots U_j$ is well defined in $D_j \times \Pi_{h_j}$ and is seen to take A_0 into

$$A_j = V_j^{-1} A_0 V_j. \tag{2.43}$$

More precisely, if we write A_0 as

$$\begin{aligned} x_1 &= x + \omega(p) + f(x, z; p), \\ z_1 &= z + g(x, z; p) \end{aligned} \tag{2.44}$$

and express V_j in the form

$$\begin{aligned} x &= \xi + p_j(\xi; p), \\ z &= \eta + q_j(\xi, \eta; p), \end{aligned} \tag{2.45}$$

then A_0 is transformed into A_j:

$$\begin{aligned}\xi_1 &= \xi + \omega_j(p) + f_j(\xi, \eta; p), \\ \eta_1 &= \eta + g_j(\xi, \eta; p),\end{aligned} \tag{2.46}$$

satisfying

$$\begin{aligned}\|f_j\|_{D_j\times\Pi_{h_j}} + \frac{1}{r_j}\|g_j\|_{D_j\times\Pi_{h_j}} &\le \epsilon_j = \alpha_j \rho_j^{\tau+2} E_j, \\ \left|\omega_{j+1}(p) - \omega_j(p)\right|_{D_j\times\Pi_{h_j}} &\le c\epsilon_j.\end{aligned} \tag{2.47}$$

In the following, we will check the assumptions in the iteration lemma to ensure that KAM step is valid for all $j \ge 0$.

Since $E_{j+1} = cE_j^{3/2}$ and $x_j = K_j\rho_j = -\ln E_j$, if E_0 is sufficiently small such that $-\ln c/\ln E_j \le 3(1-\sigma)/2$, it follows that $3/2 \le K_{j+1}/K_j \le 3/2\sigma$. Thus $h_{j+1} \le (2/3)h_j$.

By the definition of α_j, we have

$$\alpha_{j+1} = \alpha_j - \frac{\epsilon_j}{2\pi}K_j^{\tau+1} = \alpha_j\left(1 - \frac{1}{2\pi}x_j^{\tau+2}e^{-x_j}\right). \tag{2.48}$$

If E_0 is sufficiently small such that

$$\prod_{j=0}^{\infty}\left(1 - \frac{1}{2\pi}x_j^{\tau+2}e^{-x_j}\right) = 1 - O\left(\frac{1}{x_0}\right) \ge \frac{1}{2}, \tag{2.49}$$

then we obtain $\alpha_0/2 \le \alpha_j \le \alpha_0$ and so $\alpha_j/2 \le \alpha_{j+1} \le \alpha_j$, $\forall j \ge 0$.

Obviously, if E_0 is sufficiently small, the assumption (2.35) holds.

By $\sigma = (2/3)^{\delta/(\tau+1+\delta)}$, it is easy to see that $K_{j+1}^{\delta}\rho_{j+1}^{\tau+1+\delta} \ge K_j^{\delta}\rho_j^{\tau+1+\delta}$. If E_0 is sufficiently small and so x_0 is sufficiently large such that $K_0^{\delta}\rho_0^{\tau+1+\delta} = x_0^{\delta}\rho_0^{\tau+1} \ge 1$, then we have $K_j^{\delta}\rho_j^{\tau+1+\delta} \ge 1$, $\forall j \ge 0$.

Suppose $\max_{p\in\Pi_{h_j}}|\omega_j'(p)| \le T_j$. Let $T_{j+1} = T_j + 3\epsilon_j/h_j$. Then, we have $\max_{p\in\Pi_{h_{j+1}}}|\omega_{j+1}'(p)| \le T_{j+1}$.

By iteration, $T_{j+1} = T_0 + \sum_{i=0}^{j}(3\epsilon_i/h_i) \le T_0 + \sum_{j=0}^{\infty}(3T_j/\pi)x_j^{\tau+2}e^{-x_j}$. Suppose $T_j \le T+1$, then we have $\sum_{j=0}^{\infty}(3T_j/\pi)x_j^{\tau+2}e^{-x_j} \le (3(T+1)/\pi)\sum_{j=0}^{\infty}x_j^{\tau+2}e^{-x_j}$. If E_0 is sufficiently small such that $\sum_{j=0}^{\infty}x_j^{\tau+2}e^{-x_j} \le \pi/3(T+1)$, then $T_0 \le T_{j+1} \le T_0 + 1$.

Convergence of Iteration in Gevrey Space $G^{1,\tau+2+\delta}(D(s/2,r/2)\times\Pi_*)$

Now, we prove convergence of KAM iteration. Let $V_j = U_0\circ U_1\cdots\circ U_j : D_j\times\Pi_{h_j} \to D_0\times\Pi_{h_0}$, and write V_j in the form

$$\begin{aligned} x &= \xi + p_j(\xi;p), \\ z &= \eta + q_j(\xi,\eta;p). \end{aligned} \tag{2.50}$$

In the same way as in [4, 7], we have

$$\begin{aligned} \left\|W_0(V_j - V_{j-1})\right\|_{D_j\times\Pi_{h_j}} &\le \frac{c\epsilon_j}{\alpha_j\rho_j^{\tau+2}}, \\ \left\|W_0 D(V_j - V_{j-1})\right\|_{D_j\times\Pi_{h_j}} &\le \frac{c\epsilon_j}{\alpha_j\rho_j^{\tau+2}}, \end{aligned} \tag{2.51}$$

where $\|\cdot\|$ denotes the maximum of the absolute value of the elements of a matrix.

By Cauchy's estimate we have

$$\left\|W_0\partial_p^\beta(V_j - V_{j-1})\right\|_{D_j\times\Pi_j} \le \frac{cE_j\beta!}{h_j^\beta}, \tag{2.52}$$

$$\begin{aligned} \left\|W_0\partial_p^\beta D(V_j - V_{j-1})\right\|_{D_j\times\Pi_j} &\le \frac{cE_j\beta!}{h_j^\beta}, \\ \left|\partial_p^\beta(\omega_{j+1}(p) - \omega_j(p))\right|_{D_j\times\Pi_j} &\le \frac{c\epsilon_j\beta!}{h_j^\beta}. \end{aligned} \tag{2.53}$$

Let $\mathcal{P}_{j,\beta} = cE_j\beta!/h_j^\beta$ and $\mathcal{Q}_{j,\beta} = c\epsilon_j\beta!/h_j^\beta$. By $K_j^\delta\rho_j^{\tau+1+\delta}\ge 1$ and the definition of h_j, we have

$$\begin{aligned} \mathcal{P}_{j,\beta} &\le c\left(\frac{2(T_0+1)}{\pi\alpha}\right)^\beta x_j^{(\tau+1+\delta)\beta}\beta! e^{-x_j}, \\ &\le c\left(\frac{2(T_0+1)}{\pi\alpha}\right)^\beta \left(x_j^\beta e^{-\kappa x_j}\right)^{\tau+1+\delta}\beta! e^{-x_j/2}, \end{aligned} \tag{2.54}$$

where $\kappa = 1/2(\tau+1+\delta)$. It is easy to see that

$$x_j^\beta e^{-\kappa x_j} \le \beta!\kappa^{-\beta}. \tag{2.55}$$

Thus, we have

$$\mathcal{P}_{j,\beta} \le cM^{\beta}\beta!^{\tau+2+\delta}E_j^{1/2}, \tag{2.56}$$

where $M = 2^{\tau+2+\delta}(T_0+1)(\tau+1+\delta)^{\tau+1+\delta}/\pi\alpha$, c depends on n, τ, and δ.

In the same way, we have

$$\mathcal{Q}_{j,\beta} \le c\alpha M^{\beta}\beta!^{\tau+2+\delta}E_j^{1/2}\rho_j^{\tau+2}. \tag{2.57}$$

Note that $s_j \to s/2$, $r_j \to 0$, $h_j \to 0$, as $j \to \infty$. Let $D_* = D(s/2,0)$, $\Pi_* = \cap_{j\ge 0}\Pi_j$ and $V_* = \lim_{j\to\infty}V_j$. Since V_j is affine in η, these estimates (2.52)–(2.56) imply that $\partial_p^{\beta}V_j$ is uniformly convergent to $\partial_p^{\beta}V_*$ on $D(s/2,r/2)$ and satisfies

$$\left\| W_0\partial_p^{\beta}(V_* - id)\right\|_{D(s/2,r/2)\times\Pi_*} \le cM^{\beta}\beta!^{\tau+2+\delta}E_0^{1/2}. \tag{2.58}$$

Since $E_0 = (10/(1-\sigma))^{\tau+1}\gamma$, this proves (1.13).

Let $\omega_* = \lim_{j\to\infty}\omega_j$. It follows that

$$\left|\partial_p^{\beta}(\omega_*(p) - \omega(p))\right|_{\Pi_*} \le c\alpha M^{\beta}\beta!^{\tau+2+\delta}E_0^{1/2}\rho_0^{\tau+2}. \tag{2.59}$$

Moreover, we have $|(k\omega_*(p)/2\pi) - l| \ge \alpha_*/|k|^{\tau}$, $\forall (k,l) \in \mathbb{Z}\times\mathbb{Z}\setminus\{0,0\}$, $\forall p \in \Pi_*$, where $\alpha_* = \lim_{j\to\infty}\alpha_j$ with $\alpha/2 \le \alpha_* \le \alpha$. Thus (1.15) and (1.16) hold.

Whitney Extension in Gevrey Classes

In this section, we apply the Whitney extension theorem in Gevrey classes [13, 17, 18] to extend V_* as a Gevrey function of the same Gevrey index in a neighborhood of Π_*.

Denote $\mathcal{S}^j = V_j - V_{j-1}$, then for any positive integers β, γ, and $m \in \mathbb{Z}_+$ with $\beta \le m$, we denote

$$R_{p'}^{m}\left(\partial_p^{\beta}\mathcal{S}^j\right)(\xi,\eta,p) := \partial_p^{\beta}\mathcal{S}^j(\xi,\eta,p) - \sum_{\beta+\gamma\le m}\frac{(p-p')^{\gamma}\partial_p^{\beta+\gamma}\mathcal{S}^j(\xi,\eta,p')}{\gamma!}. \tag{2.60}$$

In order to apply the Whitney extension theorem in Gevrey classes for function V_*, we are going to estimate $R_{p'}^{m}(\partial_p^{\beta}\mathcal{S}^j)(\xi,0,p)$. First, we suppose that $|p-p'| \le h_j/8$, $p,p' \in \Pi_*$. Expanding in Π_j the analytic function $p \to \partial_p^{\beta}\mathcal{S}^j(\xi,0,p)$, $\xi \in \mathbb{T}^1$, in Taylor series with respect to p at p', and using the Cauchy estimate, we evaluate

$$L_{j,\beta}^{m} := \left\| W_0\left(R_{p'}^{m}\partial_p^{\beta}\mathcal{S}^j\right)(\xi,0,p)\right\|, \quad \xi \in \mathbb{T}^1,\ p,p' \in \Pi_*. \tag{2.61}$$

For $\beta \leq m+1$, we have

$$\frac{(\beta+\gamma)!}{\gamma!} \leq 2^{\beta+\gamma}\beta! \leq 2^{\beta+\gamma}\frac{(m+1)!}{(m-\beta+1)!}. \tag{2.62}$$

Then we obtain as above

$$\begin{aligned} L^m_{j,\beta} &\leq \sum_{\beta+\gamma\geq m+1} \frac{|p-p'|^{\gamma}\left\| W_0 \partial_p^{\beta+\gamma}\mathcal{S}^j(\xi,0,p')\right\|}{\gamma!} \\ &\leq \sum_{\beta+\gamma\geq m+1}\frac{c|p-p'|^{\gamma}E_j(\beta+\gamma)!2^{\beta+\gamma}}{\gamma! h_j^{\beta+\gamma}} \\ &\leq c(m+1)!\frac{|p-p'|^{m-\beta+1}}{(m-\beta+1)!}\frac{4^{m+1}E_j}{h_j^{m+1}}\sum_{\beta+\gamma\geq m+1}\left(4|p-p'|h_j^{-1}\right)^{\beta+\gamma-m-1} \end{aligned} \tag{2.63}$$

and we get

$$\begin{aligned} L^m_{j,\beta} &\leq c(m+1)!\frac{|p-p'|^{m-\beta+1}}{(m-\beta+1)!}\frac{4^{m+1}E_j}{h_j^{m+1}} \\ &\leq c(4M)^{m+1}\frac{|p-p'|^{m-\beta+1}}{(m-\beta+1)!}(m+1)!^{\tau+2+\delta}E_j^{1/2}, \end{aligned} \tag{2.64}$$

where $M = 2^{\tau+2+\delta}(T_0+1)(\tau+1+\delta)^{\tau+1+\delta}/\pi\alpha$, c depends on n, τ, and δ. Similarly, for $|p-p'| \geq h_j/8$, we obtain the same inequality.

Let $\Pi_* = \cap_{j\geq 0}\Pi_j$ and $V_* = \lim_{j\to\infty}V_j$. According to (2.64), the limit $R^m_{p'}(\partial_p^{\beta}\mathcal{S}^j)(\xi,0,p)$ satisfies that

$$\left\| W_0 R^m_{p'}\partial_p^{\beta}(V_*-id)\right\|_{D(s/2,r/2)\times\Pi_*} \leq c(4M)^{m+1}\frac{|p-p'|^{m-\beta+1}}{(m-\beta+1)!}(m+1)!^{\tau+2+\delta}E_0^{1/2}. \tag{2.65}$$

Since V_* satisfies (2.58) and (2.65), by Theorem 3.7 and Theorem 3.8 in [13], we can extend V_* as a Gevrey function of the same Gevrey index in a neighborhood of Π_*. Thus, by the definition of Gevrey function in a closed set, $V_*(x,z;p) \in G^{1,\mu}(D(s/2,r/2)\times\Pi_*)$, satisfies the estimate (1.13) and (1.15) in a neighborhood of Π_*.

Note that one can also use the inverse approximation lemma in [19] to prove the preceding Whitney extension for V_*.

Estimates of Measure for Parameters

Now we estimate the Lebesgue measure of the set Π_*, on which the small divisors condition holds in the KAM iteration. By the analyticity of $\omega(p)$ and $\omega^{(m)}(0) \neq 0$, $m > 1$, for almost all points in Π, $\omega^{(m)}(p) \neq 0$. Without loss of generality, we suppose $\omega^{(m)}(p) \neq 0$, $\forall p \in \Pi$. Then, by the KAM step, we have

$$\Pi \setminus \Pi_* = \bigcup_{j \geq 0} R_k^j, \tag{2.66}$$

where

$$R_k^j = \left\{ p \in \Pi_{j-1} \mid \left| \frac{k\omega_j(p)}{2\pi} - l \right| < \frac{\alpha_j}{|k|^{\tau}}, \forall K_{j-1} < |k| \leq K_j \right\} \tag{2.67}$$

with $K_{-1} = 0$.

By Lemma A.2, we have

$$\begin{aligned} \operatorname{meas}\left(R_k^j\right) &\leq c \sum_{K_{j-1} < |k| \leq K_j} \left(\frac{\alpha_j}{|k|^{\tau+1}} \right)^{1/m}, \\ &\leq c\alpha^{1/m} \sum_{K_{j-1} < |k| \leq K_j} \frac{1}{|k|^{\tau+1/m}}. \end{aligned} \tag{2.68}$$

Since $\tau > m - 1$, we have

$$\operatorname{meas}(\Pi \setminus \Pi_*) \leq c\alpha^{1/m} \sum_{0 \neq k \in \mathbb{Z}} \frac{1}{|k|^{(\tau+1)/m}} \leq c\alpha^{1/m}. \tag{2.69}$$

Appendix

A. Some Results on Difference Equation and Measure Estimate

In this section, we formulate some lemmas which have been used in the previous section. For detailed proofs, we refer to [1, 20].

In the construction of the transformation in Lemma 2.1, we will meet the following difference equation:

$$l(x + \omega) - l(x) = g(x). \tag{A.1}$$

Lemma A.1. *Suppose that $l(x)$, $g(x)$ are real analytic on $D(s)$, where $D(s) = \{x \in \mathbb{C}/2\pi\mathbb{Z} \mid |\operatorname{Im} x| \leq s\}$. Suppose ω satisfies the Diophantine condition $|(k\omega/2\pi) - l| \geq \alpha/|k|^\tau$, $\forall (k,l) \in \mathbb{Z}\times\mathbb{Z} \setminus \{0,0\}$, then for any $0 < s' < s$, the difference equation* (A.1) *has the unique solution $l(x) \in D(s')$ satisfying*

$$\|l(x)\|_{s'} \leq \frac{c}{\alpha(s-s')^{\tau+1}} \|g(x)\|_s. \tag{A.2}$$

Moreover, if $g(-x-\omega) = g(x)$, then $l(x)$ is odd in x if $g(-x-\omega) = -g(x)$, $l(x)$ is even in x.

Lemma A.2. *Suppose $g(x)$ is mth differentiable function on the closure $\overline{I}$ of I, where $I \subset \mathbb{R}$ is an interval. Let $I_h = \{x \mid |g(x)| < h, x \in I\}$, $h > 0$. If $|g^{(m)}(x)| \geq d > 0$ for all $x \in I$, where d is a constant, then*

$$\operatorname{meas}(I_h) \leq ch^{1/m}, \tag{A.3}$$

where $c = 2(2 + 3 + \cdots + m + d^{-1})$.

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