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Research Article

Solutions to Fractional Differential Equations with Nonlocal Initial Condition in Banach Spaces

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A new existence and uniqueness theorem is given for solutions to differential equations involving the Caputo fractional derivative with nonlocal initial condition in Banach spaces. An application is also given.

1. Introduction

Fractional differential equations have played a significant role in physics, mechanics, chemistry, engineering, and so forth. In recent years, there are many papers dealing with the existence of solutions to various fractional differential equations; see, for example, [1–6].

In this paper, we discuss the existence of solutions to the nonlocal Cauchy problem for the following fractional differential equations in a Banach space E:

$$\begin{gathered} {}^cD^\alpha x(t) = f(t, x(t)), \quad 0 \le t \le 1, \\ x(0) = \int_0^1 g(s)x(s)ds, \end{gathered} \tag{1.1}$$

where ${}^cD^\alpha$ is the standard Caputo's derivative of order $0 < \alpha < 1$, $g \in L^1([0,1], R_+)$, $g(t) \in [0,1)$, and f is a given E-valued function.

2. Basic Lemmas

Let E be a real Banach space, and θ the zero element of E. Denote by $C([0,1],E)$ the Banach space of all continuous functions $x : [0,1] \rightarrow E$ with norm $\|x\|_c = \sup_{t\in[0,1]}\|x(t)\|$. Let $L^1([0,1],E)$ be the Banach space of measurable functions $x : [0,1] \rightarrow E$ which are Lebesgue integrable, equipped with the norm $\|x\|_{L^1} = \int_0^1\|x(s)\|ds$. Let $R_+ = [0,+\infty)$, $R^+ = (0,+\infty)$, and $\mu = \int_0^1 g(s)ds$. A function $x \in C([0,1],E)$ is called a solution of (1.1) if it satisfies (1.1).

Recall the following defenition

Definition 2.1. Let B be a bounded subset of a Banach space X. The *Kuratowski measure of noncompactness* of B is defined as

$$\alpha(B) := \inf\{\gamma > 0;\ B \text{ admits a finite cover by sets of diameter} \leq \gamma\}. \tag{2.1}$$

Clearly, $0 \leq \alpha(B) < \infty$. For details on properties of the measure, the reader is referred to [2].

Definition 2.2 (see [7, 8]). The fractional integral of order q with the lower limit t_0 for a function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}f(s)ds, \quad t > t_0,\ q > 0, \tag{2.2}$$

where Γ is the gamma function.

Definition 2.3 (see [7, 8]). Caputo's derivative of order q with the lower limit t_0 for a function f can be written as

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)}\int_{t_0}^{t}(t-s)^{n-q-1}f^{(n)}(s)ds, \quad t > t_0,\ q > 0,\ n = [q]+1. \tag{2.3}$$

Remark 2.4. Caputo's derivative of a constant is equal to θ.

Lemma 2.5 (see [7]). *Let $\alpha > 0$. Then we have*

$${}^cD^q\left(I^q f(t)\right) = f(t). \tag{2.4}$$

Lemma 2.6 (see [7]). *Let $\alpha > 0$ and $n = [\alpha] + 1$. Then*

$$I^{\alpha}\left({}^cD^{\alpha} f(t)\right) = f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}t^k. \tag{2.5}$$

Lemma 2.7 (see [9]). *If $H \subset C([0,1],E)$ is bounded and equicontinuous, then*

(a) $\alpha_C(H) = \alpha(H([0,1]))$;

(b) $\alpha(H([0,1])) = \max_{t\in[0,1]}\alpha(H(t))$, *where* $H([0,1]) = \{x(t) : x \in H, t \in [0,1]\}$.

Lemma 2.8.

$$\frac{Q(\tau)}{\Gamma(\alpha)} < e, \qquad \frac{\int_0^t (t-s)^{\alpha-1}ds}{\Gamma(\alpha)} < e, \tag{2.6}$$

where $Q(\tau) = \int_\tau^1 g(s)(s-\tau)^{\alpha-1}ds$, $t,\tau \in [0,1]$.

Proof. A direct computation shows

$$\begin{aligned}\frac{Q(\tau)}{\Gamma(\alpha)} &= \frac{\int_\tau^1 g(s)(s-\tau)^{\alpha-1}ds}{\int_0^\infty s^{\alpha-1}e^{-s}ds} \\ &< \frac{\int_\tau^1 (s-\tau)^{\alpha-1}ds}{\int_0^\infty s^{\alpha-1}e^{-s}ds} \\ &= \frac{\int_0^{1-\tau} s^{\alpha-1}ds}{\int_0^\infty s^{\alpha-1}e^{-s}ds} \\ &\le \frac{e\int_0^{1-\tau} s^{\alpha-1}e^{-s}ds}{\int_0^\infty s^{\alpha-1}e^{-s}ds} \\ &< e\end{aligned} \tag{2.7}$$

and

$$\frac{\int_0^t (t-s)^{\alpha-1}ds}{\Gamma(\alpha)} = \frac{\int_0^t s^{\alpha-1}ds}{\int_0^\infty s^{\alpha-1}e^{-s}ds} \le \frac{e\int_0^t s^{\alpha-1}e^{-s}ds}{\int_0^\infty s^{\alpha-1}e^{-s}ds} < e. \tag{2.8}$$

□

3. Main Results

(H_1) $f \in ([0,1]\times E, E)$, and there exist $M > 0$, $p_f(t) \le M$ for $t \in [0,1]$, $p_f \in L^1([0,1],R^+)$ such that $\|f(t,x)\| \le p_f(t)\|x\|$ for $t \in [0,1]$ and each $x \in E$.

(H_2) For any $t \in [0,1]$ and $R > 0$, $f(t,B_R) = \{f(t,x) : x \in B_R\}$ is relatively compact in E, where $B_R = \{x \in C([0,1],E), \|x\|_C \le R\}$and

$$\Lambda_1 = \frac{(2-\mu)e}{1-\mu}M < 1. \tag{3.1}$$

Lemma 3.1. *If* (H_1) *holds, then the problem* (1.1) *is equivalent to the following equation:*

$$x(t)=\frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau)f(\tau,x(\tau))d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))ds. \tag{3.2}$$

Proof. By Lemma 2.6 and (1.1), we have

$$x(t)=x(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))ds. \tag{3.3}$$

Therefore,

$$\begin{aligned}x(0)&=\int_0^1 g(s)x(s)ds\\&=\int_0^1 g(s)\left[x(0)+\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}f(\tau,x(\tau))d\tau\right]ds\\&=\int_0^1 g(s)ds\,x(0)+\frac{1}{\Gamma(\alpha)}\int_0^1 g(s)\int_0^s(s-\tau)^{\alpha-1}f(\tau,x(\tau))d\tau ds.\end{aligned} \tag{3.4}$$

So,

$$\begin{aligned}x(0)&=\frac{1}{\left(1-\int_0^1 g(s)ds\right)\Gamma(\alpha)}\int_0^1 g(s)\int_0^s(s-\tau)^{\alpha-1}f(\tau,x(\tau))d\tau ds\\&=\frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 f(\tau,x(\tau))\left[\int_\tau^1(s-\tau)^{\alpha-1}g(s)ds\right]d\tau\\&=\frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau)f(\tau,x(\tau))d\tau,\end{aligned} \tag{3.5}$$

and then

$$x(t)=\frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau)f(\tau,x(\tau))d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))ds. \tag{3.6}$$

Conversely, if x is a solution of (3.2), then for every $t \in [0,1]$, according to Remark 2.4 and Lemma 2.5, we have

$$\begin{aligned} {}^cD^\alpha x(t) &= {}^cD^\alpha\left[\frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau)f(\tau,x(\tau))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s))ds\right] \\ &= {}^cD^\alpha\left[\frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau)f(\tau,x(\tau))d\tau\right] \\ &\quad + {}^cD^\alpha\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s))ds\right] \\ &= \theta + {}^cD^\alpha\left(I^\alpha f(t,x(t))\right) \\ &= f(t,x(t)). \end{aligned} \tag{3.7}$$

It is obvious that $x(0) = \int_0^1 g(s)x(s)ds$. This completes the proof. □

Theorem 3.2. *If* (H_1) *and* (H_2) *hold, then the initial value problem* (1.1) *has at least one solution.*

Proof. Define operator $A : C([0,1],E) \to C([0,1],E)$, by

$$(Ax)(t) = \frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau)f(\tau,x(\tau))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s))ds. \tag{3.8}$$

Clearly, the fixed points of the operator A are solutions of problem (1.1).

It is obvious that B_R is closed, bounded, and convex.

Step 1. We prove that A is continuous.

Let

$$x_n, \overline{x} \in C([0,1],E), \quad \|x_n - \overline{x}\|_c \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.9}$$

Then $r = \sup_n \|x_n\|_C < \infty$ and $\|\overline{x}\|_C \le r$. For each $t \in [0,1]$,

$$\begin{aligned} \|(Ax_n)(t) - (A\overline{x})(t)\| \le{}& \frac{e}{1-\mu}\int_0^1 \left\|f(\tau,x_n(\tau)) - f(\tau,\overline{x}(\tau))\right\|d\tau \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)}\left\|f(s,x_n(s)) - f(s,\overline{x}(s))\right\|ds. \end{aligned} \tag{3.10}$$

It is clear that

$$\begin{gathered} f(t,x_n(t)) \longrightarrow f(t,\overline{x}(t)), \quad \text{as } n \longrightarrow \infty,\ t \in [0,1], \\ \left\|f(t,x_n(t)) - f(t,\overline{x}(t))\right\| \le 2Mr. \end{gathered} \tag{3.11}$$

It follows from (3.11) and the dominated convergence theorem that

$$\|(Ax_n)-(A\overline{x})\|_C \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.12}$$

Step 2. We prove that $A(B_R) \subset B_R$.

Let $x \in B_R$. Then for each $t \in [0,1]$, we have

$$\begin{aligned}
\|(Ax)(t)\| &\leq \frac{1}{1-\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(\alpha)}\|f(\tau,x(\tau))\|d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s))\|ds \\
&\leq \frac{1}{1-\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(\alpha)}p_f(\tau)\|x(\tau)\|d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p_f(s)\|x(s)\|ds \\
&\leq \left(\frac{e}{1-\mu}M + eM\right)\|x\|_C \\
&< R.
\end{aligned} \tag{3.13}$$

Step 3. We prove that $A(B_R)$ is equicontinuous.

Let $t_1, t_2 \in [0,1]$, $t_1 < t_2$, and $x \in B_R$. We deduce that

$$\begin{aligned}
&\|(Ax)(t_2)-(Ax)(t_1)\| \\
&\quad = \frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,x(s))ds - \int_0^{t_1}(t_1-s)^{\alpha-1}f(s,x(s))ds\right\| \\
&\quad \leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|\|f(s,x(s))\|ds \\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|f(s,x(s))\|ds \\
&\quad \leq \left[\int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right]\frac{MR}{\Gamma(\alpha)} \\
&\quad \leq \left[2(t_2-t_1)^{\alpha} + \left(t_2^{\alpha}-t_1^{\alpha}\right)\right]\frac{MR}{\Gamma(\alpha+1)}.
\end{aligned} \tag{3.14}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

Step 4. We prove that $A(B_R)$ is relatively compact.

Let $5 \subset B_R$ be arbitrarily given. Using the formula

$$\int_a^b y(t)dt \in (b-a)\overline{\mathrm{co}}\{y(t) : t \in [0,1]\} \tag{3.15}$$

for $y \in C([a,b],E)$ and (H_2), we obtain

$$
\begin{aligned}
\alpha((AV)(t)) &\le \alpha\left(\overline{\mathrm{co}}\left\{\frac{Q(s)}{(1-u)\Gamma(\alpha)}f(s,x(s)) : s \in [0,1],\ x \in V\right\}\right) \\
&\quad + \alpha\left(\overline{\mathrm{co}}\left\{\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s)) : s \in [0,t],\ t \in [0,1],\ x \in V\right\}\right) \\
&\le \left\{\frac{Q(s)}{(1-u)\Gamma(\alpha)}\alpha(f(s,V(s))) : s \in [0,1]\right\} \\
&\quad + \left\{\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\alpha(f(s,V(s))) : s \in [0,t],\ t \in [0,1]\right\} \\
&= 0.
\end{aligned}
\tag{3.16}
$$

It follows from (3.16) that $\alpha((AV)(t)) = 0 \ \ for \ \ t \in [0,1]$. This, together with Lemma 2.7, yields that

$$\alpha_C(AV) = 0. \tag{3.17}$$

From (3.17), we see that $A(B_R)$ is relatively compact. Hence, $A : B_R \to B_R$ is completely continuous. Finally, the Schauder fixed point theorem guarantees that A has a fixed point in B_R.

□

Theorem 3.3. *Besides the hypotheses of Theorem 3.2, we suppose that there exists a constant L such that*

$$0 < L < \Lambda_2, \tag{3.18}$$

$$\|f(t,u) - f(t,w)\| \le L\|u - w\|, \quad \textit{for every } u,w \in B_R, \tag{3.19}$$

where

$$\Lambda_2 = \frac{1-\mu}{(2-\mu)e}. \tag{3.20}$$

Then, the solution $x(t)$ of (1.1) *is unique in B_R.*

Proof. From Theorem 3.2, we know that there exists at least one solution $x(t)$ in B_R. We suppose to the contrary that there exist two different solutions $u(t)$ and $w(t)$ in B_R. It follows from (3.8) that

$$\begin{aligned}\|u(t)-w(t)\| &\leq \frac{e}{1-\mu}\int_0^1 \left\| f(\tau,u(\tau))-f(\tau,w(\tau))\right\| d\tau \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\| f(s,u(s))-f(s,w(s))\right\| ds \\ &\leq \frac{e}{1-\mu}\int_0^1 L\|u(\tau)-w(\tau)\| d\tau \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} L\|u(s)-w(s)\| ds.\end{aligned} \tag{3.21}$$

Therefore, we get

$$\|u-w\|_C \leq \frac{2-\mu}{1-\mu} eL\|u-w\|_C. \tag{3.22}$$

By (3.18), we obtain $\|u-w\|_C = 0$. So, the two solutions are identical in B_R. □

4. Example

Let

$$E = c_0 = \{x=(x_1,\ldots,x_n,\ldots) : x_n \longrightarrow 0\} \tag{4.1}$$

with the norm $\|x\| = \sup_n |x_n|$. Consider the following nonlocal Cauchy problem for the following fractional differential equation in E:

$$\begin{gathered}{}^{c}D^{\alpha} x_n(t) = \frac{1+t}{100n^2} x_n(t), \quad t\in[0,1],\ 0<\alpha<1, \\ x_n(0) = \int_0^1 \frac{1}{2} x_n(s) ds.\end{gathered} \tag{4.2}$$

Conclusion. Problem (4.2) has only one solution on $[0,1]$.

Proof. Write

$$\begin{gathered}f_n(t,x) = \frac{1+t}{100n^2} x_n, \qquad f=(f_1,\ldots,f_n,\ldots), \\ g(s) = \frac{1}{2}, \qquad p_f(t) = \frac{1+t}{100n}.\end{gathered} \tag{4.3}$$

Then it is clear that

$$
\begin{aligned}
&f \in C([0,1] \times E, E), \qquad p_f(t) \le \frac{1}{50} = M, \\
&p_f \in L([0,1], R^+), \qquad \|f(t,x)\| \le p_f \|x\|.
\end{aligned}
\tag{4.4}
$$

So, (H_1) is satisfied.

In the same way as in Example 3.2.1 in [9], we can prove that $f(t, B_R)$ is relatively compact in c_0.

By a direct computation, we get

$$
\Lambda_1 = \frac{(2-\mu)e}{1-\mu} M \le \frac{(2-\mu)e}{1-\mu} \frac{1}{50} = \frac{3e}{50} < 1. \tag{4.5}
$$

Hence, condition (H_2) is also satisfied.

Moreover, we have

$$
\left| f_n(t,u) - f_n(t,w) \right| = \left| \frac{1+t}{100n^2} u_n - \frac{1+t}{100n^2} w_n \right| \le \frac{1}{50} |u_n - w_n|, \tag{4.6}
$$

so

$$
\|f(t,u) - f(t,w)\| \le \frac{1}{50} \|u - w\|. \tag{4.7}
$$

Clearly,

$$
\Lambda_2 = \frac{1-\mu}{(2-\mu)e} = \frac{1-1/2}{3e/2} = \frac{1}{3e}. \tag{4.8}
$$

Therefore, $L = 1/50 < 1/3e$. Thus, our conclusion follows from Theorem 3.3. □

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